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ON SOME SCHUR ALGEBRAS

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ABSTRACT. A Schur algebra was generalized to projective Schur algebra by admitting twisted group algebra. A Schur algebra is a projective Schur algebra with trivial 2-cocycle. In this paper we study situations that Schur algebra is a projective Schur algebra with nontrivial cocycle, and we find a criterion for a projective Schur algebra to be a Schur algebra.

1. Introduction

Let R be a commutative ring and $B(R)$ be the Brauer group of equivalence classes of Azumaya R-algebras A. If an Azumaya algebra A is the homomorphic image of a group ring RG for some finite group G then A is called a Schur algebra. Equivalence classes of Schur algebras form a Schur subgroup $S(R)$ of $B(R)$. An Azumaya algebra is called a projective Schur algebra if it is an image of a twisted group ring $R^\alpha G$ with a finite group G and a 2-cocycle $\alpha \in Z^2(G, U(R))$, where $U(R)$ is the set of units of R. The classes of similar algebras form a group under tensor product, called projective Schur group $PS(R)$ (refer to [1], [3], [8] and [9]).

The purpose of the paper is to study relationships between Schur and projective Schur algebras. Every Schur algebra is a projective Schur algebra by taking a trivial 2-cocycle. Besides the trivial case, we study situations that a Schur algebra can be a projective Schur algebra with nontrivial 2-cocycle. We prove that if a Schur algebra A is an image of RG and if G has a nontrivial center then A is a projective Schur algebra represented by a twisted group ring with nontrivial cocycle. Conversely, if a projective Schur algebra A is represented by $R^\alpha G$ and if α is of

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finite order then A is a Schur algebra. We give some examples for the situation. On the other hand, in Section 4 we study projective character in order to express projective Schur algebra as a simple component of a twisted group algebra, and have a finer result.

Throughout the paper, G is a finite group, Q the field of rational numbers, Z the ring of integers and $U(R)$ is the set of units in a commutative ring R.

2. Schur and projective Schur algebras

Let $K = Q(\sqrt{2})$ and let D be the quaternion K-algebra $(-1,-1)/K = K \oplus Ki \oplus Kj \oplus Kij$ with $i^2 = j^2 = -1$ and $ij = -ji$. Then D is a central simple K-algebra of order 2 in $B(K)$ and is a homomorphic image of KQ_8 where Q_8 is the quaternion group of order 8, thus D is a Schur algebra.

Write $Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, yx = x^{-1}y \rangle$, and consider a sequence

$$1 \to \langle -1 \rangle \cong \langle x^2 \rangle \to Q_8 \xrightarrow{\pi} Q_8/\langle x^2 \rangle \to 1.$$

The $Q_8/\langle x^2 \rangle = V_4$ is generated by $\bar{x}$ and $\bar{y}$ ($\bar{x}$ means the modulus of x by $\langle x^2 \rangle$), and the sequence yields a factor set f such that $f(\bar{y}, \bar{x}) = -1$ and all the other values are 1. The twisted group algebra $K^f V_4$ with basis $\{u_g | \ g \in V_4\}$ such that $u_{g_1} u_{g_2} = f(g_1, g_2) u_{g_1 g_2}$ represents D under the map $P : K^f V_4 \to D$ defined by $u_{\bar{x}} \mapsto i, \quad u_{\bar{y}} \mapsto j$ and $u_{\bar{x}\bar{y}} \mapsto ij$. Thus D is a projective Schur algebra with respect to nontrivial f. This example motivates the next theorem.

