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HAUSDORFF INTERVAL VALUED FUZZY FILTERS

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ABSTRACT. The notion of Interval Valued Fuzzy Sets (IVF sets) was introduced by T. K. Mondal. In this paper a notion of IVF filter is introduced and studied. A new notion of Hausdorffness, which can not be defined in crisp theory of filters, is defined on IVF filters and their properties are studied.

Introduction

The concept of fuzzy sets was introduced by Zadeh [5]. The theory of fuzzy filters has been studied in [1], [3], et al. Interval valued fuzzy sets are introduced and studied in [4]. In this paper the notion of interval valued fuzzy filter (IVF filter) is defined and studied in Section 2 and a new notion of Hausdorffness on IVF filters is introduced and studied in Section 3.

1. Preliminaries

Here we give a brief review of preliminaries.

DEFINITION 1.1 ([4]). Let D be the set of all closed subintervals of the interval [0, 1]. Let X be a given nonempty set. A function $\tilde{\mu} : X \to D$ is called an interval valued fuzzy set (briefly IVF set) on X. Singletons $\{a\}$ in [0,1] are also considered as closed subintervals of the form $[a, a]$.

NOTE 1.1. For each $x \in X, \tilde{\mu}(x)$ is a closed interval $[\tilde{\mu}^L x, \tilde{\mu}^U(x)]$ and if $a \in [0, 1]$ then $\tilde{a}$ is an IVF set defined by $\tilde{a}(x) = [a, a]$ for all $x \in X$.

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DEFINITION 1.2 ([4]). Let $\tilde{\mu}$ be an IVF set on X. Then supp $\tilde{\mu} = \{x \in X \mid \tilde{\mu}^U(x) > 0\}$. An IVF point is an IVF set which has singleton support.

DEFINITION 1.3 ([4]). Let $\tilde{\mu}, \tilde{\nu} \in D^X$. Then

(i) $\tilde{\mu} = \tilde{\nu} \Rightarrow \tilde{\mu}^L(x) = \tilde{\nu}^L(x)$ and $\tilde{\mu}^U(x) = \tilde{\nu}^U(x)$ for all $x \in X$.
(ii) $\tilde{\mu} \subseteq \tilde{\nu} \Rightarrow \tilde{\mu}^L(x) \leq \tilde{\nu}^L(x)$ and $\tilde{\mu}^U(x) \leq \tilde{\nu}^U(x)$ for all $x \in X$.
(iii) The complement $\tilde{\mu}^c$ of $\tilde{\mu}$ is defined by $\tilde{\mu}^c(x) = [1-\tilde{\mu}^U(x), 1-\tilde{\mu}^L(x)]$ for all $x \in X$.

Let $\mathcal{A}$ be an indexed family of IVF sets. Then

(a) $(\vee_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu})(x) = [\sup_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu}^L(x), \sup_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu}^U(x)]$.
(b) $(\wedge_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu})(x) = [\inf_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu}^L(x), \inf_{\tilde{\mu} \in \mathcal{A}} \tilde{\mu}^U(x)]$.

NOTE 1.2 ([4]). Let $f : X_1 \to X_2$ be a map. Let $\tilde{\mu} \in D^{X_1}$ be an IVF set of X_1. Then $f(\tilde{\mu})$ is an IVF set of X_2 defined by $f(\tilde{\mu})(y) = [\sup_{x \in f^{-1}(y)} \tilde{\mu}^L(x), \sup_{x \in f^{-1}(y)} \tilde{\mu}^U(x)]$ for all $y \in X_2$ and $f(\tilde{\mu})(y) = [0, 0]$ if $f^{-1}(y)$ is empty. Let $\tilde{\nu}$ be an IVF set of X_2. Then $f^{-1}(\tilde{\nu})$ is an IVF set of X_1 defined by $f^{-1}(\tilde{\nu})(x) = [\tilde{\nu}^L(f(x)), \tilde{\nu}^U(f(x))]$ for all $x \in X_1$.

DEFINITION 1.4 ([4]). Let $f : X \to Y$ be a map. An IVF set $\tilde{\mu}$ is said to be f - invariant if $f(x) = f(y) \Rightarrow \tilde{\mu}(x) = \tilde{\mu}(y)$.

THEOREM 1.1 ([4]). *Let $f : X \to Y$ be a function. Then*

(i) $f^{-1}(\tilde{\nu}^c) = [f^{-1}(\tilde{\nu})]^c$ *for all* $\tilde{\nu} \in D^Y$,
(ii) $[f(\tilde{\mu})]^c \subseteq f(\tilde{\mu}^c)$ *for all* $\tilde{\mu} \in D^X$,
(iii) $\tilde{\nu}_1 \subseteq \tilde{\nu}_2 \Rightarrow f^{-1}(\tilde{\nu}_1) \subseteq f^{-1}(\tilde{\nu}_2)$, *where* $\tilde{\nu}_1, \tilde{\nu}_2 \in D^Y$,
(iv) $\tilde{\mu}_1 \subseteq \tilde{\mu}_2 \Rightarrow f(\tilde{\mu}_1) \subseteq f(\tilde{\mu}_2)$, *where* $\tilde{\mu}_1, \tilde{\mu}_2 \in D^X$,
(v) $f(f^{-1}(\tilde{\nu})) \subseteq \tilde{\nu}$ *for all* $\tilde{\nu} \in D^Y$,
(vi) $\tilde{\mu} \subseteq f^{-1}(f(\tilde{\mu}))$ *for all* $\tilde{\mu} \in D^X$,
(vii) *Let $f : X \to Y$ and $g : Y \to Z$ be maps. Then* $(g \circ f)^{-1}(\tilde{\gamma}) = f^{-1}(g^{-1}(\tilde{\gamma}))$ *for all* $\tilde{\gamma} \in D^Z$, *where $g \circ f$ is the composition of g and f.*

NOTATION. Let $X = \{x_1, x_2, \cdots, x_n\}$. Then $\tilde{\mu} = ([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n])$ denotes an IVF set of X such that $\tilde{\mu}^L(x_i) = a_i$ and $\tilde{\mu}^U(x_i) = b_i$ for all $i = 1, 2, \cdots, n$.

2. IVF filters

DEFINITION 2.1. A collection $\mathcal{F}$ of interval valued fuzzy sets is said to be a fuzzy filter of IVF sets or an IVF filter if
(1) $\tilde{0} \notin \mathcal{F}$
(2) If $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}$, then $\tilde{\mu} \wedge \tilde{\nu} \in \mathcal{F}$
(3) If $\tilde{\mu} \in \mathcal{F}$, and $\tilde{\nu} \geq \tilde{\mu}$, then $\tilde{\nu} \in \mathcal{F}$.

DEFINITION 2.2. A collection $\mathcal{B}$ of interval valued fuzzy sets is said to be a base for an IVF filter if
(1) $\tilde{0} \notin \mathcal{B}$.
(2) $\tilde{\mu}, \tilde{\nu} \in \mathcal{B} \Rightarrow \exists \tilde{\gamma} \in \mathcal{B}$ such that $\tilde{\gamma} \leq \tilde{\mu} \wedge \tilde{\nu}$.

