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TOPOLOGIES AND INCIDENCE STRUCTURE ON R^n-GEOMETRIES

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Abstract. An R^n-geometry $(\mathcal{P}^n, \mathcal{L})$ is a generalization of the Euclidean geometry on R^n (see Def. 1.1). We can consider some topologies (see Def. 2.2) on the line set $\mathcal{L}$ such that the join operation $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow \mathcal{L}$ is continuous. It is a notable fact that in the case $n = 2$ the introduced topologies on $\mathcal{L}$ are same and the join operation $\vee : \mathcal{P}^2 \times \mathcal{P}^2 \setminus \triangle \longrightarrow \mathcal{L}$ is continuous and open [10, 11]. It is a fundamental topological property of plane geometry, but in the cases $n \geq 3$, it is no longer true. There are counter examples [2]. Hence, it is a fundamental problem to find suitable topologies on the line set $\mathcal{L}$ in an R^n-geometry $(\mathcal{P}^n, \mathcal{L})$ such that these topologies are compatible with the incidence structure of $(\mathcal{P}^n, \mathcal{L})$. Therefore, we need to study the topologies of the line set $\mathcal{L}$ in an R^n-geometry $(\mathcal{P}^n, \mathcal{L})$. In this paper, the relations of such topologies on the line set $\mathcal{L}$ are studied.

1. Introduction

We shall use $\mathcal{P}^n$ to denote a topological space which is homeomorphic to R^n.

Definition 1.1. Let $n \geq 2$. Let $\mathcal{L}$ be a system of subsets of $\mathcal{P}^n$. The elements of $\mathcal{P}^n$ are called points, and the elements of $\mathcal{L}$ are called lines. We shall say that $(\mathcal{P}^n, \mathcal{L})$ is an *R^n-geometry* if it satisfies the following axioms:

(1) The incidence structure $(\mathcal{P}^n, \mathcal{L})$ is a linear space, i.e., any two distinct points p, q lie on a unique line $p \vee q \in \mathcal{L}$.
(2) Each line is closed in the topological space $\mathcal{P}^n$ and homeomorphic to R.

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An R^n-geometry $(\mathcal{P}^n, \mathcal{L})$ is called *topological* if $\mathcal{L}$ is a topological space and the join operation

$$\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow \mathcal{L}$$

is continuous, where $\triangle = \{(p,p) | p \in \mathcal{P}^n\}$ is the diagonal. An R^2-geometry $(\mathcal{P}^2, \mathcal{L})$ is called also R^2-plane or a Salzmann-plane.

We start with some basic definitions and lemmas in general topology. Let X be a topological space and $(A_n)_{n \in N}$ be a sequence of subsets of X. Denote by $\liminf A_n$ the set of all limit points of sequences $(a_n)_{n \in N}$ with $a_n \in A_n$, and denote by $\limsup A_n$ the set of all accumulation points of such sequences. The sequence $(A_n)_{n \in N}$ is Hausdorff-convergent to $A \subseteq X$ if and only if $\liminf A_n = \limsup A_n = A$ (written by $\lim A_n = A$ or $A_n \longrightarrow A$).

Hausdorff metric: Let $\mathcal{U}$ be the set of the non-empty closed subsets of $\mathcal{P}^n$. We define the metric on $\mathcal{U}$ by

$$\delta : \mathcal{U} \times \mathcal{U} \longrightarrow R : (A, B) \longrightarrow \sup\{|d(x, A) - d(x, B)| e^{-d(p,x)} | x \in \mathcal{P}^n\},$$

where d is the metric on $\mathcal{P}^n$ and $p \in \mathcal{P}^n$. Then δ is a metric on $\mathcal{U}$. Let $(A_n)_{n \in N}$ be a sequence in $\mathcal{U}$ and $A \in \mathcal{U}$. Then $(A_n)_{n \in N}$ converges to A in $(\mathcal{U}, \delta)$ if and only if $\lim A_n \longrightarrow A$ ([4, Chapter 1.3]).

Let X be a set and $\mathcal{T}$ a topology of X. We use often the couple $(X, \mathcal{T})$ to denote the given topology on X. We need the following basic lemmas which are well known.

LEMMA 1.2. *Let Y be an arbitrary set, X a topological space, and $f : X \to Y$ a surjection. Let $\mathcal{T}_1$ be the final topology on Y and the topology $\mathcal{T}_2$ on Y be generated by the subbasis $\sum = \{f(U) | U$ is open in $X\}$. Then:*

(1) $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
(2) *Let $\mathcal{T}$ be a topology on Y. Then $f : X \longrightarrow (Y, \mathcal{T})$ is a continuous open mapping if and only if $\mathcal{T}_1 = \mathcal{T} = \mathcal{T}_2$. That is, if $f : X \longrightarrow Y$ is a continuous open surjection, then the topology on Y is uniquely determined.*
(3) *1°, 2° countability and local compactness are invariant under continuous open surjections.*

Let X be a topological space. It is well known that $A \subset X$ is closed if and only if together with any filterbase consisting of subsets of A it contains all its limits. This shows that all the basic topological concepts can be expressed by filterbases, that is, closed sets are characterized as the notation of filterbase. Now, let us consider the two classes of spaces in which closed sets are characterized as the notation of sequence. A topological space X is called a *sequential space* if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits. A topological space X is called a *Fréchet space* if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence $x_1, x_2, \ldots$ of points of A converging to x. Every 1° countable space is a Fréchet space and every Fréchet space is a sequential space ([6, Chapter 1.6.14]). Every $\mathcal{L}^*$-space is also a sequential space (Definition 2.7).

LEMMA 1.3. *Let $(X, \mathcal{T}_1)$ be a sequential space and $(X, \mathcal{T}_2)$ a topological space. Let (x_n) be an arbitrary sequence and $x \in X$. Then*

$$[(x_n \longrightarrow x \text{ in } (X, \mathcal{T}_1)) \Rightarrow (x_n \longrightarrow x \text{ in } (X, \mathcal{T}_2))] \Longleftrightarrow \mathcal{T}_2 \subseteq \mathcal{T}_1.$$

2. Comparison of some topologies on $\mathcal{L}$

LEMMA 2.1. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry and let (L_n) be a sequence in $\mathcal{L}$ with* $\limsup L_n \neq \emptyset$. *Then* $|\limsup L_n| = \infty$.

Proof. [12, Lemma 2]. □

Generalizing [10], we introduce some topologies on the line set $\mathcal{L}$.

DEFINITIONS 2.2. The *final topology* F on $\mathcal{L}$ is the largest topology on $\mathcal{L}$ for which the mapping $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow \mathcal{L}$ is continuous. The *open join topology* OJ is generated by the subbasic elements $O_1 \vee O_2 = \{p \vee q \in \mathcal{L} | p \in O_1, q \in O_2, p \neq q\}$, where O_1, O_2 are open sets in $\mathcal{P}^n$. The *open meet topology* OM is defined by the subbasic sets $M_O = \{L \in \mathcal{L} | L \cap O \neq \emptyset\}$, where O is an open set in $\mathcal{P}^n$. The *hausdorff topology* H is the induced topology of $(\mathcal{U}, \delta)$ on $\mathcal{L}$.

The following results can be founded in [11, 31.19 Theorem] for $n = 2$, and for arbitrary n we can find similar results in [12, Theorem 7].

