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ON A CENTRAL LIMIT THEOREM FOR A STATIONARY MULTIVARIATE LINEAR PROCESS GENERATED BY LINEARLY POSITIVE QUADRANT DEPENDENT RANDOM VECTORS

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ABSTRACT. For a stationary multivariate linear process of the form $\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j}$, where $\{\mathbb{Z}_t : t = 0, \pm 1, \pm 2, \cdots\}$ is a sequence of stationary linearly positive quadrant dependent m-dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$, we prove a central limit theorem.

1. Introduction

Lehmman [8] introduced a simple and natural definition of positive dependence : A sequence $\{Y_t : t = 0, 1, 2, \cdots\}$ of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real α_i, α_j and $i \neq j$ $P\{Y_i > \alpha_i, Y_j > \alpha_j\} \geq P\{Y_i > \alpha_i\}P\{Y_j > \alpha_j\}$. A concept stronger than PQD was introduced by Newman [10]: A sequence $\{Y_t\}$ of random variables is said to be linearly positive quadrant dependent (LPQD) if for any disjoint A, B and positive $r_j's$, $\sum_{i \in A} r_i Y_i$ and $\sum_{j \in B} r_j Y_j$ are PQD.

Two m-variate random vectors $\mathbb{Z}_1$, $\mathbb{Z}_2$ are said to be positive quadrant dependent (PQD) if Z_{1i}, Z_{2j} are PQD for all $i, j = 1, \cdots, m$, where Z_{1i}, Z_{2j} are components of $\mathbb{Z}_1$, $\mathbb{Z}_2$, respectively.

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Let $(\mathbb{Z}_1, \mathbb{Z}_2, \cdots, \mathbb{Z}_t)$ be m-variate random vectors. We say that $(\mathbb{Z}_1, \mathbb{Z}_2, \cdots, \mathbb{Z}_t)$ is linearly positive quadrant dependent if for any disjoint A, $B \subset \{1, \cdots, t\}$ and for any real vectors a_r with nonnegative components,

$$\sum_{s \in A} a_s \mathbb{Z}_s \text{ and } \sum_{r \in B} a_r \mathbb{Z}_r \text{ are PQD.} \tag{1}$$

Let $\{\mathbb{X}_t, t = 0, \pm 1, \cdots\}$ be an m-variate linear process of the form

$$\mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u} \tag{2}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathbb{Z}_t\}$ is a sequence of stationary m-variate LPQD random vectors with $E\mathbb{Z}_t = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$ and positive definite covariance matrix $\Gamma_{m \times m}$. Throughout this paper we shall assume that

$$\sum_{u=0}^{\infty} \|A_u\| < \infty \text{ and } \sum_{u=0}^{\infty} A_u \neq \mathbb{O}_{m \times m}, \tag{3}$$

where for any $m \times m$, $m \geq 1$, matrix $A = (a_{ij})$, $\|A\| = \sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|$ and $\mathbb{O}_{m \times m}$ denotes the $m \times m$ zero matrix. Further, let

$$T = \left(\sum_{j=0}^{\infty} A_j\right) \Gamma \left(\sum_{j=0}^{\infty} A_j\right)',$$

where the prime denotes transpose, and the matrix $\Gamma = [\sigma_{kj}]$ with

$$\sigma_{kj} = E(Z_{1k} Z_{1j}) + \sum_{t=2}^{\infty} \left(E(Z_{1k} Z_{tj}) + E(Z_{1j} Z_{tk})\right). \tag{4}$$

Further, let $\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t, (n \geq 0; \mathbb{S}_0 = \mathbb{O})$.

Fakhre-Zakeri and Lee [4] proved a central limit theorem for multivariate linear processes generated by independent multivariate random vectors and Fakhre-Zakeri and Lee [5] also derived a functional central limit theorem for multivariate linear processes generated by multivariate random vectors with martingale difference sequence.

In this note we prove a central limit theorem for an m-variate linear process generated by m-variate LPQD random vectors.

THEOREM 1.1. *Let $\{\mathbb{Z}_t, t = 0, \pm 1, \cdots\}$ be a strictly stationary LPQD sequence of m-dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$ and positive definite covariance matrix Γ as in (4). Let $\{\mathbb{X}_t\}$ be an m-variate linear process defined as in (2). Assume that*

$$E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(Z_{1i}Z_{ti}) = \sigma^2 < \infty, \tag{5}$$

$$\sum_{t=n+1}^{\infty} E\|Z_{1i}Z_{ti}\| = O(n^{-\rho}) \quad \text{for some} \quad \rho > 0, \tag{6}$$

and

$$E\|\mathbb{Z}_t\|^s < \infty \quad \text{for some} \quad s > 2. \tag{7}$$

Then, the multivariate linear process $\{\mathbb{X}_t\}$ fulfills the central limit theorem, that is, $n^{-\frac{1}{2}}\mathbb{S}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$.