DEFINITION 2.3. A collection $\mathcal{S}$ of IVF sets is said to be a subbase for an IVF filter $\mathcal{F}$ if the finite intersections of members of $\mathcal{S}$ forms a base for $\mathcal{F}$.

THEOREM 2.1. *Let $\mathcal{F}$ be an IVF filter on X. Let $Y \subseteq X$. Then $\mathcal{F}|Y$ is an IVF filter on Y, if no element of $\mathcal{F}$ vanishes on Y.*

Proof. (i) Since no element of $\mathcal{F}$ vanishes on Y, $\tilde{0} \notin \mathcal{F} \mid Y$.

(ii) Let $\tilde{\mu} \mid Y, \tilde{\nu} \mid Y \in \mathcal{F}| Y$. Clearly, $(\tilde{\mu} \wedge \tilde{\nu}) \mid Y = \tilde{\mu} \mid Y \wedge \tilde{\nu} \mid Y$. Since $\mathcal{F}$ is an IVF filter we have $(\tilde{\mu} \wedge \tilde{\nu}) \in \mathcal{F}$ and hence $\tilde{\mu} \mid Y \wedge \tilde{\nu} \mid Y \in \mathcal{F} \mid Y$.

(iii) Let $\tilde{\mu} \mid Y \in \mathcal{F} \mid Y$. Let $\tilde{\nu} \in D^Y$ such that $\tilde{\nu} \geq \tilde{\mu} \mid Y$. Choose $\tilde{\gamma} \in D^X$ such that $\tilde{\gamma}(z) \geq \tilde{\mu}(z)$ for all $z \notin Y$ and $\tilde{\gamma}(z) = \tilde{\nu}(z)$ for all $z \in Y$. Clearly, $\tilde{\gamma} \in D^X$ such that $\tilde{\gamma} \geq \tilde{\mu}$ with $\tilde{\gamma} \mid Y = \tilde{\nu}$. Since $\tilde{\mu} \in \mathcal{F}$ and $\mathcal{F}$ is an IVF filter, it follows $\tilde{\gamma} \in \mathcal{F}$ and hence $\tilde{\gamma} \mid Y = \tilde{\nu} \in \mathcal{F} \mid Y$. Therefore $\mathcal{F} \mid Y$ is an IVF filter on Y. □

THEOREM 2.2. (i) *Let A be any indexed family of IVF filters on X. Then*

(a) $\cap_{\mathcal{F} \in A}$ *is also an IVF filter.*
(b) $\cup_{\mathcal{F} \in A}$ *is also an IVF filter if A is directed family of IVF filters under inclusion and hence* $\cup_{\mathcal{F} \in A}$ *is also an IVF filter if A is totally ordered under inclusion.*

(ii) *Let $\mathcal{B}_1, \mathcal{B}_2$ be two IVF filter bases. Then $\mathcal{F}_{\mathcal{B}1} \subseteq \mathcal{F}_{\mathcal{B}2}$ if and only if for all $\tilde{\mu} \in \mathcal{B}_1$, there exists $\tilde{\nu} \in \mathcal{B}_2$ such that $\tilde{\nu} \leq \tilde{\mu}$.*

Proof is easy as in crisp setup.

THEOREM 2.3. *Let $f : X \to Y$ be a map. Let $\mathcal{F}$ be an IVF filter on X. Then $f(\mathcal{F}) = \{f(\tilde{\mu}) \mid \tilde{\mu} \in \mathcal{F}\}$ forms a base for an IVF filter on Y.*

Proof. We know that $f(\tilde{\mu})^L(z) = \sup_{x\in f^{-1}(z)} \tilde{\mu}^L(x)$ and $f(\tilde{\mu})^U(z) = \sup_{x\in f^{-1}(z)} \tilde{\mu}^U(x)$.

(i) Clearly, $f(\tilde{\mu})^U(y) \neq 0$ for at least one $y \in Y$. Otherwise, $\tilde{\mu}^U(x) = 0$ for all $x \in f^{-1}(t)$ for $t \in Y$. So $\tilde{\mu}^U(x) = 0$ for all $x \in X$ and hence $\tilde{0} \in \mathcal{F}$, which contradicts IVF filterness of $\mathcal{F}$. Therefore $f(\tilde{\mu}) \neq \tilde{0}$ and so $\tilde{0} \notin f(\mathcal{F})$.

(ii) Let $f(\tilde{\mu}), f(\tilde{\nu}) \in f(\mathcal{F})$. Since $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}$, it follows $\tilde{\mu} \wedge \tilde{\nu} \in \mathcal{F}$. We now claim that $f(\tilde{\mu} \wedge \tilde{\nu}) \leq f(\tilde{\mu}) \wedge f(\tilde{\nu})$. We have to prove that $f(\tilde{\mu} \wedge \tilde{\nu})(z) \leq f(\tilde{\mu}) \wedge f(\tilde{\nu})(z)$ for all $z \in Y$. First, observe that $f(\tilde{\mu} \wedge \tilde{\nu})^L(z) = \sup_{x\in f^{-1}(z)} \min(\tilde{\mu}^L(x), \tilde{\nu}^L(x))$. Clearly, $\min(\tilde{\mu}^L(x),\ \tilde{\nu}^L(x)) \leq \tilde{\mu}^L(x)$ and $\min(\tilde{\mu}^L(x),\ \tilde{\nu}^L(x)) \leq \tilde{\nu}^L(x)$. Hence sup min $(\tilde{\mu}^L(x),\ \tilde{\nu}^L(x))$ $\leq$ sup $\tilde{\mu}^L(x)$ and sup min $(\tilde{\mu}^L(x),\ \tilde{\nu}^L(x)) \leq$ sup $\tilde{\nu}^L(x)$. Hence sup min $(\tilde{\mu}^L(x),\ \tilde{\nu}^L(x)) \leq$ min (sup $\tilde{\mu}^L(x)$, sup $\tilde{\nu}^L(x)) = f(\tilde{\mu})\ \wedge f(\tilde{\nu})^L(z)$ for all $z \in Y$. Similarly, $f(\tilde{\mu} \wedge \tilde{\nu})^U(z) \leq f(\tilde{\mu}) \wedge f(\tilde{\nu})^U(z)$ for all $z \in Y$. Hence the claim is proved. Consequently, $f(\mathcal{F})$ forms an IVF filter base. □

THEOREM 2.4. *Let $f : X \to Y$ be an onto map. Let $\mathcal{G}$ be an IVF filter on Y. Then $f^{-1}(\mathcal{G}) = \{f^{-1}(\tilde{\mu}) \mid \tilde{\mu} \in \mathcal{G}\}$ forms a base for an IVF filter on X.*

Proof. We know that $f^{-1}(\tilde{\mu})^L(x) = \tilde{\mu}^L(f(x))$ and $f^{-1}(\tilde{\mu})^U(x) = \tilde{\mu}^U(f(x))$.

(i) Clearly, $f^{-1}(\tilde{\mu})^U(z) \neq 0$ for at least one $z \in X$. For, if $\tilde{\mu}^U(f(x)) = 0$ for all $x \in X$, then by surjective of f, we have $\tilde{\mu}^U(z) = 0$ for all $z \in Y$. Hence $\tilde{\mu} = \tilde{0} \in \mathcal{G}$, which contradicts $\mathcal{G}$ is an IVF filter.