LEMMA 2.3. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Then $F \subseteq OJ = OM \subseteq H$.*

Proof. By Lemma 1.2, we have $F \subseteq OJ$. Let O_1, O_2 be open in $\mathcal{P}^n$. Since $O_1 \vee O_2 = M_{O_1} \cap M_{O_2}$ and $M_{O_1} = O_1 \vee O_1$, hence we have $OJ = OM$. If O is open in $\mathcal{P}^n$, then $\mathcal{L} \setminus M_O = \{L \in \mathcal{L} | L \cap O = \emptyset\}$ is H-closed. Assume that $\mathcal{L} \setminus M_O$ is not H-closed. Then there exists a sequence (L_n) in $\mathcal{L} \setminus M_O$ which converges to $L \in M_O$. Choose a point

$a \in L \cap O$. Then there exists a sequence (a_n) with $a_n \in L_n$ which converges to a. For sufficiently large $n \in N$, a_n lie in O, a contradiction to $L_n \in \mathcal{L} \setminus M_O$. Consequently, $OM \subseteq H$. □

COROLLARY 2.4. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Then:*

(1) *Let $\mathcal{T}$ be a topology on $\mathcal{L}$. Then the join map $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow (\mathcal{L}, \mathcal{T})$ is continuous open if and only if $\mathcal{T} = F = OJ(= OM)$.*
(2) *$(\mathcal{L}, OM)$ is a T_1-space and 2° countable.*
(3) *If $F = OJ$, then the line space $(\mathcal{L}, OJ)$ is a locally compact space.*

Proof. (1) and (3): By Lemma 1.2, the proofs are clear.

(2) Let L_1, L_2 be two distinct lines. Choose a point $a \in L_1$ with $a \notin L_2$. Since $\mathcal{P}^n$ is regular, there is an open nbd O of a such that $O \cap L_2 = \emptyset$. Hence, M_O is an open nbd of L_1 not containing L_2. Since $\mathcal{P}^n$ has a countable basis and a countable union of countable sets is countable, $(\mathcal{L}, OM)$ is 2° countable. □

LEMMA 2.5. *Let $(P^n, \mathcal{L})$ be an R^n-geometry. Let (L_n) be a sequence in $\mathcal{L}$ and $L \in \mathcal{L}$. Then $L_n \longrightarrow L$ in $OM(= OJ)$ if and only if $L \subseteq \liminf L_n$.*

Proof. [12, Theorem 7]. □

DEFINITION 2.6. Let X be a set. A *sequential convergence* λ on X is a relation between sequences (x_n) of elements of X and elements x of X, denoted $x_n \xrightarrow{\lambda} x$, such that

(L1) If $x_n = x$ for all n, then $x_n \xrightarrow{\lambda} x$.
(L2) If $x_n \xrightarrow{\lambda} x$, then $x_{n_m} \xrightarrow{\lambda} x$ for every subsequence (x_{n_m}) of (x_n).

We say that (x_n) *converges* to x if $x_n \xrightarrow{\lambda} x$. If λ is a sequential convergence on a set X, then one can generate a topology by taking as the family of closed sets all sets containing together with any convergent sequence the limit of this sequence; this topology is called the *sequential topology* induced by the limit operator λ. Let T^λ be the induced topology on X by the sequential convergence λ. Then we have the following implication:

$$x_n \xrightarrow{\lambda} x \Rightarrow x_n \xrightarrow{T^\lambda} x.$$

DEFINITION 2.7. Let λ be a sequential convergence on X such that

(L3) If a sequence (x_n) does not converge to x (i.e., it is false that $x_n \xrightarrow{\lambda} x$), then it contains a subsequence (x_{n_m}) such that no subsequence of (x_{n_m}) converges to x.

(L4) If $x_n \xrightarrow{\lambda} x_1$ and $x_n \xrightarrow{\lambda} x_2$, then $x_1 = x_2$.

Then the sequential convergence λ is called an $\mathcal{L}^*$-convergence and the pair (X, T^λ) an $\mathcal{L}^*$-space.

We note that every $\mathcal{L}^*$-space with the sequential topology is a T_1-space. The following basic theorem relating sequential convergence and topology was proved by J. Kisyński [7]: for a sequence (x_n) of points of an $\mathcal{L}^*$-space with the sequential topology, $x_n \xrightarrow{T^\lambda} x$ if and only if $x_n \xrightarrow{\lambda} x$, i.e., that the convergence *a posteriori* is equivalent to the convergence *a priori.*

The following sequential convergence on the line set $\mathcal{L}$ was defined by D. Simon in [12, 1.2.1].

DEFINITION 2.8. Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. We define on the line set $\mathcal{L}$ the sequential convergence to be

$$L_n \xrightarrow{*} L :\Longleftrightarrow \forall (L_{n_m}): \quad \emptyset \neq \limsup L_{n_m} \subseteq L.$$

Then $*$ is an $\mathcal{L}^*$-convergence in [12, 1.2.1, Lemma 3]. Let T^* be the sequential topology on $\mathcal{L}$ induced by the $\mathcal{L}^*$-convergence $*$.

LEMMA 2.9. *Let $(P^n, \mathcal{L})$ be an R^n-geometry. Let (L_n) be a sequence in $\mathcal{L}$ and $L \in \mathcal{L}$. Then:*

(1) *If $L_n \longrightarrow L$ in $(\mathcal{L}, T^*)$, then $L_n \longrightarrow L$ in the final topology $(\mathcal{L}, F)$.*
(2) *If $L_n \longrightarrow L$ in $(\mathcal{L}, H)$, then $L_n \longrightarrow L$ in the sequential topology $(\mathcal{L}, T^*)$.*
(3) *$F \subseteq T^* \subseteq H$.*
(4) *If $OM = T^*$, then $OM = T^* = H$.*

Proof. (1) Let (L_n) converge to L in $(\mathcal{L}, T^*)$. Assume that (L_n) does not converge to L in $(\mathcal{L}, F)$. Hence there exist an open nbd U of L and a subsequence (L_{n_m}) such that $[\cup_{n_m}\{L_{n_m}\}] \cap U = \emptyset$. Hence we have $\emptyset = \vee^{-1}([\cup_{n_m}\{L_{n_m}\}] \cap U) = [\cup_{n_m}(L_{n_m} \times L_{n_m} - \triangle)] \cap \vee^{-1}(U)$. Since $\emptyset \neq \limsup L_{n_m} \subseteq L$, choose a point $a \in \limsup L_{n_m} \subseteq L$. Hence there exists a sequence $(a_{n_{m_k}})$ with $a_{n_{m_k}} \in L_{n_{m_k}}$ such that $a_{n_{m_k}} \longrightarrow a$. We have also

$$a \in \liminf L_{n_{m_k}} \subseteq \limsup L_{n_{m_k}}.$$

By Lemma 2.1, choose a point $b \in \limsup L_{n_{m_k}}$ with $a \neq b$. Hence there exists a subsequence $(L_{n_{m_{k_q}}})$ of (L_{n_m}) such that $a, b \in \liminf L_{n_{m_{k_q}}}$. We get also

$$(a,b) \in \liminf L_{n_{m_{k_q}}} \times \liminf L_{n_{m_{k_q}}} - \triangle \subseteq L \times L - \triangle \subset \vee^{-1}(U).$$

Since $\vee^{-1}(U)$ is open in $\mathcal{P}^n \times \mathcal{P}^n - \triangle$ and $a \neq b$, choose two disjoint open sets V_1, V_2 in $\mathcal{P}^n$ such that $(a,b) \in V_1 \times V_2 \subset \vee^{-1}(U)$. We get

$$\emptyset \neq [\cup_{n_{m_{k_q}}}(L_{n_{m_{k_q}}} \times L_{n_{m_{k_q}}} - \triangle)] \cap (V_1 \times V_2) \subseteq [\cup_{n_m}(L_{n_m} \times L_{n_m} - \triangle)] \cap \vee^{-1}(U),$$

this leads to a contradiction. Hence (L_n) converges to L in $(\mathcal{L}, F)$.