REMARK. For $m = 1$, Kim and Baek [7] showed that the central limit theorem holds for the linear processes generated by an LPQD process.

2. Proofs

Note that Newman [10] has proved the central limit theorem for LPQD random variables (see Theorem 12 of [10]). Thus by means of the simple device due to Cramer Wold the following result holds.

LEMMA 2.1. *Let $\{\mathbb{Z}_t\}$ be a sequence of stationary LPQD m-variate random vectors with $E(Z_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$. If (5) holds then*

$$n^{-\frac{1}{2}}\sum_{t=1}^{n}\mathbb{Z}_t \xrightarrow{\mathcal{D}} N(\mathbb{O}, \Gamma),$$

where $\Gamma = [\sigma_{kj}]$ is defined as in (4); that is, $\{\mathbb{Z}_t\}$ satisfies the central limit theorem.

LEMMA 2.2. *Let $\{\mathbb{Z}_t\}$ be a sequence of stationary LPQD random vectors with $E(Z_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$. Let $\tilde{\mathbb{X}}_t = (\sum_{j=0}^{\infty} A_j)\mathbb{Z}_t$ and $\tilde{\mathbb{S}}_k =$*

$\sum_{t=1}^{k} \tilde{\mathbb{X}}_k$. *Assume that* (5), (6) *and* (7) *hold. Then*

$$(8) \qquad n^{-\frac{1}{2}} \max_{1 \le k \le n} \|\tilde{\mathbb{S}}_k - \mathbb{S}_k\| = o_p(1).$$

Proof. See Appendix. □

Proof of Theorem 1.1. As in Lemma 2.2, set $\tilde{\mathbb{X}}_t = (\sum_{j=0}^{\infty} A_j)\mathbb{Z}_t$ and $\tilde{\mathbb{S}}_n = \sum_{t=1}^{n} \tilde{\mathbb{X}}_t$. First note that

$$E\|\tilde{\mathbb{X}}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(\tilde{X}_{1i}\tilde{X}_{ti})$$

$$(9) \qquad = (\sum_{j=1}^{\infty} A_j)^2 (E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(Z_{1i}Z_{ti})).$$

Since $\tilde{\mathbb{X}}_t$ is LPQD, by Lemma 2.1 $\{\tilde{\mathbb{X}}_t\}$ satisfies the central limit theorem, that is,

$$(10) \qquad n^{-\frac{1}{2}}\tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T).$$

According to Lemma 2.2 we also have

$$(11) \qquad n^{-\frac{1}{2}}|\tilde{\mathbb{S}}_n - \mathbb{S}_n| = o_p(1).$$

Hence from (10) and (11) the desired conclusion follows by Theorem 4.1 of [1]. □

Appendix

Proof of Lemma 2.2. We prove Lemma 2.2 using the ideas in the proof of Lemma 3 of [5] and Lemma 2 of [7]. First observe that

$$(A.1) \qquad \sum_{t=n+1}^{\infty} E(\tilde{X}_{1i}\tilde{X}_{ti}) = (\sum_{j=0}^{\infty} A_j)^2 \sum_{t=n+1}^{\infty}\sum_{i=1}^{m} E(Z_{1i}Z_{ti}) = O(n^{-\rho})$$

and that

$$(A.2) \qquad E\|\tilde{\mathbb{X}}_t\|^s = (\sum_{j=0}^{\infty} A_j)^s E\|\mathbb{Z}_t\|^s < \infty \quad \text{for some} \quad s > 2.$$

By Lemma 3 of [7], it follows from (A.1) and (A.2) that

$$(A.3) \qquad E(\max_{1\geq k<n} \|\tilde{\mathbb{X}}_1 + \cdots + \tilde{\mathbb{X}}_k\|^r) \geq Bn^{\frac{r}{2}}$$

for some $r > 2$ and a constant B.

Next, we observe that

$$\begin{aligned}
\tilde{\mathbb{S}}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} A_j\right)\mathbb{Z}_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\
&= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} A_j \mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t
\end{aligned}$$

and thus

$$\begin{aligned}
\tilde{\mathbb{S}}_k - \mathbb{S}_k &= -\sum_{t=1}^{k}\sum_{j=t}^{\infty} A_j \mathbb{Z}_{t-j} + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\
&= I_1 \; + \; I_2 \; (say).
\end{aligned}$$