(ii) Let $f^{-1}(\tilde{\mu}), f^{-1}(\tilde{\nu}) \in f^{-1}(\mathcal{G})$. We claim that $f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu}) = f^{-1}(\tilde{\mu} \wedge \tilde{\nu})$. Now

$$\begin{aligned}
[f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu})]^L(z) &= \min\{f^{-1}(\tilde{\mu})^L(z), f^{-1}(\tilde{\nu})^L(z)\} \\
&= \min\{\tilde{\mu}^L(f(z)), \tilde{\mu}^L(f(z))\} \\
&= (\tilde{\mu} \wedge \tilde{\nu})^L(f(z)) \\
&= f^{-1}(\tilde{\mu} \wedge \tilde{\nu})^L(z).
\end{aligned}$$

Similarly, $[f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu})]^U(z) = f^{-1}(\tilde{\mu} \wedge \tilde{\nu})^U(z)$. Since $\mathcal{G}$ is an IVF filter, we have $\tilde{\mu} \wedge \tilde{\nu} \in \mathcal{G}$ and hence $f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu}) \in f^{-1}(\mathcal{G})$. Therefore $f^{-1}(\mathcal{G})$ forms a basis for an IVF filter. □

DEFINITION 2.4. A map $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is said to be IVF filter continuous if for every $\tilde{\mu} \in \mathcal{F}_2, f^{-1}(\tilde{\mu}) \in \mathcal{F}_1$.

EXAMPLE 2.1. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $\mathcal{B}_1 = \{([a_1, a_2], [a_1, a_2], [b_1, b_2])\}$ and $\mathcal{B}_2 = \{([a_1, a_2], [b_1, b_2], [c_1, c_2])\}$ be IVF filterbases on X and Y, respectively. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the IVF filters generated by $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $f : X \to Y$ be a map defined by $a \mapsto x$, $b \mapsto x$ and $c \mapsto y$. By definition, $f^{-1}(\mathcal{B}_2) = \mathcal{B}_1$ and hence f is IVF filter continuous.

NOTE 2.1. Let $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ be a constant function. Then f need not be IVF filter continuous.

EXAMPLE 2.2. Let $X = \{a, b, c\}$ and $Y = \{e, f, g\}$. Let $\mathcal{B}_1 = \{([r_1, r_2], [r_1, r_2], [r_1, r_2])\}$ and $\mathcal{B}_2 = \{([r_1, r_2], [s_1, s_2], [t_1, t_2])\}$, where $s_i < r_i$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be IVF filters generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Define $h : X \to Y$ by $h(z) = f$ for all $z \in X$. Choose y_i such that $s_i < y_i < r_i$. Clearly, $\tilde{\mu} = ([r_1, r_2], [y_1, y_2], [t_1, t_2]) \in \mathcal{F}_2$. But $h^{-1}(\tilde{\mu}) = ([y_1, y_2], [y_1, y_2], [y_1, y_2]) \notin \mathcal{F}_1$. Hence h is not IVF filter continuous.

The following note is an immediate consequence of definitions.

NOTE 2.2. (i) Let f, g be IVF filter continuous maps. Then so is $f \circ g$, whenever the composition is defined.

(ii) The identity function on an IVF filter is IVF filter continuous.

(iii) Let $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ be an IVF filter continuous map. Let $Z \subseteq X_1$ such that no member of $\mathcal{F}_1$ vanishes on Z. Then $f \mid Z : (Z, \mathcal{F}_1 \mid Z) \to (X_2, \mathcal{F}_2)$ is also an IVF filter continuous function.

THEOREM 2.5. *A map $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ is an IVF filter continuous if and only if for every IVF point $\tilde{p}$ in X_1 and $\tilde{\nu} \in \mathcal{F}_2$ such that $f(\tilde{p}) \in \tilde{\nu}$, there exists $\tilde{\mu} \in \mathcal{F}_1$ such that $\tilde{p} \in \tilde{\mu}$ and $f(\tilde{\mu}) \leq \tilde{\nu}$.*

Proof. Let f be an IVF filter continuous function. Let $\tilde{p} \in D^X$ be an IVF point. Let supp $\tilde{p} = \{x\}$. Let $\tilde{\nu} \in \mathcal{F}_2$ such that $f(\tilde{p}) \in \tilde{\nu}$. We know that $f(\tilde{p})^U(z) = \tilde{p}^U(x)$ if $z = f(x)$ and $f(\tilde{p})^U(z) = 0$ if $z \neq f(x)$. By filter continuity, $f^{-1}(\tilde{\nu}) \in \mathcal{F}_1$. Clearly, $f^{-1}(\tilde{\nu})^L(x) > \tilde{p}^L(x)$ and $f^{-1}(\tilde{\nu})^U(x) > \tilde{p}^U(x)$. Hence $\tilde{p} \in f^{-1}(\tilde{\nu})$. By (v) of Theorem 1.1, $f(f^{-1}(\tilde{\nu})) \leq \tilde{\nu}$. If $\tilde{\mu} = f^{-1}(\tilde{\nu}), \tilde{\mu}$ satisfies our requirements.

Conversely, let $\tilde{\nu} \in \mathcal{F}_2$. Let $\tilde{p} \in f^{-1}(\tilde{\nu})$. Clearly, $f(\tilde{p}) \in f(f^{-1}(\tilde{\nu})) \leq \tilde{\nu}$. Hence by hypothesis, there exists $\tilde{\mu}_p \in \mathcal{F}_1$ such that $\tilde{p} \in \tilde{\mu}_p$ and $f(\tilde{\mu}_p) \leq \tilde{\nu}$. By (iii) of Theorem 1.1, $f^{-1}(f(\tilde{\mu}_p)) \leq f^{-1}(\tilde{\nu})$. By (vi) of Theorem 1.1, $\tilde{\mu}_p \leq f^{-1}(f(\tilde{\mu}_p))$ and hence $\tilde{\mu}_p \leq f^{-1}(\tilde{\nu})$. Since $\tilde{\mu}_p \in \mathcal{F}_1, f^{-1}(\tilde{\nu}) \in \mathcal{F}_1$. Hence f is an IVF filter continuous. $\square$

Now we generalize the definition of characteristic set of a fuzzy filter $\mathcal{F}$ with respect to a fuzzy set μ, $\mathcal{C}^{\mu}(\mathcal{F}) = \{a \in [0,1] \mid$ for all $\nu \in \mathcal{F}$, there exists $x \in X$ such that $\nu(x) > \mu(x) + a\}$ and the supremum of $\mathcal{C}^{\mu}(\mathcal{F})$ is the characteristic value of $\mathcal{F}$ with respect to μ in [3] as follows.

DEFINITION 2.5. Let $\mathcal{F}$ be an IVF filter. Let $\tilde{\mu}$ be an IVF set. Then the characteristic set of $\mathcal{F}$ with respect to $\tilde{\mu}$ is given by $\mathcal{C}^{\tilde{\mu}}(\mathcal{F}) = \{[a,b] \in D \mid$ for all $\tilde{\nu} \in \mathcal{F}$, there exists $x \in X$ such that $\tilde{\nu}^L(x) > \tilde{\mu}^L(x) + a, \tilde{\nu}^U(x) > \tilde{\mu}^U(x) + b\}$ and $c^{\tilde{\mu}}(\mathcal{F}) = [\sup a, \sup b]$, where the supremum is taken over all $[a,b] \in \mathcal{C}^{\tilde{\mu}}(\mathcal{F})$ is the characteristic value of $\mathcal{F}$ with respect to $\tilde{\mu}$.