(2) Assume that (L_n) converges to L in $(\mathcal{L}, H)$. Then for all subsequence (L_{n_m}), we have $L = \liminf L_{n_m} = \limsup L_{n_m}$. Hence (L_n) converges to L in $(\mathcal{L}, T^*)$.

(3) By Lemma 1.3, the assertion is obvious.

(4) It is sufficient to show that if the sequence (L_n) converges to L in $(\mathcal{L}, OM = T^*)$, then the sequence (L_n) converges to L in $(\mathcal{L}, H)$. By Lemma 2.5 and Definition 2.8, we have $L \subseteq \liminf L_n \subseteq \limsup L_n \subseteq L$, hence (L_n) converges to L in $(\mathcal{L}, H)$. □

LEMMA 2.10. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $F = OM$. Then the following properties are equivalent:*

(1) *$(\mathcal{L}, OM)$ is a T_2-space.*
(2) *If a sequence of lines (L_n) converges to L in OM, then the sequence (L_n) converges to L in the sequential topology T^*.*

Proof. (1) ⇒ (2). Assume that a sequence (L_n) converges to L in OM. Let (L_{n_m}) be a subsequence of (L_n). By Lemma 2.5, we have $L \subseteq \liminf L_{n_m} \subseteq \limsup L_{n_m}$, in particular, $\emptyset \neq \limsup L_{n_m}$. Next we will show that $\limsup L_{n_m} \subseteq L$. Let $a \in \limsup L_{n_m}$, then there exists a sequence $(a_{n_{m_k}}), a_{n_{m_k}} \in L_{n_{m_k}}$ such that $a_{n_{m_k}} \longrightarrow a$. Assume that $a \notin L$. Choose a point $b \in L$. Since $L \subseteq \liminf L_{n_{m_k}}$, there is a sequence $(b_{n_{m_k}}), b_{n_{m_k}} \in L_{n_{m_k}}$ such that $b_{n_{m_k}} \longrightarrow b$. Since $F = OM$, the join operation is topological, hence we have $a_{n_{m_k}} \vee b_{n_{m_k}} \longrightarrow a \vee b$ in OM. On the other hand, we have $L_{n_{m_k}} = a_{n_{m_k}} \vee b_{n_{m_k}} \longrightarrow L$ in OM. By the assumption (1), we get $L = a \vee b$ in OM, this leads to contradiction, hence $a \in L$.

(2) $\Rightarrow$ (1). Let a sequence of lines (L_n) converges to some two lines L and L_1 in OM. Since $L \subseteq \liminf L_n \subseteq \limsup L_n \subseteq L$ and $L_1 \subseteq \liminf L_n \subseteq \limsup L_n \subseteq L_1$, we get $L_1 = L$. Hence, $(\mathcal{L}, OM)$ is a T_2-space. □

COROLLARY 2.11. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $F = OM$. Then:*

(1) *$(\mathcal{L}, OM)$ is a T_2-space if and only if $OM = H = T^*$.*
(2) *If $F = H$, then the line space $(\mathcal{L}, H)$ is a locally compact hausdorff space.*

Proof. (1) Assume that $OM = H$, then it is clear that the space $(\mathcal{L}, OM)$ is a T_2-space. Conversely, assume that $(\mathcal{L}, OM)$ is a T_2-space. Let U be open in the hausdorff topology H. We have to show that $\mathcal{L} - U$ is closed in OM. Assume that $\mathcal{L} - U$ is not closed in OM. Since $(\mathcal{L}, OM)$ is 2° countable, there exists a sequence (L_n) in $\mathcal{L} - U$ which converges to $L \in U$ in OM. By Lemma 2.10, (L_n) converges to L in the sequential topology T^*. Therefore, (L_n) converges to L in the hausdorff topology H. For sufficiently large $n \in N$, we have $L_n \in U$, this leads to a contradiction. Consequently, $H \subseteq OM$. Hence we have $F = OM = H$. By Lemma 2.9 (3), we have also $F = OM = T^* = H$.

(2) By Lemma 1.2, the proof is obvious. □

Now we want to consider two topologies on the line set $\mathcal{L}$. Let $\mathcal{H}$ be the set of all parametrizations of lines, i.e., $\mathcal{H}$ consists of all maps $f : R \longrightarrow L \subset \mathcal{P}^n$ such that f is a homeomorphism onto some line $L \in \mathcal{L}$. Endow $\mathcal{H}$ with the *compact-open topology* and the *point-open topology* respectively (compare Dugundji XII.1 [5]).

DEFINITION 2.12. The COT topology on $\mathcal{L}$ is the quotient topology induced by the compact-open topology on $\mathcal{H}$ obtained by identifying maps which have the same image. The POT *topology* on $\mathcal{L}$ is the quotient topology induced by the point-open topology on $\mathcal{H}$.

Let $l : R \longrightarrow L$ be a homeomorphism onto some line $L \in \mathcal{L}$, that is, $l \in \mathcal{H}$. Then we use $[\ ] : \mathcal{H} \longrightarrow \mathcal{H}/\sim : l \rightarrow [l]$ to denote the quotient mapping from $\mathcal{H}$ to $\mathcal{H}/\sim$. We define a mapping from $\mathcal{H}/\sim$ to $\mathcal{L}$ to be $\varphi : \mathcal{H}/\sim \longrightarrow \mathcal{L} : [l] \rightarrow \varphi([l]) = l(R)$. Let us denote by η the composition of two mappings $[\ ]$ and φ, that is, $\eta(l) = \varphi([l])$ for $l \in \mathcal{H}$. We also call the mapping η the *quotient mapping* from $\mathcal{H}$ to $\mathcal{L}$. From now on, we shall use only the closed intervals $I = [a, b]$, where a, b are not necessarily distinct. The family of all sets (I,V), where $I \subset R$ is a closed interval and $V \subset \mathcal{P}^n$ is open, will still be a subbasis for the compact-open topology on $\mathcal{H}$ [5, Chapter XII].

LEMMA 2.13. *Let $(\mathcal{P}^n,\mathcal{L})$ be an R^n-geometry. Then:*

(1) *$\eta : \mathcal{H} \longrightarrow (\mathcal{L}, COT)$ is a continuous open surjection.*

(2) *$OM \subsetneq POT$. Hence the inclusions hold: $F \subsetneq OM(= OJ) \subsetneq POT \subsetneq COT$.*

Proof. (1) It is sufficient to show that η is an open mapping. Let $\cap_1^n(I_i,V_i)$ be a basic open set in $\mathcal{H}$. Then we get $\eta(\cap_1^n(I_i,V_i)) = \{K \in \mathcal{L} :$ there is a representation $k : R \longrightarrow K$ with $k \in \cap_1^n(I_i,V_i)\}$. We have to show that $\eta^{-1}(\eta(\cap_1^n(I_i,V_i)))$ is open in $\mathcal{H}$. Let $l \in \eta^{-1}(\eta(\cap_1^n(I_i,V_i)))$, then $\eta(l) \in \eta(\cap_1^n(I_i,V_i))$. Hence there is a representation $l' : R \longrightarrow \eta(l)$ with $l' \in \cap_1^n(I_i,V_i)$. Let $l^{-1}(l'(I_i)) = I'_i$, then $l \in \cap_1^n(I'_i,V_i)$. Next we shall show that $l \in \cap_1^n(I'_i,V_i) \subset \eta^{-1}(\eta(\cap_1^n(I_i,V_i)))$. Let $\alpha = l^{-1} \circ l' : R \longrightarrow R$, then $\alpha(I_i) = I'_i$. For $g \in \cap_1^n(I'_i,V_i)$ let $g' = g\circ\alpha$, then $g'(I_i) = g(\alpha(I_i)) = g(I'_i) \subsetneq V_i$. Therefore, $\eta(g) \in \eta(\cap_1^n(I_i,V_i))$ and $g \in \eta^{-1}(\eta(\cap_1^n(I_i,V_i)))$. Hence, $\eta : H \longrightarrow (\mathcal{L}, COT)$ is an open mapping.