To prove

$$(A.4) \qquad n^{-\frac{1}{2}} \max_{1\leq k\leq n} \|I_1\| \xrightarrow{P} 0,$$

we observe that for $r > 2$

$$\begin{aligned}
& n^{-\frac{r}{2}} \, E \max_{1\leq k\leq n} \left\| \sum_{t=1}^{k} \sum_{j=t}^{\infty} A_j \, \mathbb{Z}_{t-j} \right\|^r \\
= \; & n^{-\frac{r}{2}} \, E \max_{1\leq k\leq n} \left\| \sum_{j=1}^{\infty} \sum_{t=1}^{j\wedge k} A_j \, \mathbb{Z}_{t-j} \right\|^r \\
\leq \; & n^{-\frac{r}{2}} \left(\sum_{j=1}^{\infty} \|A_j\| \left\{ E \max_{1\leq k\leq n} \left\| \sum_{t=1}^{j\wedge k} \mathbb{Z}_{t-j} \right\|^r \right\}^{\frac{1}{r}} \right)^r \\
\leq \; & K \left[\sum_{j=1}^{\infty} \|A_j\| \left(\frac{j\wedge k}{n} \right)^{\frac{1}{2}} \right]^r
\end{aligned}$$

for some positive constant K, where we have used Lemma 2 in [7] for LPQD random variables. By the dominated convergence theorem the last term above tends to zero as $n \longrightarrow \infty$ from which (A4) follows.

Next, we show that

$$(A.5) \qquad n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_2\| = o_p(1).$$

Write

$$I_2 = I_{21} + I_{22}, \text{ where}$$
$$I_{21} = A_1\mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)$$

and

$$I_{22} = (A_{k+1} + A_{k+2} + \cdots)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).$$

Let p_n be a sequence of positive integers such that

$$(A.6) \qquad p_n \longrightarrow \infty \quad \text{and} \quad p_n/n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Note that

$$\begin{aligned} n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_{22}\| \le & \left(\sum_{i=0}^{\infty} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le p_n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ & + \left(\sum_{i>p_n} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ = & \ III + IV \text{ (say)}. \end{aligned}$$

It follows from (3) and (A.6) that for some $r > 2$

$$III \le \left(\sum_{i=0}^{\infty} \|A_i\|\right)^r B_1 (p_n/n)^{\frac{r}{2}} \xrightarrow{P} 0$$

and

$$IV \le \left(\sum_{i>p_n} \|A_i\|\right)^r B_2 \xrightarrow{P} 0,$$

by Lemma 2 of [7]. It remains to prove that

$$Y_n := n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_{21}\| = o_p(1).$$

To this end, define for each $l \ge 1$

$$I_{21,l} = B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k + \mathbb{Z}_{k+1}) + \cdots + B_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l \\ \mathbb{O}_{m\times m}, & k > l. \end{cases}$$

Let $Y_{n,l} = n^{-\frac{1}{2}} \max_{1 \le k \le n} \|I_{21,l}\|$. Clearly, for each $l \ge 1$,

$$(A.7) \qquad Y_{n,l} = \circ_p(1).$$

On the other hand,

$$(Y_{n,l} - Y_n) \le n^{-\frac{1}{2}} \max_{1 \le k \le n} \left\| \sum_{i=1}^{k} (A_i - B_i)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \right\|$$

$$\le n^{-\frac{1}{2}} \max_{l < k \le n} \left(\sum_{i=l+1}^{k} \|A_i\| \max_{l < i \le n} \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \right)$$

$$\le n^{-\frac{1}{2}} \sum_{i > l} \|A_i\| \max_{l < k \le n} \max_{l < i \le k} (\|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| + \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}\|)$$

$$\le n^{-\frac{1}{2}} \sum_{i > l} \|A_i\| \,(\max_{l < k \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k|$$

$$+ \max_{l < k \le n} \max_{l < i \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}|)$$

$$\le n^{-\frac{1}{2}} \sum_{i > l} \|A_i\| \left(\max_{1 \le j \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\| + \max_{1 \le k \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\| \right)$$

$$= 2n^{-\frac{1}{2}} \sum_{i > l} \|A_i\| \max_{1 \le j \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|.$$

From this result and Lemma 2 of [7], for any $\delta > 0$,

$$\lim_{l \to \infty} \lim_{n \to \infty} \sup P(|Y_{n,l} - Y_n|^2 > \delta)$$

$$(A.8) \qquad \le \lim_{l \to \infty} 2^r \delta^{-r} \left(\sum_{i > l} \|A_i\| \right)^r \lim_{n \to \infty} n^{-\frac{r}{2}} \max_{1 \le j \le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|^r$$

$$\le B \lim_{l \to \infty} \delta^{-r} 2^r \left(\sum_{i > l} \|A_i\| \right)^r = 0.$$

In view of (A.7) and (A.8), it follows from Theorem 4.2 of [1, p.25] that $Y_n = \circ_p(1)$. This completes the proof of Lemma 2.2. □

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