THEOREM 2.6. *Let $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ be an IVF filter continuous function. Then $\mathcal{C}^{\tilde{\mu}}(\mathcal{F}_1) \subseteq \mathcal{C}^{f(\tilde{\mu})}(\mathcal{F}_2)$ if $\tilde{\mu} \in D^X$ is f-invariant.*

Proof. Let $[a,b] \in \mathcal{C}^{\tilde{\mu}}(\mathcal{F}_1)$. So for every $\tilde{\nu} \in \mathcal{F}_1$, there exists $x \in X_1$ such that $\tilde{\nu}^L(x) > \tilde{\mu}^L(x) + a$ and $\tilde{\nu}^U(x) > \tilde{\mu}^U(x) + b$. To prove that $[a,b] \in \mathcal{C}^{f(\tilde{\mu})}(\mathcal{F}_2)$, let $\tilde{\gamma} \in \mathcal{F}_2$. Clearly, $f^{-1}(\tilde{\gamma}) \in \mathcal{F}_1$. Hence there exists $x \in X$ such that $f^{-1}(\tilde{\gamma})^L(x) > \tilde{\mu}^L(x) + a$ and $f^{-1}(\tilde{\gamma})^U(x) > \tilde{\mu}^U(x) + b$. Hence $\tilde{\gamma}^L(f(x)) > \tilde{\mu}^L(x) + a$ and $\tilde{\gamma}^U(f(x)) > \tilde{\mu}^U(x) + b$. Since $\tilde{\mu}$ is f-invariant, $f(x) = f(y)$ implies $\tilde{\mu}(x) = \tilde{\mu}(y)$. Hence $f(\tilde{\mu})^L(f(x)) = \sup_{z \in f^{-1}(f(x))} \tilde{\mu}^L(z) = \tilde{\mu}^L(x)$ and similarly, $f(\tilde{\mu})^U(f(x)) = \tilde{\mu}^U(x)$. Hence $\tilde{\gamma}^L(f(x)) > f(\tilde{\mu})^L(f(x)) + a$ and $\tilde{\gamma}^U(f(x)) > f(\tilde{\mu})^U(f(x)) + b$. Hence the proof is complete. □

DEFINITION 2.6. A map $f : (X_1, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is said to be an IVF filter open map if for every $\tilde{\mu} \in \mathcal{F}_1, f(\tilde{\mu}) \in \mathcal{F}_2$. In addition if f is IVF filter continuous and 1-1, then f is said to be an IVF filter homeomorphism.

NOTE 2.3. $\mathcal{C}^{f(\tilde{\mu})}(\mathcal{F}_2) \subseteq \mathcal{C}^{\tilde{\mu}}(\mathcal{F}_1)$ holds if $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ is an injective IVF filter open map.

Proof. Let $[a,b] \in \mathcal{C}^{f(\tilde{\mu})}(\mathcal{F}_2)$. Let $\tilde{\nu} \in \mathcal{F}_1$. Clearly by IVF filter openness of f, $f(\tilde{\nu}) \in \mathcal{F}_2$. Hence there exists $y \in X_2$ such that

$$(1) \quad f(\tilde{\nu})^L(y) > f(\tilde{\mu})^L(y) + a \quad \text{and} \quad f(\tilde{\nu})^U(y) > f(\tilde{\mu})^U(y) + b.$$

Clearly, $f^{-1}(y)$ is not empty, otherwise $f(\tilde{\nu})^L(y) = 0$. Since f is an injective map, clearly $f^{-1}(y)$ is singleton. Hence $f(\tilde{\nu})^L(y) = \tilde{\nu}^L(f^{-1}(y))$ and so by (1), $\tilde{\nu}^L(f^{-1}(y)) > \tilde{\mu}^L(f^{-1}(y)) + a$ and similarly, $\tilde{\nu}^U(f^{-1}(y)) > \tilde{\mu}^U(f^{-1}(y)) + b$. Therefore $[a,b] \in \mathcal{C}^{\tilde{\mu}}(\mathcal{F}_1)$. □

The following corollaries are immediate.

COROLLARY 2.1. *If $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ are IVF filter homeomorphic, then $\mathcal{C}^{\tilde{\mu}}(\mathcal{F}_1) = \mathcal{C}^{f(\tilde{\mu})}(\mathcal{F}_2)$.*

COROLLARY 2.2. *Let $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ be a f-invariant IVF filter continuous map. Let $\tilde{\mu} \in D^{X_1}$. Then $c^{\tilde{\mu}}(\mathcal{F}_1) \leq c^{f(\tilde{\mu})}(\mathcal{F}_2)$.*

COROLLARY 2.3. *If $f : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ is an IVF filter homeomorphism, then $c^{\tilde{\mu}}(\mathcal{F}_1) = c^{f(\tilde{\mu})}(\mathcal{F}_2)$.*

3. Hausdorff IVF filters

DEFINITION 3.1. Two fuzzy sets $\mu, \nu \in I^X$ are said to intersect if $\mu(z) + \nu(z) > 1$ for some $z \in X$. Otherwise μ and ν are said to be disjoint.

Now we extend the above definition to IVF sets as follows:

Two IVF sets $\tilde{\mu}$, $\tilde{\nu} \in D^X$ are said to intersect at $z \in X$ if $\tilde{\mu}^L(z) + \tilde{\nu}^U(z) > 1$ or $\tilde{\mu}^U(z) + \tilde{\nu}^L(z) > 1$. Otherwise $\tilde{\mu}$ and $\tilde{\nu}$ do not intersect at z. $\tilde{\mu}$ and $\tilde{\nu}$ are said to be disjoint if $\tilde{\mu}$ and $\tilde{\nu}$ do not intersect anywhere.

NOTE 3.1. In crisp theory, two disjoint members cannot be members of a filter. Hence one can not speak about Hausdorffness on a filter. But in fuzzy setup, we have a definition of intersection such that two disjoint members can be members of a fuzzy filter.

For example, Let $X = \{a, b, c\}$ and $\mathcal{B} = \{(x_0, 3/4, z_0), (3/4, y_0, z_0), (x_0, y_0, 3/4), (x_0, y_0, z_0)\}$, where x_0, y_0, $z_0 \in (0, 1/4]$. Clearly, $\mathcal{B}$ forms a base for a filter. Let $\mathcal{F}$ be the fuzzy filter generated by $\mathcal{B}$. Under the above definition of intersection, $\mathcal{F}$ contains members which are disjoint.

So one can speak about Hausdorffness on fuzzy filters and Hausdorffness can be extended as follows.

DEFINITION 3.2. An IVF filter $(X, \mathcal{F})$ is said to be a Hausdorff IVF filter if for all pair $x, y \in X$ such that $x \neq y$, there exists $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}$ such that $\tilde{\mu}^U(x) > 1/2$, $\tilde{\nu}^U(y) > 1/2$ and $\tilde{\mu}^L(z) + \tilde{\nu}^U(z) \leq 1$ and $\tilde{\mu}^U(z) + \tilde{\nu}^L(z) \leq 1$ for all $z \in X$.