(2) Let O be open in $\mathcal{P}^n$ and $L \in M_O$. Since $\eta : H \longrightarrow (\mathcal{L}, POT)$ is an open mapping, let $l : R \longrightarrow L$ be a representation of L with $l(x) \in O$. Then $L = \eta(l) \in \eta((x,O)) \subset M_O$. □

3. The sequential topology induced by the sequential convergence η^*

DEFINITION 3.1. Let Y be an arbitrary set, X a 1° countable space, and $f : X \longrightarrow Y$ a surjection. We define on the set Y the sequential convergence to be

$$y_n \xrightarrow{f^*} y :\Longleftrightarrow$$

for each $x \in X$ with $f(x) = y$, there exists a sequence (x_n) in X such that $x_n \longrightarrow x$ in X and $f(x_n) = y_n$.

We can also define on Y the notation of accumulation (we say that (y_n) accumulates at y) to be

$$y_n \overset{f^*}{\succ\!\!-\!\!\circ}\, y :\Longleftrightarrow$$

for each $x \in X$ with $f(x) = y$, there exists a sequence (x_n) in X such that $x_n \succ\!\!-\!\!\circ\, x$ in X and $f(x_n) = y_n$, where the symbol $x_n \succ\!\!-\!\!\circ\, x$ means that the sequence (x_n) accumulates at x.

LEMMA 3.2. *Let Y be an arbitrary set, X a 1° countable space, and $f : X \longrightarrow Y$ a surjection. Let (y_n) be a sequence in Y and $y \in Y$ Then:*

(1) *If $y_n \overset{f^*}{\longrightarrow} y$, then $y_n \overset{f^*}{\rightarrowtail\!\circ}\, y$.*
(2) *Let (y_{n_m}) be a subsequence of (y_n). Then:*
  (a) $[y_n \overset{f^*}{\longrightarrow} y] \Rightarrow [y_{n_m} \overset{f^*}{\longrightarrow} y]$.
  (b) $[y_{n_m} \overset{f^*}{\rightarrowtail\!\circ}\, y] \Rightarrow [y_n \overset{f^*}{\rightarrowtail\!\circ}\, y]$.

THEOREM 3.3. *Let Y be an arbitrary set, X a 1° countable space, and $f : X \longrightarrow Y$ a surjection. Then f^* is a sequential convergence satisfying the condition $(L3)$.*

Proof. It is easy to check that f^* satisfies the conditions (L1) and (L2). We prove that f^* satisfies the condition (L3). Let (y_n) be a given sequence in Y and $y \in Y$ with the following property:

$$\forall (y_{n_m})\ \exists (y_{n_{m_k}}) : y_{n_{m_k}} \overset{f^*}{\longrightarrow} y.$$

Then we will show that $y_n \overset{f^*}{\longrightarrow} y$. Let $x \in X$ with $f(x) = y$. For each $n \in N$ define $A_n := f^{-1}(y_n)$, then (A_n) is a sequence of subsets of X. We call (A_{n_m}) a subsequence of (A_n) if $n_1 < n_2 < \cdots < n_m \cdots$.

Step 1. First we show that for all nbds U of x and for all subsequence (A_{n_m}) of (A_n), there exists an element $A_{n_m} \in \{A_{n_1}, A_{n_2}, A_{n_3}, \cdots\}$ such that $A_{n_m} \cap U \neq \emptyset$.
Let U be a given nbd of x and a given subsequence (A_{n_m}) of (A_n). Since $f(A_{n_m}) = f[f^{-1}(y_{n_m})] = y_{n_m}$, we have a subsequence (y_{n_m}) of (y_n). By the given condition, there exists a subsequence $(y_{n_{m_k}})$ of (y_{n_m}) such that $y_{n_{m_k}} \overset{f^*}{\longrightarrow} y$. By definition, there exists a sequence $(x_{n_{m_k}})$ in X such that $x_{n_{m_k}} \longrightarrow x$ in X and $f(x_{n_{m_k}}) = y_{n_{m_k}}$. Since $x_{n_{m_k}} \longrightarrow x$ in X with $x_{n_{m_k}} \in f^{-1}(y_{n_{m_k}}) = A_{n_{m_k}} \in \{A_{n_1}, A_{n_2}, A_{n_3}, \cdots\}$ and U is a nbd of x, the assertion is clear.

Step 2. Next we show that for all nbds U of x and for all subsequence (A_{n_m}) of (A_n), there exists an integer N such that

$$A_{n_m} \cap U \neq \emptyset \quad \text{for all} \quad n_m > N.$$

Let U be a given nbd of x and a given subsequence (A_{n_m}) of (A_n). Assume that there are no integers N such that $A_{n_m} \cap U \neq \emptyset$ for all $n_m > N$. We start with $N = n_1$, then there exists $n_{m_1} > n_1$ such that

$A_{n_{m_1}} \cap U = \emptyset$. Let $N = n_{m_1}$, then there exists $n_{m_2} > n_{m_1}$ such that $A_{n_{m_2}} \cap U = \emptyset$, and so on. Hence, we have a subsequence $(A_{n_{m_k}})$ of (A_n) such that $A_{n_{m_k}} \cap U = \emptyset$ for all $n_{m_k} \in N$, this leads to a contradiction to Step 1.

Step 3. Since X is 1° countable, let $\mathcal{C} = \{U_n | U_{n+1} \subset U_n \quad \text{for} \quad n \in N\}$ be a decreasing countable basis at the point x. For U_1 we may assume that $A_n \cap U_1 \neq \emptyset$ for all $n \in N$. For U_2 there exists an integer k_1 such that $A_n \cap U_2 \neq \emptyset$ for all $n > k_1$. For $n = 1$ to k_1 choose $x_1, \cdots, x_{k_1} \in U_1$. For U_3 and the subsequence $\{A_n\}_{n \in N} - \{A_1, \cdots, A_{k_1}\}$, there exists an integer k_2 such that $A_n \cap U_3 \neq \emptyset$ for all $n > k_2$. Similarly, for $n = k_1 + 1$ to k_2, choose $x_{k_1+1}, \cdots, x_{k_2}$, and so on. By this process, we have a sequence (x_n) which converges to x and $f(x_n) = y_n$. □

In the proof of Theorem 3.3 we have the following corollary.

COROLLARY 3.4. *Let Y be an arbitrary set, X a 1° countable space, and $f : X \longrightarrow Y$ a surjection. Let (y_n) be a sequence in Y and $y \in Y$. Let $x \in X$ with $f(x) = y$, and for each $n \in N$ let $A_n = f^{-1}(y_n)$. Then:*

(1) $y_n \xrightarrow{f^*} y \iff \forall\, U : \textit{nbd of } x, \quad \exists N \quad \forall n \geq N : A_n \cap U \neq \emptyset.$

(2) $y_n \overset{f^*}{\rightarrowtail\!\!\circ}\, y \iff \forall\, U : \textit{nbd of } x, \quad \forall N \quad \exists n \geq N : A_n \cap U \neq \emptyset.$

DEFINITION 3.5. Let Y be an arbitrary set, X a 1° countable topological space, and $f : X \longrightarrow Y$ a surjection. Then the topology T^{f^*} on Y is the sequential topology induced by the sequential convergence f^*.