EXAMPLE 3.1. Let $X = \{a, b, c\}$ and $\mathcal{F}$ be the IVF filter generated by $\mathcal{B} = \{([x_0, x_1], [y_0, 3/4], [z_0, z_1]), ([x_0, 3/4], [y_0, y_1], [z_0, z_1]), ([x_0, x_1], [y_0, y_1], [z_0, 3/4]), ([x_0, x_1], [y_0, y_1], [z_0, z_1])\}$, where $x_0, y_0, z_0 \in (0, 1/4)$ and $x_0 + x_1 \leq 1, y_0 + y_1 \leq 1, z_0 + z_1 \leq 1$. Clearly, $(X, \mathcal{F})$ is a Hausdorff IVF filter.

DEFINITION 3.3. A sequence $\{x_n\}$ of $(X,\mathcal{F})$ is said to converge filterly to x if for every $\mu \in \mathcal{F}$ such that $\mu(x) > 1/2$, there exists an integer N such that $\mu(x_n) > 1/2$ for all $n \geq N$, or equivalently, $\mu^c(x_n) < 1/2$ for all $n \geq N$.

The above definition is extended in IVF filter as follows.

DEFINITION 3.4. A sequence $\{x_n\}$ of $(X,\mathcal{F})$ is said to converge IVF filterly to x ($\{x_n\} \rightharpoonup_{ivf} x$) if for every $\tilde{\mu} \in \mathcal{F}$ such that $\tilde{\mu}^U(x) > 1/2$, there exists N such that $(\tilde{\mu}^c)^U(x_n) < 1/2$ for all $n \geq N$, or equivalently, $\tilde{\mu}^L(x_n) > 1/2$ for all $n \geq N$.

THEOREM 3.1. *Let $(X,\mathcal{F})$ be a Hausdorff IVF filter. Let $Y \subseteq X$. Then $(Y, \mathcal{F} \mid Y)$ is also a Hausdorff IVF filter if no element of $\mathcal{F}$ vanishes on Y.*

Proof. By Theorem 2.1, $(Y,\mathcal{F} \mid Y)$ is an IVF filter on Y. Now we prove that $(Y,\mathcal{F} \mid Y)$ is Hausdorff. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $Y \subseteq X$ and $(X,\mathcal{F})$ is Hausdorff, we have $\tilde{\mu},\tilde{\nu} \in \mathcal{F}$ such that $\tilde{\mu}^U(y_1) > 1/2$, $\tilde{\nu}^U(y_2) > 1/2$, $\tilde{\mu}^L(z) + \tilde{\nu}^U(z) \leq 1$, and $\tilde{\mu}^U(z) + \tilde{\nu}^L(z) \leq 1$ for all $z \in X$. Hence $(\tilde{\mu} \mid Y)^U(y_1) > 1/2$, $(\tilde{\nu} \mid Y)^U)(y_2) > 1/2$, $(\tilde{\mu} \mid Y)^L(z) + (\tilde{\nu} \mid Y)^U(z) \leq 1$, and $(\tilde{\mu} \mid Y)^L(z) + (\tilde{\nu} \mid Y)^U(z) \leq 1$ for all $z \in Y$. Hence (Y, $\mathcal{F} \mid Y$) is also a Hausdorff IVF filter. □

THEOREM 3.2. *In a Hausdorff IVF filter $(X, \mathcal{F})$, any sequence of points of X converges uniquely if it converges IVF filterly.*

Proof. Let $\{x_n\}$ be a sequence of X. Assume that $\{x_n\}$ converges IVF filterly to x and y of X such that $x \neq y$. Since $x, y \in X$ such that $x \neq y$ and $(X,\mathcal{F})$ is Hausdorff, we have $\tilde{\mu},\tilde{\nu} \in \mathcal{F}$ such that $\tilde{\mu}^U(x) > 1/2$, $\tilde{\nu}^U(y) > 1/2$ and

$$\tilde{\mu}^L(z) + \tilde{\nu}^U(z) \leq 1, \tilde{\mu}^U(z) + \tilde{\nu}^L(z) \leq 1 \text{ for all } z \in X.$$

So, $\{x_n\} \rightharpoonup_{ivf} x \Rightarrow \tilde{\mu}^L(x_n) > 1/2$ for all $n \geq N_1$, for some N_1. Similarly, $\{x_n\} \rightharpoonup_{ivf} y \Rightarrow \tilde{\nu}^L(x_n) > 1/2$ for all $n \geq N_2$, for some N_2, and hence $\tilde{\nu}^U(x_n) > 1/2$ for all $n \geq N_2$, for some N_2.

Clearly, for all $n \geq N = \max\{N_1, N_2\}$, we have $\tilde{\mu}^L(x_n) + \tilde{\nu}^U(x_n) > 1$, a contradiction. □

THEOREM 3.3. *Let $f : (X_1,\mathcal{F}_1) \rightarrow (X_2,\mathcal{F}_2)$ be a bijective IVF filter open map. Then $(X_2,\mathcal{F}_2)$ is a Hausdorff IVF filter if $(X_1,\mathcal{F}_1)$ is a Hausdorff IVF filter.*

Proof. Let $y_1 \neq y_2 \in X_2$. By hypothesis there exist unique $x_1 \neq x_2 \in X_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $x_1 \neq x_2$ and $(X_1, \mathcal{F}_1)$ is Hausdorff, there exist $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}_1$ such that $\tilde{\mu}^U(x_1) > 1/2$, $\tilde{\nu}^U(x_2) > 1/2$,

$$\tilde{\mu}^L(z) + \tilde{\nu}^U(z) \leq 1 \quad \text{and} \quad \tilde{\mu}^L(z) + \tilde{\nu}^U(z) \leq 1 \text{ for all } z \in X_1.$$

Since f is IVF filter open, $f(\tilde{\mu}), f(\tilde{\nu}) \in \mathcal{F}_2$. Clearly, $f(\tilde{\mu})^U(y_1) = \tilde{\mu}^U(x_1) > 1/2$ and $f(\tilde{\nu})^U(y_2) = \tilde{\nu}^U(x_2) > 1/2$. Suppose $f(\tilde{\mu})^L(z) + f(\tilde{\nu})^U(z) > 1$ for some $z \in X_2$. By hypothesis there exists unique $x \in X_1$ such that $f(x) = z$. Hence $\tilde{\mu}^L(x) + \tilde{\nu}^U(x) > 1$, a contradiction. □

NOTE 3.2. We cannot drop any one of the condition in the above theorem.

THEOREM 3.4. *Let $f : (X_1, \mathcal{F}_1) \to X_2$ be a bijective map. Let $\mathcal{F} = \{\tilde{\mu} \in D^{X_2} \mid f^{-1}(\tilde{\mu}) \in \mathcal{F}_1\}$. If $(X_1, \mathcal{F}_1)$ is a Hausdorff IVF filter, then $(X_2, \mathcal{F})$ is a Hausdorff IVF filter.*

Proof. We first prove the following lemma.

LEMMA. *Let $f : (X_1, \mathcal{F}_1) \to X_2$ be a surjective map. Then $\mathcal{F} = \{\tilde{\mu} \in D^{X_2} \mid f^{-1}(\tilde{\mu}) \in \mathcal{F}_1\}$ is an IVF filter on X_2.*

Proof of the lemma. (i) Clearly $\tilde{0} \notin \mathcal{F}$. Otherwise $f^{-1}(\tilde{0}) = \tilde{0} \in \mathcal{F}_1$ contradicts $\tilde{0} \notin \mathcal{F}_1$.

(ii) Let $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}$. Hence $f^{-1}(\tilde{\mu}), f^{-1}(\tilde{\nu}) \in \mathcal{F}_1$, and so $f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu}) \in \mathcal{F}_1$. Clearly, $f^{-1}(\tilde{\mu}) \wedge f^{-1}(\tilde{\nu}) = f^{-1}(\tilde{\mu} \wedge \tilde{\nu})$ and hence $\tilde{\mu} \wedge \tilde{\nu} \in \mathcal{F}$.

(iii) Let $\tilde{\mu} \in \mathcal{F}$ and $\tilde{\nu} \geq \tilde{\mu}$. Since $\tilde{\mu} \in \mathcal{F}$, $f^{-1}(\tilde{\mu}) \in \mathcal{F}_1$. Clearly, $f^{-1}(\tilde{\nu}) \geq f^{-1}(\tilde{\mu})$ by (iii) of Theorem 1.1, and hence $f^{-1}(\tilde{\nu}) \in \mathcal{F}_1$. Hence $\tilde{\nu} \in \mathcal{F}$. □

DEFINITION 3.5. The IVF filter defined in the above lemma is called as Quotient IVF filter determined by the surjective map f.

Proof of Theorem 3.4. Let $f : (X_1, \mathcal{F}_1) \to X_2$ be a bijective map. Let $\mathcal{F} = \{\tilde{\mu} \in D^{X_2} \mid f^{-1}(\tilde{\mu}) \in \mathcal{F}_1\}$. To prove $(X_2, \mathcal{F})$ is a Hausdorff IVF filter, it is enough to prove that f is an IVF filter open map. Let $\tilde{\nu} \in \mathcal{F}_1$. Since f is bijective, $f^{-1}(f(\tilde{\nu}))^L(x) = \tilde{\nu}^L(x)$ and $f^{-1}(f(\tilde{\nu}))^U(x) = \tilde{\nu}^U(x)$ and hence $f^{-1}(f(\tilde{\nu})) = \tilde{\nu} \in \mathcal{F}_1$. Hence f is IVF filter open. Hence by Theorem 3.3, $(X_2, \mathcal{F})$ is a Hausdorff IVF filter. □

DEFINITION 3.6. Let $(X_\alpha, \mathcal{F}_\alpha)$ be an indexed family of IVF filters. Let $X = \prod X_\alpha$. Now the product IVF filter $\mathcal{F} = \Pi\mathcal{F}_\alpha$ is the smallest IVF filter for which the projection maps $p_\alpha : X \to X_\alpha$ defined by $p_\alpha((x_\alpha)) = x_\alpha$ are IVF filter continuous.

THEOREM 3.5. *Let $(X_\alpha, \mathcal{F}_\alpha)$ be an indexed family of IVF filters. Let $X = \prod X_\alpha$. Then $\mathcal{S} = \{p_\alpha{}^{-1}(\tilde{\mu}_\alpha) \mid \tilde{\mu}_\alpha \in \mathcal{F}_\alpha\}$ forms a subbasis for the product IVF filter.*

Proof. Clearly, $\tilde{0} \notin \mathcal{S}$. Now we prove that the IVF filter $\mathcal{F}$ $(\mathcal{S})$ generated by $\mathcal{S}$ is the smallest IVF filter in which all projection maps are IVF filter continuous. Let $\mathcal{F}_0$ be any IVF filter in which all projection maps are IVF filter continuous. Let $\tilde{\mu} \in \mathcal{F}(\mathcal{S})$. Hence there exist $p_{\alpha_1}{}^{-1}(\tilde{\mu}_{\alpha_1}), p_{\alpha_2}{}^{-1}(\tilde{\mu}_{\alpha_2}), \cdots, p_{\alpha_n}{}^{-1}(\tilde{\mu}_{\alpha_n}) \in \mathcal{S}$ such that $p_{\alpha_1}{}^{-1}(\tilde{\mu}_{\alpha_1}) \wedge p_{\alpha_2}{}^{-1}(\tilde{\mu}_{\alpha_2}) \wedge \cdots \wedge p_{\alpha_n}{}^{-1}(\tilde{\mu}_{\alpha_n}) \leq \tilde{\mu}$ for some $\tilde{\mu}_{\alpha_i} \in \mathcal{F}_{\alpha_i}$ $(i = 1, 2, .., n)$. Clearly, by IVF filter continuity of p_{α_i}, $p_{\alpha_i}{}^{-1}(\tilde{\mu}_{\alpha_i}) \in \mathcal{F}_0$ and hence $\tilde{\mu} \in \mathcal{F}_0$. Hence the theorem is proved. □

NOTE 3.3. Let $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ be IVF filters. Let $X = X_1 \times X_2$. Then the product IVF filter $\mathcal{F}_1 \times \mathcal{F}_2$ is generated by $\mathcal{B} = \{\tilde{\mu} \times \tilde{\nu} \mid \tilde{\mu} \in \mathcal{F}_1, \ \tilde{\nu} \in \mathcal{F}_2\}$.

Proof. First we check $\mathcal{B}$ forms an IVF filter base.

(i) We know that $(\tilde{\mu} \times \tilde{\nu})^U(x_1, x_2) = \min\{\tilde{\mu}^U(x_1), \ \tilde{\nu}^U(x_2)\}$. Since $\tilde{\mu} \in \mathcal{F}_1$ and $\tilde{\nu} \in \mathcal{F}_2$, we can see $\tilde{\mu} \neq \tilde{0}$ and $\tilde{\nu} \neq \tilde{0}$ and hence $\tilde{\mu}^U(x) \neq 0$, for some $x \in X_1$ and $\tilde{\nu}^U(y) \neq 0$ for some $y \in X_2$. Hence $(\tilde{\mu} \times \tilde{\nu})^U(x, y) = \min\{\tilde{\mu}^U(x), \ \tilde{\nu}^U(y)\} \neq 0$ and hence $\tilde{\mu} \times \tilde{\nu} \neq \tilde{0}$, for every $\tilde{\mu} \in \mathcal{F}_1$, $\tilde{\nu} \in \mathcal{F}_2$. Hence $\tilde{0} \notin \mathcal{B}$.