LEMMA 3.6. *Let $(Y, \mathcal{T})$ be a T_2-space, X a 1° countable space, and $f : X \longrightarrow (Y, \mathcal{T})$ a continuous surjection. Then the sequential convergence f^* is an $\mathcal{L}^*$-convergence, i.e., f^* satisfies the condition $(L4)$.*

Proof. Assume that $a_n \xrightarrow{f^*} a$ and $a_n \xrightarrow{f^*} b$. Let $f(x) = a$, then there exists a sequence (x_n) in X such that $x_n \longrightarrow x$ in X and $f(x_n) = a_n$. Let $f(y) = b$, then there exists a sequence (y_n) in X such that $y_n \longrightarrow y$ in X and $f(y_n) = a_n$. Since $f : X \longrightarrow (Y, \mathcal{T})$ is continuous, we have $f(x_n) = a_n \xrightarrow{\mathcal{T}} f(x) = a$ and $f(y_n) = a_n \xrightarrow{\mathcal{T}} f(y) = b$. Since $(Y, \mathcal{T})$ is a T_2-space, we have $a = b$. □

THEOREM 3.7. *Let Y be an arbitrary set, X a 1° countable space, and $f : X \longrightarrow Y$ a surjection. Let $\mathcal{T}_2$ be the topology on Y be generated by the subbasis $\sum = \{f(U) | U$ is open in $X\}$. Let T^{f^*} be the sequential* *topology on Y. Assume that the sequential convergence f^* is an $\mathcal{L}^*$-convergence, i.e., f^* satisfies the condition $(L4)$. Then $f : X \longrightarrow (Y, T^{f^*})$ is an open mapping, that is, $\mathcal{T}_2 \subseteq T^{f^*}$.*

Proof. We prove that f is an open mapping. Let U be open in X. Then we show that $Y - f(U)$ is closed in (Y, f^*). Assume that $Y - f(U)$ is not closed in (Y, f^*). Hence, there exists a sequence (y_n) in $Y - f(U)$ which converges to $y \in f(U)$. Let $x \in U$ with $f(x) = y$. Since $y_n \xrightarrow{f^*} y$ in Y, there is a sequence (x_n) in X such that $x_n \longrightarrow x$ and $f(x_n) = y_n$. Therefore, $x_n \in U$ for sufficiently large n, consequently we have $f(x_n) \in f(U)$. This leads to a contradiction to (y_n) in $Y - f(U)$. Hence, $f(U)$ is open in (Y, T^{f^*}). □

COROLLARY 3.8. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry and $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow \mathcal{L}$ the join mapping. Let $T^{\vee^*}$ be the sequential topology on $\mathcal{L}$ induced by the sequential convergence $\vee^*$. Assume that $F = OM$. Then the following two properties are equivalent:*

1. *The sequential convergence $\vee^*$ is an $\mathcal{L}^*$-convergence, i.e., $\vee^*$ satisfies the condition $(L4)$.*
2. *$F = OM = T^{\vee^*}$ and $(\mathcal{L}, OM)$ is a T_2-space.*

Proof. (1) $\Rightarrow$ (2). By Theorem 3.7, the join mapping $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow (\mathcal{L}, T^{\vee^*})$ is an open mapping. It is sufficient to show that $\vee : \mathcal{P}^n \times \mathcal{P}^* \setminus \triangle \longrightarrow (\mathcal{L}, T^{\vee^*})$ is continuous. Assume that a sequence $((a_n, b_n))$ converges to (a, b) in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. We have to show that $a_n \vee b_n \xrightarrow{\vee^*} a \vee b$. Let $(x, y) \in \mathcal{P}^n \times \mathcal{P}^* \setminus \triangle$ with $x \vee y = a \vee b$. Since $F = OM$, by Lemma 2.5, we have $a \vee b \subseteq \liminf(a_n \vee b_n)$, hence we have a sequence $((x_n, y_n))$ in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$ such that $(x_n, y_n) \longrightarrow (x, y)$ and $x_n \vee y_n = a_n \vee b_n$. Hence $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow (\mathcal{L}, T^{\vee^*})$ is continuous. Since $(\mathcal{L}, OM)$ is a 2° countable space, the condition $(L4)$ implies that $(\mathcal{L}, OM)$ is a T_2-space.

(2) $\Rightarrow$ (1). If $L_n \xrightarrow{\vee^*} L_1$ and $L_n \xrightarrow{\vee^*} L_2$, then $L_n \xrightarrow{T^{\vee^*}} L_1$ and $L_n \xrightarrow{T^{\vee^*}} L_2$. Since $(\mathcal{L}, T^{\vee^*})$ is a T_2- and 2° space, we have $L_1 = L_2$. □

Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $\mathcal{H}$ be the set of all parametrizations of lines with the compact-open topology and $\eta : \mathcal{H} \longrightarrow \mathcal{L}$ the quotient mapping. Since $\mathcal{H}$ is a 2° countable space, we consider the following sequential convergence on the line set $\mathcal{L}$:

DEFINITION 3.9. Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $\eta : \mathcal{H} \longrightarrow \mathcal{L}$ be the quotient mapping. We define on the line set $\mathcal{L}$ a sequential convergence to be

$$L_n \xrightarrow{\eta^*} L :\Longleftrightarrow$$

for each $l \in \mathcal{H}$ with $\eta(l) = L$, there exists a sequence (l_n) in $\mathcal{H}$ such that $l_n \longrightarrow l$ in $\mathcal{H}$ and $\eta(l_n) = L_n$. That is, let $l : R \longrightarrow L$ be an arbitrarily given representation of the line L. Then for each $n \in N$ there is a representation $l_n : R \longrightarrow L_n$ such that $l_n \longrightarrow l$ in the compact-open topology of $(\mathcal{L}, \mathcal{H})$.

For $\alpha \in R^R$ and $l \in \mathcal{H}$, the composition $l \circ \alpha \in \mathcal{H}$, so that $T(\alpha, l) = l \circ \alpha$ defines a mapping $R^R \times \mathcal{H} \longrightarrow \mathcal{H}$ which is continuous in [5, 2.2 Theorem]. Note that if $L_n \xrightarrow{\eta^*} L$, then for a given representation $l : R \longrightarrow L$ there are representations l_n of L_n such that $l_n \longrightarrow l$ in the compact-open topology of $\mathcal{H}$. Let $l' : R \longrightarrow L$ be another representation of the line L, then we can find representations l'_n such that $l'_n \longrightarrow l'$ in the compact-open topology of $\mathcal{H}$. For $l^{-1} : L \longrightarrow R$ let $\varphi := l^{-1} \circ l' : R \longrightarrow R$. Since the mapping $T(\varphi, \ ) : \mathcal{H} \longrightarrow \mathcal{H}$ is continuous with $T(\varphi, l) = l'$, hence we have $l'_n = T(\varphi, l_n) \longrightarrow l' = T(\varphi, l)$.