(ii) Let $\tilde{\mu}_1 \times \tilde{\nu}_1$, $\tilde{\mu}_2 \times \tilde{\nu}_2 \in \mathcal{B}$. Then we have $(\tilde{\mu}_1 \times \tilde{\nu}_1) \wedge (\tilde{\mu}_2 \times \tilde{\nu}_2) = \tilde{\mu}_1 \wedge \tilde{\mu}_2 \times \ \tilde{\nu}_1 \wedge \tilde{\nu}_2$: indeed,

$$\begin{aligned}
& [(\tilde{\mu}_1 \times \tilde{\nu}_1) \wedge (\tilde{\mu}_2 \times \tilde{\nu}_2)]^L(x, y) \\
= \ & \min\{(\tilde{\mu}_1 \times \tilde{\nu}_1)^L(x, y), (\tilde{\mu}_2 \times \tilde{\nu}_2)^L(x, y)\} \\
= \ & \min\{\min(\tilde{\mu}_1^L(x), \tilde{\nu}_1^L(y)), \min(\tilde{\mu}_2^L(x), \tilde{\nu}_2^L(y))\} \\
= \ & \min\{\tilde{\mu}_1^L(x), \tilde{\nu}_1^L(y)), \tilde{\mu}_2^L(x), \tilde{\nu}_2^L(y)\} \\
= \ & \min\{\min(\tilde{\mu}_1^L(x), \tilde{\mu}_2^L(x)), \min(\tilde{\nu}_1^L(x), \tilde{\nu}_2^L(y))\} \\
= \ & [(\tilde{\mu}_1 \wedge \tilde{\mu}_2) \times (\tilde{\nu}_1 \wedge \tilde{\nu}_2)]^L(x, y).
\end{aligned}$$

Similarly, $[(\tilde{\mu}_1 \times \tilde{\nu}_1) \wedge (\tilde{\mu}_2 \times \tilde{\nu}_2)]^U(x, y) = [(\tilde{\mu}_1 \wedge \tilde{\mu}_2) \times (\tilde{\nu}_1 \wedge \tilde{\nu}_2)]^U(x, y)$. Hence $\mathcal{B}$ forms an IVF filterbase.

We know that $\mathcal{S} = \{p_i{}^{-1}(\tilde{\mu}_i) \mid \tilde{\mu}_i \in \mathcal{F}_i\}$ forms a subbasis for the product IVF filter. Let $\mathcal{D}$ = finite intersections of members of $\mathcal{S}$. Clearly, $p_1{}^{-1}(\tilde{\mu}_1) \wedge p_1{}^{-1}(\tilde{\mu}_2) \wedge \cdots \wedge p_1{}^{-1}(\tilde{\mu}_m) = p_1{}^{-1}(\tilde{\mu}_1 \wedge \tilde{\mu}_2 \wedge \cdots \wedge \tilde{\mu}_m) \in \mathcal{D}$, where $\mu_i \in \mathcal{F}_1$, $i = 1, 2, \cdots, m$. Similarly, $p_2{}^{-1}(\tilde{\nu}_1) \wedge p_2{}^{-1}(\tilde{\nu}_2) \wedge \cdots \wedge p_2{}^{-1}(\tilde{\nu}_n) = p_2{}^{-1}(\tilde{\nu}_1 \wedge \tilde{\nu}_2 \wedge \cdots \wedge \tilde{\nu}_n) \in \mathcal{D}$, where $\nu_j \in \mathcal{F}_2$, $j = 1, 2, \cdots n$. Now we claim that $p_1{}^{-1}(\tilde{\mu}) \wedge p_2{}^{-1}(\tilde{\nu}) = \tilde{\mu} \times \tilde{\nu}$. Clearly, $[p_1{}^{-1}(\tilde{\mu}) \wedge p_2{}^{-1}(\tilde{\nu})]^L(x, y) = \min\{p_1{}^{-1} \ (\tilde{\mu})^L \ (x, y), \ p_2{}^{-1} \ (\tilde{\nu})^L(x, y)\} = \min\{\tilde{\mu}^L(x), \ \tilde{\nu}^L(y)\} = (\tilde{\mu} \times$

$\tilde{\nu})^L(x, y)$. Similarly, $[p_1{}^{-1}(\tilde{\mu}) \wedge p_2{}^{-1}(\tilde{\nu})]^U(x, y) = (\tilde{\mu} \times \tilde{\nu})^U(x, y)$. We know that $p_1{}^{-1}(\tilde{\mu}) = \tilde{\mu} \times \tilde{1}$ and $p_2{}^{-1}(\tilde{\nu}) = \tilde{1} \times \tilde{\nu}$. Hence $\mathcal{D} = \mathcal{B}$. So the product IVF filter $\mathcal{F}_1 \times \mathcal{F}_2$ is generated by $\mathcal{B} = \{\tilde{\mu} \times \tilde{\nu} \mid \tilde{\mu} \in \mathcal{F}_1, \tilde{\nu} \in \mathcal{F}_2\}$. □

THEOREM 3.6. *Arbitrary product of Hausdorff IVF filters is Hausdorff IVF filter.*

Proof. Let $(X_\alpha, \mathcal{F}_\alpha)$ be an indexed family of Hausdorff IVF filters, let $X = \prod X_\alpha$, and let $\mathcal{F} = \prod \mathcal{F}_\alpha$ be the product IVF filter in which each $p_\alpha : (X, \mathcal{F}) \to (X_\alpha, \mathcal{F}_\alpha)$ is filter continuous. We know that $\mathcal{S} = \{p_\alpha{}^{-1}(\tilde{\mu}_\alpha) \mid \tilde{\mu}_\alpha \in \mathcal{F}_\alpha\}$ forms a subbasis for the product IVF filter.

To prove that $(X, \mathcal{F})$ is Hausdorff IVF filter, consider $x = (x_\alpha), y = (y_\alpha) \in X$ such that $x \neq y$. So we have at least one β such that $x_\beta \neq y_\beta$. Since $(X_\beta, \mathcal{F}_\beta)$ is Hausdorff IVF filter, there exist $\tilde{\mu}_\beta, \tilde{\nu}_\beta \in \mathcal{F}_\beta$ such that $\tilde{\mu}^U_\beta(x_\beta) > 1/2, \tilde{\nu}^U_\beta(y_\beta) > 1/2$ and $\tilde{\mu}^L_\beta(z_\beta) + \tilde{\nu}^U_\beta(z_\beta) \leq 1$ for every $z_\beta \in X_\beta$ and $\tilde{\mu}^U_\beta(z_\beta) + \tilde{\nu}^L_\beta(z_\beta) \leq 1$ for every $z_\beta \in X_\beta$. Clearly, $p_\beta{}^{-1}(\tilde{\mu}_\beta)$ and $p_\beta{}^{-1}(\tilde{\nu}_\beta)$ are members of $\mathcal{S}$ and hence elements of $\mathcal{F}$. Now $p_\beta{}^{-1}(\tilde{\mu}_\beta)^U(x) = \tilde{\mu}^U_\beta(p_\beta(x)) = \tilde{\mu}^U(x_\beta) > 1/2$, and similarly $p_\beta{}^{-1}(\tilde{\nu}_\beta)^U(y) > 1/2$. Now we claim that $p_\beta{}^{-1}(\tilde{\mu}_\beta)^L(z) + p_\beta{}^{-1}(\tilde{\nu}_\beta)^U(z) \leq 1$ for every $z \in X$. Suppose that $p_\beta{}^{-1}(\tilde{\mu}_\beta)^L(z) + p_\beta{}^{-1}(\tilde{\nu}_\beta)^U(z) > 1$ for some $z \in X$. By definition, $p_\beta{}^{-1}(\tilde{\mu}_\beta)^L(z) + p_\beta{}^{-1}(\tilde{\nu}_\beta)^U(z) = \tilde{\mu}^L_\beta(p_\beta(z)) + \tilde{\nu}^L_\beta(p_\beta(z)) = \tilde{\mu}^L_\beta(z_\beta) + \tilde{\mu}^L_\beta(z_\beta) > 1$, where $z_\beta \in X_\beta$, a contradiction. Similarly, $p_\beta{}^{-1}(\tilde{\mu}_\beta)^U(z) + p_\beta{}^{-1}(\tilde{\nu}_\beta)^L(z) \leq 1$ for every $z \in X$. Hence $(X, \mathcal{F})$ is a Hausdorff IVF filter. □

THEOREM 3.7. *Let $(X, \mathcal{F})$ be an IVF filter on X. Let R be an equivalence relation. Let X/R denote the collection of all disjoint equivalence classes. Let p be the identification map from $X \to X/R$. Let $Q(\mathcal{F})$ be the quotient IVF filter on X/R determined by p. Then $(X/R, Q(\mathcal{F}))$ is a Hausdorff IVF filter if $\tilde{1}_R \in \mathcal{F} \times \mathcal{F}$ and p is an IVF filter open map.*

Proof. Let $p(x) \neq p(y) \in X/R$. Hence x is not related to y. By definition, $\tilde{1}_R(x, y) = [1, 1]$. By hypothesis, $\tilde{1}_R \in \mathcal{F} \times \mathcal{F}$. So there exists $\tilde{\mu} \times \tilde{\nu} \in \mathcal{B}$ such that $\tilde{\mu} \times \tilde{\nu} \leq \tilde{1}_R$.

We can choose $\tilde{\mu} \times \tilde{\nu}$ such that $(\tilde{\mu} \times \tilde{\nu})^L(x, y) > 0$. For, Suppose $(\tilde{\mu} \times \tilde{\nu})^L(x, y) = 0$ such that $\tilde{\mu} \times \tilde{\nu} \leq \tilde{1}_R$, $\tilde{\mu}^L(x) = 0$ or $\tilde{\nu}^L(y) = 0$ or both are zero. Choose $\tilde{\mu}_1$ such that $\tilde{\mu}^L_1(z) = \tilde{\mu}^L(z)$ if $z \neq x$ and $\tilde{\mu}^L_1(x) > 0$ and $\tilde{\mu}^U_1(z) = \tilde{\mu}^U(z)$ if $z \neq x$ and $\tilde{\mu}^U_1(x) > \max\{\tilde{\mu}^L_1(x), \tilde{\mu}^U(x)\}$. Clearly, $\tilde{\mu}_1 \geq \tilde{\mu}$ and hence $\tilde{\mu}_1 \in \mathcal{F}$. Similarly choose $\tilde{\nu}_1 \in \mathcal{F}$. Clearly $\tilde{\mu}_1 \times \tilde{\nu}_1 \in \mathcal{B}$ such that $(\tilde{\mu}_1 \times \tilde{\nu}_1)^L(x, y) > 0$ and $(\tilde{\mu}_1 \times \tilde{\nu}_1) \leq \tilde{1}_R$.

So choose $\tilde{\mu} \times \tilde{\nu} \in \mathcal{B}$ such that $(\tilde{\mu} \times \tilde{\nu})^L(x, y) > 0$ and $(\tilde{\mu} \times \tilde{\nu}) \leq \tilde{1}_R$. Define $\tilde{\mu}_2$ such that $\tilde{\mu}_2^L(z) = \tilde{\mu}^L(z)$ if $\tilde{\mu}^L(z) > 1/2$ or $\tilde{\mu}^L(z) = 0$ and $\tilde{\mu}_2^L(z) > 3/4$ otherwise and $\tilde{\mu}_2^U(z) = \tilde{\mu}^U(z)$ if $\tilde{\mu}^U(z) > 3/4$ or $\tilde{\mu}^U(z) = 0$ and $\tilde{\mu}_2^U(z) = 7/8$ otherwise. Clearly, $\tilde{\mu}_2 \geq \tilde{\mu}$. Similarly, define $\tilde{\nu}_2$ such that $\tilde{\nu}_2 \geq \tilde{\nu}$. Since $\tilde{\mu}, \tilde{\nu} \in \mathcal{F}$, $\tilde{\mu}_2, \tilde{\nu}_2 \in \mathcal{F}$.

Since $\tilde{\mu} \times \tilde{\nu} \leq \tilde{1}_R$, we have $(\tilde{\mu} \times \tilde{\nu})^L(s, t) + \tilde{1}_R^U(s, t) \leq 1$ for all (s, t). Clearly if s is related with t, then $(\tilde{\mu}_2 \times \tilde{\nu}_2)^L(s, t) = 0$ and $(\tilde{\mu}_2 \times \tilde{\nu}_2)^U(s, t) = 0$. So if s is related with t, then $(\tilde{\mu}_2 \times \tilde{\nu}_2)(s, t) = [0, 0]$ and if instead s is not related with t, then $\tilde{1}_R(s, t) = [0, 0]$, hence $(\tilde{\mu}_2 \times \tilde{\nu}_2)^L(s, t) + \tilde{1}_R^U(s, t) \leq 1$, and $(\tilde{\mu}_2 \times \tilde{\nu}_2)^U(s, t) + \tilde{1}_R^L(s, t) \leq 1$ for every $(s, t) \in X \times X$.

Since p is IVF filter open, we have $p(\tilde{\mu}_2), p(\tilde{\nu}_2) \in Q(\mathcal{F})$. Clearly, $p(\tilde{\mu}_2)^U(p(x)) = \sup_{z \in p^{-1}p(x)} \tilde{\mu}_2^U(z) \geq \tilde{\mu}_2^U(x) > 1/2$, since by our choice, $\tilde{\mu}^U(x) > 0$. Similarly, $p(\tilde{\nu}_2)^U(p(y)) > 1/2$.

It is enough to prove that $p(\tilde{\mu}_2)^L(p(z)) + p(\tilde{\nu}_2)^U(p(z)) \leq 1$ for all $z \in X$. Suppose that $p(\tilde{\mu}_2)^L(p(z_0)) + p(\tilde{\nu}_2)^U(p(z_0)) > 1$ for some $z_0 \in X$. So $p(\tilde{\mu}_2)^L(p(z_0)) > 0$ and $p(\tilde{\nu}_2)^U(p(z_0)) > 0$. Hence there exists $z_1 \in p^{-1}(p(z_0))$ such that $\tilde{\mu}_1^L(z_2) > 0$. Similarly there exists $z_2 \in p^{-1}(p(z_0))$ such that $\tilde{\nu}_2^U(z_2) > 0$. Since $z_1, z_2 \in p^{-1}(p(z_0))$, z_1 is related with z_2. We have $(\tilde{\mu}_2 \times \tilde{\nu}_2)^U(z_1, z_2) > 0$ and hence $(\tilde{\mu}_2 \times \tilde{\nu}_2)^U(z_1, z_2) + \tilde{1}_R^U(z_1, z_2) > 1$, a contradiction. Similarly, $p(\tilde{\mu}_2)^U(p(z)) + p(\tilde{\nu}_2)^L(p(z)) \leq 1$ for all $z \in X$. Hence the theorem is proved. □

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