LEMMA 3.10. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $\eta : \mathcal{H} \longrightarrow \mathcal{L}$ be the quotient mapping, η^* the sequential convergence and T^{η^*} the sequential topology on $\mathcal{L}$. Then the following properties are equivalent:*

(1) *The sequential convergence η^* is an $\mathcal{L}^*$-convergence.*
(2) *$(\mathcal{L}, T^{\eta^*})$ is a T_2-space.*

Proof. (1) $\Rightarrow$ (2). By Theorem 3.7, $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, T^{\eta^*})$ is an open mapping. Next we show that $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous. Let $l_n \longrightarrow l$ in $\mathcal{H}$. Let $l' \in \mathcal{H}$ with $\eta(l') = \eta(l)$. For $l^{-1} : L \longrightarrow R$ let $\varphi = l^{-1} \circ l' : R \longrightarrow R$. Then we have $l'_n = T(\varphi, l_n) \longrightarrow l' = T(\varphi, l)$ and $\eta(l'_n) = \eta(l_n)$. Therefore, η is continuous mapping. By Lemma 1.2 and Lemma 2.13, $COT = \eta^*$. Since $(\mathcal{L}, T^{\eta^*})$ is 2° countable space and every sequence in $(\mathcal{L}, T^{\eta^*})$ has at most one limit, hence $(\mathcal{L}, T^{\eta^*})$ is a T_2-space.

(2) $\Rightarrow$ (1). Assume that $(\mathcal{L}, T^{\eta^*})$ is a T_2-space. It is easy to check that the sequential convergence η^* satisfies the condition (L4). □

THEOREM 3.11. *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $\eta : \mathcal{H} \longrightarrow \mathcal{L}$ be the quotient mapping. Assume that the sequential convergence η^* is an $\mathcal{L}^*$-convergence. Then:*

(1) *$(\mathcal{L}, T^{\eta^*})$ is a T_2-space.*

(2) $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, T^{\eta^*})$ *is a continuous open surjection and* $COT = T^{\eta^*}$.

(3) $(\mathcal{L}, T^{\eta^*})$ *is a* $2°$*-countable space.*

THEOREM 3.12. *Let* $(\mathcal{P}^n, \mathcal{L})$ *be a topological* R^n*-geometry with the* T^{η^*} *topology. Then the following properties are equivalent:*

(1) *The sequential convergence* η^* *is an* $\mathcal{L}^*$*-convergence.*

(2) $(\mathcal{L}, T^{\eta^*})$ *is a* T_2*-space.*

(3) *If a sequence of lines* (L_n) *converges to* L *in* T^{η^*}*, then the sequence* (L_n) *converges to* L *in the hausdorff topology* H*.*

(4) $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, H)$ *is continuous.*

Proof. By Lemma 3.10, the assertions (1) and (2) are equivalent.

(2) $\Rightarrow$ (3). Assume that a sequence of lines (L_n) converges to L in T^{η^*}. Let l be a representation of L, and for each $n \in N$ let l_n be a representation of L_n such that (l_n) converges to l in the compact-open topology of $\mathcal{H}$. We have to show that $L \subseteq \liminf L_n \subseteq \limsup L_n \subseteq L$. Since $OM \subseteq T^{\eta^*}$, we have the first inclusion $L \subseteq \liminf L_n$. Let $y \in \limsup L_n$, then there exist a subsequence $(y_{n_m}), y_{n_m} \in L_{n_m}$ such that $y_{n_m} \longrightarrow y$. Assume that $y \notin L$. Choose a point $y_1 \in L$ and let $l(x_1) = y_1$, then $l_{n_m}(x_1) \longrightarrow l(x_1) = y_1$. Since the join operation is topological, we have $l_{n_m}(x_1) \vee y_{n_m} \longrightarrow l(x_1) \vee y$ in T^{η^*}. On the other hand, we have $L_{n_m} = l_{n_m}(x_1) \vee y_{n_m} \longrightarrow L$ in T^{η^*}. By the assumption (2), we get $L = l(x_1) \vee y$ in T^{η^*}, this leads to contradiction, hence $y \in L$.

(3) $\Rightarrow$ (4). Let $l_n \longrightarrow l$ in $\mathcal{H}$. Since $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous, we have $\eta(l_n) \xrightarrow{T^{\eta^*}} \eta(l)$. By (3), we have $\eta(l_n) \longrightarrow \eta(l)$ in $(\mathcal{L}, H)$. Hence, $\eta : \mathcal{H} \longrightarrow (\mathcal{L}, H)$ is continuous.

(4) $\Rightarrow$ (2). By (3), $H \subseteq T^{\eta^*}$ and $(\mathcal{L}, H)$ is a T_2-space, hence $(\mathcal{L}, T^{\eta^*})$ is a T_2-space. □

We use $T(d)$ to denote the usual topology of $\mathcal{P}^n$. Denote by $T(d^2)$ the usual product topology on $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. Let $p, q \in R$ with $p < q$. For an R^n-geometry $(\mathcal{P}^n, \mathcal{L})$, we define the two mappings as follows:

$$\phi_{p,q} : \mathcal{H} \longrightarrow \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle : l \rightarrow (l(p), l(q)),$$

$$\phi_p : \mathcal{H} \longrightarrow \mathcal{P}^n : l \rightarrow l(p).$$

It is clear that two mappings $\phi_{p,q}, \phi_p$ are continuous surjections and $\eta = \vee \circ \phi_{p,q}$. Furthermore, the evaluation mapping

$$e : \mathcal{H} \times R \longrightarrow \mathcal{P}^n : e(l, x) = l(x) \quad \text{and}$$

the mapping

$$E:\mathcal{H}\times(R\times R\setminus\triangle_R)\longrightarrow\mathcal{P}^n\times\mathcal{P}^n\setminus\triangle : E(l,x,y)=(l(x),l(y))$$

are continuous surjections, where $\triangle_R=\{(x,x)|x\in R\}$ is the diagonal.

LEMMA 3.13. *Let $p,q\in R$ with $p<q$. Let $(\mathcal{P}^n,\mathcal{L})$ be an R^n-geometry. Then:*

(1) *The sequential convergences $\phi^*_{p,q}$, ϕ^*_p, e^* and E^* are $\mathcal{L}^*$-convergences.*

(2) *Let $T^{\phi^*_{p,q}}, T^{E^*}$ be the sequential topologies on $\mathcal{P}^n\times\mathcal{P}^n\setminus\triangle$ induced by the $\mathcal{L}^*$-convergences $\phi^*_{p,q}, E^*$ respectively. Then $T(d^2)\subseteq T^{\phi^*_{p,q}}=T^{E^*}$.*

Proof. (1) By Lemma 3.6, the convergences $\phi^*_{p,q}$, ϕ^*_p, e^* and E^* satisfy the condition (L4).

(2) Since two mappings $\phi_{p,q}$ and E are continuous, we have $T(d^2)\subseteq T^{\phi^*_{p,q}}$ and $T(d^2)\subseteq T^{E^*}$. Let $((a_n,b_n))$ be a sequence in $\mathcal{P}^n\times\mathcal{P}^n\setminus\triangle$ and $(a,b)\in\mathcal{P}^n\times\mathcal{P}^n\setminus\triangle$. Assume that $(a_n,b_n)\xrightarrow{\phi^*_{p,q}}(a,b)$. We have to show that $(a_n,b_n)\xrightarrow{E^*}(a,b)$. Let $(l,u,v)\in\mathcal{H}\times(R\times R\setminus\triangle_R)$ with $(l(u),l(v))=(a,b)$. Define a function $\alpha:R\longrightarrow R$ to be

$$\alpha(x)=\frac{u-v}{p-q}x+u-\frac{u-v}{p-q}p.$$

Let $l'=l\circ\alpha$, then $(l'(p),l'(q))=(a,b)$. Since $(a_n,b_n)\xrightarrow{\phi^*_{p,q}}(a,b)$, there exists a sequence (l'_n) in $\mathcal{H}$ such that $l'_n\longrightarrow l'$ in $\mathcal{H}$ and $\phi_{p,q}(l'_n)=(l'_n(p),l'_n(q))=(a_n,b_n)$. Let $l_n=l'_n\circ\alpha^{-1}$. Hence we have a sequence $((l_n,u,v))$ in $\mathcal{H}\times(R\times R\setminus\triangle_R)$ such that $(l_n,u,v)\longrightarrow(l,u,v)$ in $\mathcal{H}\times(R\times R\setminus\triangle_R)$ and $E(l_n,u,v)=(a_n,b_n)$.

Conversely, assume that $(a_n,b_n)\xrightarrow{E^*}(a,b)$. We have to show that $(a_n,b_n)\xrightarrow{\phi^*_{p,q}}(a,b)$. Let $l\in\mathcal{H}$ with $\phi_{p,q}(l)=(l(p),l(q))=(a,b)$. Since $E(l,p,q)=\phi_{p,q}(l)=(a,b)$, by assumption, there exists a sequence $((l_n,p_n,q_n))$ in $\mathcal{H}\times(R\times R\setminus\triangle_R)$ such that $(l_n,p_n,q_n)\longrightarrow(l,p,q)$ in $\mathcal{H}\times(R\times R\setminus\triangle_R)$ and $E(l_n,p_n,q_n)=(a_n,b_n)$. We define a function $\alpha_n:R\longrightarrow R$ to be

$$\alpha_n(x)=\frac{p_n-q_n}{p-q}x+p_n-\frac{p_n-q_n}{p-q}p$$

for each $n\in N$. We can check that the sequence (α_n) converges to identity function $i:R\longrightarrow R$ in R^R. Since the mapping $T:R^R\times\mathcal{H}\longrightarrow\mathcal{H}$ is continuous, we have $T(\alpha_n,l_n)=l_n\circ\alpha_n\longrightarrow T(i,l)$ in $\mathcal{H}$ and $\phi_{p,q}(T(\alpha_n,l_n))=(a_n,b_n)$. □

From now on, we assume that the sequential convergence η^* is an $\mathcal{L}^*$-convergence (see Definition 3.9 and Lemma 3.10).

THEOREM 3.14. *Let $p, q \in R$ with $p < q$. Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Let $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2))$ be the continuous mapping defined above. Then:*

(1) *Let $T^{\phi^*_{p,q}}$ be the sequential topology on $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$ induced by the $\mathcal{L}^*$-convergence $\phi^*_{p,q}$. Then $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T^{\phi^*_{p,q}})$ is a continuous open mapping. In particular, the join mapping $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T^{\phi^*_{p,q}}) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is a continuous mapping.*
(2) *If $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2))$ is an open mapping, then $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2)) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is a continuous open mapping. In particular, $T(d^2) = \phi^*_{p,q}$ and the topologies $F, OJ = OM, POT, COT = T^{\eta^*}$ and H for $\mathcal{L}$ coincide.*

Proof. (1) By Theorem 3.7, $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T^{\phi^*_{p,q}})$ is an open mapping. We have to show that $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T^{\phi^*_{p,q}})$ is also continuous. Assume that $l_n \longrightarrow l$ in $\mathcal{H}$. Then we have to show that $(l_n(p), l_n(q)) \xrightarrow{\phi^*_{p,q}} (l(p), l(q))$. Let $g \in \mathcal{H}$ with $\phi_{p,q}(g) = (g(p), g(q)) = (l(p), l(q))$. For $l^{-1} : L \longrightarrow R$ let $\varphi = l^{-1} \circ g$. Since the mapping $T(\varphi, \ ) : \mathcal{H} \longrightarrow \mathcal{H}$ is continuous, hence we have $T(\varphi, l_n) = l_n \circ \varphi \longrightarrow T(\varphi, l) = g$ and $\phi_{p,q}(T(\varphi, l_n)) = (l_n(p), l_n(q))$. Since $\phi_{p,q} : \mathcal{H} \longrightarrow (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T^{\phi^*_{p,q}})$ is a continuous open mapping, then $\phi_{p,q}$ is an identification map. By [5, Chapter VI, Theorem 3.1], $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, \phi^*_{p,q}) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous.

(2) By [5, Chapter VI, Theorem 3.1], $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2)) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous. By Lemma 3.12, the topologies $F, OJ = OM, POT, COT = T^{\eta^*}$ and H for $\mathcal{L}$ coincide. □

Lemma 3.13 implies the following facts. Let (a_n) and (b_n) be two convergent sequences with limits a and b respectively. Assume that $a_n \vee b_n \longrightarrow a \vee b$ in $(\mathcal{L}, T^{\eta^*})$. Hence, for $l \in \mathcal{H}$ with $\eta(l) = a \vee b$, there exists a sequence (l_n) in $\mathcal{H}$ such that $l_n \longrightarrow l$ in $\mathcal{H}$ and $\eta(l_n) = a_n \vee b_n$. Let $l(p) = a, l(q) = b, l_n(p_n) = a_n$ and $l(q_n) = b_n$. Assume that $p_n \longrightarrow p$ and $q_n \longrightarrow q$ in R. We define a function $\alpha_n : R \longrightarrow R$ to be

$$\alpha_n(x) = \frac{p_n - q_n}{p - q} x + p_n - \frac{p_n - q_n}{p - q} p$$

for each $n \in N$. We can check that the sequence (α_n) converges to identity function $i : R \longrightarrow R$ in R^R. Since the mapping $T : R^R \times \mathcal{H} \longrightarrow \mathcal{H}$ is continuous, we have $T(\alpha_n, l_n) = l_n \circ \alpha_n \longrightarrow T(i, l)$ in $\mathcal{H}$. Since $\phi_{p,q}$ is continuous, we have $\phi_{p,q}(T(\alpha_n, l_n)) = (a_n, b_n) \longrightarrow \phi_{p,q} = (a, b)$. Hence, we have a sequence $(l'_n = l_n \circ \alpha_n)$ such that $l'_n \longrightarrow l$ in $\mathcal{H}$ with $l'_n(p) = a_n$ and $l'_n(q) = b_n$.

For an R^n-geometry $(\mathcal{P}^n, \mathcal{L})$, we assume that the join mapping $\vee : \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle \longrightarrow (\mathcal{L}, T^{\eta^*})$ is a continuous open mapping. Let (L_n) converge to L in the topology T^{η^*}. Let l, l_n be representations of lines L and L_n such that $l_n \longrightarrow l$ in $\mathcal{H}$. By [5, Chapter XII, 7.4, 7.5], for each $x \in R$ and sequence $x_n \longrightarrow x$ we have $l_n(x_n) \longrightarrow l(x)$, that is, the evaluation mapping $e : R \times \mathcal{H} \longrightarrow \mathcal{P}^n : e(x, l) = l(x)$ is continuous. Conversely, for each $y \in \liminf L_n$ we have a sequence (y_n) converging to y. Since $(\mathcal{L}, T^{\eta^*})$ is a T_2-space, we have $y \in L$. Let $x_n, x \in R$ such that $l_n(x_n) = y_n$ and $l(x) = y$. Does the sequence (x_n) converge to x in R?

4. Order- and bounded-condition

Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Since each line is homeomorphic to R, there is a natural notion of intervals in lines. If $L \in \mathcal{L}$ is a line and $p, q \in L$ are two (not necessarily distinct) points on L, then we shall denote the interval which consists of all points on L between p and q by the symbol $[p, q]$. The open interval between p and q is defined as $(p, q) := [p, q] \setminus \{p, q\}$.

The following axioms are introduced by H. Klein in [8]

DEFINITION 4.1. Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Given two subsets $A, B \subseteq \mathcal{P}^n$, we define

$$[A, B] := \bigcup_{a \in A, b \in B.} [a, b],$$

i.e. $[A, B]$ is the set of all points between A and B.

Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Here we shall introduce the following additional axioms:

(B) (Bounded-axiom) If $A, B \subseteq \mathcal{P}^n$ are compact, then $\overline{[A, B]}$ is also compact.

(O) (Order-axiom) Let $((a_n, c_n))$ be a sequence converging to (a, c) in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. If a sequence (b_n) converges to b in $\mathcal{P}^n$ with $b_n \in [a_n, c_n] \subset a_n \vee b_n$ for all $n \in N$, then we have also that $b \in [a, c] \subset a \vee b$.

THEOREM 4.2. (Order-condition) *Let* $(\mathcal{P}^n, \mathcal{L})$ *be an* R^n*-geometry. Then the following two properties are equivalent:*

(1) $T(d^2) = T^{\phi^*_{p,q}}$.
(2) *The join mapping* $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2)) \longrightarrow (\mathcal{L}, T^{\eta^*})$ *is continuous and the given* R^n*-geometry* $(\mathcal{P}^n, \mathcal{L})$ *satisfies Order-axiom.*

Proof. (1) $\Rightarrow$ (2). By the assumption $T(d^2) = \phi^*_{p,q}$ and Theorem 3.14(1), the join mapping $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2)) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous. Next we show that the R^n-geometry $(\mathcal{P}^n, \mathcal{L})$ satisfies Order-axiom. Let $((a_n, c_n))$ be a sequence converging to (a, c) in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. Assume that a sequence (b_n) converges to b in $\mathcal{P}^n$ with $b_n \in [a_n, c_n] \subset a_n \vee b_n$ for all $n \in N$. Note that we consider Order-axiom under usual topologies $T(d)$ and $T(d^2)$. By the assumption $T(d^2) = \phi^*_{p,q}$, we have $(a_n, b_n) \overset{\phi^*_{p,q}}{\longrightarrow} (a, b)$. Let $l \in \mathcal{H}$ with $\phi_{p,q}(l) = (l(p), l(q)) = (a, c)$. Then we have a sequence (l_n) in $\mathcal{H}$ such that $l_n \longrightarrow l$ in $\mathcal{H}$ and $\phi_{p,q}(l_n) = (l_n(p), l_n(q)) = (a_n, c_n)$ Let $I = [p, q]$, hence we have $l_n(I) = [a_n, c_n] \subset a_n \vee b_n$ for all $n \in N$ and $l(I) = [a, c] \subset a \vee b$. Since $b_n \in [a_n, c_n]$ for all $n \in N$, there exists a sequence (x_n) in I with $l_n(x_n) = b_n$. Since I is compact, there exists a subsequence (x_{n_m}) of (x_n) which converges to $x \in I$. We have also $l_{n_m}(x_{n_m}) = b_{n_m} \longrightarrow l(x) = b$, hence $b \in [a, c]$.

(2) $\Rightarrow$ (1). We show that $T^{\phi^*_{p,q}} \subseteq T(d^2)$. Let $((a_n, b_n))$ be a sequence in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$ and $(a, b) \in \mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. We have to show that if $(a_n, b_n) \longrightarrow (a, b)$ in $(\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2))$, then we have also $(a_n, b_n) \overset{\phi^*_{p,q}}{\longrightarrow} (a, b)$. Since the join mapping $\vee : (\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle, T(d^2)) \longrightarrow (\mathcal{L}, T^{\eta^*})$ is continuous, we have $a_n \vee b_n \overset{\eta^*}{\longrightarrow} a \vee b$. Let $l \in \mathcal{H}$ with $\phi_{p,q}(l) = (l(p), l(q)) = (a, b)$. Hence there exists a sequence (l_n) in $\mathcal{H}$ such that $l_n \longrightarrow l$ in $\mathcal{H}$ and $\eta(l_n) = a_n \vee b_n$.

Step 1. Let $l_n(p_n) = a_n, l_n(q_n) = b_n$ for all $n \in N$. We shall show that $p_n \longrightarrow p, q_n \longrightarrow q$ in R respectively. Assume that the sequence (p_n) does not converge to p in R. There exists an open interval (u, v) in R with $p \in (u, v)$ and a subsequence (p_{n_m}) such that $\{p_{n_m} | n_m \in N\} \cap (u, v) = \emptyset$. Since $l_{n_m} \longrightarrow l$ in $\mathcal{H}$, we have $(l_{n_m}(u), l_{n_m}(v)) \longrightarrow (l(u), l(v))$ in $\mathcal{P}^n \times \mathcal{P}^n \setminus \triangle$. On the other hand, we have $l_{n_m}(p_{n_m}) = a_{n_m} \notin l_{n_m}((u, v)) = (l_{n_m}(u), l_{n_m}(v)) \subset a_{n_m} \vee b_{n_m}$ and $a_{n_m} \longrightarrow a \in (l(u), l(v)) \subset a \vee b$, this leads to a contradiction to Order-axiom.

Step 2. By Lemma 3.13, we have another sequence (l'_n) in $\mathcal{H}$ such that $l'_n \longrightarrow l$ and $\phi_{p,q}(l'_n) = (a_n, b_n)$. Consequently, we have $(a_n, b_n) \overset{\phi^*_{p,q}}{\longrightarrow} (a, b)$. □

THEOREM 4.3. (Bounded-condition) *Let $(\mathcal{P}^n, \mathcal{L})$ be an R^n-geometry. Assume that $T(d^2) = \phi^*_{p,q}$. Then $(\mathcal{P}^n, \mathcal{L})$ satisfies also bounded-axiom.*

Proof. Let $A, B \subset \mathcal{P}^n$ be compact. Assume that $\overline{[A, B]}$ is not compact. Then there exists a sequence $((a_n, b_n))$ in $A \times B$ and a sequence (p_n), $p_n \in [a_n, b_n] \setminus \{a_n, b_n\}$ such that (p_n) is unbounded in $\mathcal{P}^n$.. Since $A \times B$ is compact, there exists a convergent subsequence $((a_{n_m}, b_{n_m}))$ which converges to a point $(a, b) \in A \times B$. Let $a \neq b$. We may assume that $a_{n_m} \neq b_{n_m}$ for all $n_m \in N$. Since $a_{n_m} \vee b_{n_m} \longrightarrow a \vee b$ in T^{η^*}, for $l \in \mathcal{H}$ with $\phi_{p,q} = (l(p), l(q)) = (a, b)$, there exists a sequence (l_{n_m}) in $\mathcal{H}$ such that $l_{n_m} \longrightarrow l$ in $\mathcal{H}$ and $\phi_{p,q}(l_{n_m}) = (l_{n_m}(p), l_{n_m}(q)) = (a_{n_m}, b_{n_m})$. Choose a relative compact open set U which contains $l([p, q]) = [a, b]$. there exists N such that for all $m \geq N$ $\cup([a_{n_m}, b_{n_m}]) \cup [a, b] \subseteq U$. Consequently, $\cup([a_{n_m}, b_{n_m}]) \cup [a, b]$ is bounded, a contradiction. Let $a = b$. Without loss of generality, let the sequence $((a_n, b_n))$ converges to (a, a). We may assume that $a_n \neq b_n$ for sufficiently large $n \in N$. By lemma 2.1, there exists a point $q \in \limsup(a_n \vee b_n)$ with $q \neq a$. Hence there exists a sequence $q_{n_m} \in a_{n_m} \vee b_{n_m}$ which converges to q. Therefore, $a_{n_m} \vee q_{n_m} = b_{n_m} \vee q_{n_m} \longrightarrow a \vee q$. Therefore, we have the same argument in the case $a \neq b$. □

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