THEOREM 1. *Let A be a Schur R-algebra which is a homomorphic image of RG with finite group G. Suppose that the center $Z(G) \neq 1$. Then there is a finite group H and a 2-cocycle $\alpha \in Z^2(H, U(R))$ which is not necessarily trivial such that A is a homomorphic image of $R^\alpha H$.*

Proof. Let $\psi : RG \to A$ be the surjective homomorphism. Since $Z(G)$ is not trivial, by considering a central group extension $1 \to Z(G) \to G \to G/Z(G) \to 1$, we have a nontrivial factor set $f \in Z^2(G/Z(G), Z(G))$ satisfying

$$f(\bar{s}, \bar{t}) st = s \cdot t \quad \text{for } \bar{s}, \ \bar{t} \in G/Z(G),$$

where $s \cdot t$ is a product of s and t in G and st is an element mapped on $\bar{s} \cdot \bar{t}$. Regarding f as an element in $Z^2(G/Z(G), U(RZ(G)))$, it was proved in [9] that

$$RZ(G)^f (G/Z(G)) \cong RG, \quad \sigma \ w_{\bar{s}} \mapsto \sigma s$$

for $\sigma \in Z(G)$, $s \in G$, $\bar{s} \in G/Z(G)$ and $w_{\bar{s}}$ is a basis for $RZ(G)^f(G/Z(G))$. Thus we use the same notation ψ for the surjection $\psi : RZ(G)^f\ (G/Z(G)) \to A$.

It is obvious that for any $\sigma \in Z(G)$, $\psi(\sigma)$ is central in A, hence is contained in $U(R)$. Therefore $\psi(f(\bar{s},\bar{t})) \in U(R)$ for any $\bar{s}$, $\bar{t} \in G/Z(G)$. If we define

$$\alpha :\ G/Z(G) \times G/Z(G) \to U(R) \ \text{ by } \ \alpha(\bar{s},\bar{t}) = \psi(f(\bar{s},\bar{t}))$$

then it is routine to check that α is a 2-cocycle in $Z^2(G/Z(G), U(R))$. Thus we have a twisted group algebra $R^\alpha(G/Z(G))$ and there is a surjection from $R^\alpha(G/Z(G))$ to A defined by $w_{\bar{s}} \mapsto \psi(s)$. This completes the proof. □

We now ask the converse question that when projective Schur algebras can be Schur algebras. For the purpose, we add a simple lemma.

LEMMA 2. *Let k be a field of any characteristic p and α be a 2-cocycle in $Z^2(G,k^*)$ of finite order n. Then k contains a primitive n-th root of unity ε_n.*

Proof. If characteristic of k is $p > 0$, p can not divide n because k contains no primitive p-th roots of unity. For any $g, x \in G$, $\alpha(g,x)^n = 1$ thus $\alpha(g,x)$ can be written as $\alpha(g,x) = \varepsilon_n^l$ for some $l > 0$ and $\langle \alpha(g,x) |\ g, x \in G\rangle \subseteq \langle \varepsilon_n \rangle$. If we suppose that $|\langle \alpha(g,x) |\ g,x \in G\rangle| = n_1 < n$, then $\alpha(g,x)^{n_1} = 1$ for all $g, x \in G$, contrary to the minimality of n. Therefore, $n_1 = n$ thus it follows that ε_n belongs to k^*. □

THEOREM 3. *Let A be a projective Schur algebra which is a homomorphic image of a twisted group algebra $k^\alpha G$ where G is a finite group and $\alpha \in Z^2(G,k^*)$. If the order of α is finite then A is a Schur algebra over k.*

Proof. Let $\{a_g |\ g \in G\}$ with $a_1 = 1$ be the k-basis for $k^\alpha G$ such that $a_g a_x = \alpha(g,x) a_{gx}$ for all $g, x \in G$. Let the order of α be $n < \infty$.

Consider the α-covering group $G(\alpha)$ of G which is generated by $\varepsilon_n^i a_g$ for all $g \in G$ and $i \in Z$ (refer to [2] or [10]). Then $|G(\alpha)| = n|G|$ and there is a surjection $\pi :\ G(\alpha) \to G$ defined by $\pi(\varepsilon_n^i a_g) = g$ for all $g \in G$.

Let $T = \langle \varepsilon_n^i a_1 |\ i \in Z\rangle$. Then $|T| = n$ and T is a subgroup of the center of $G(\alpha)$, thus we have a central cyclic group extension of T by G

$$1 \to T \to G(\alpha) \to G \to 1.$$

Let $f \in Z^2(G,T)$ be the factor set corresponding to the above sequence. Since $\varepsilon_n \in k^*$ by Lemma 2, T is a subset of k^*, hence f is contained in $Z^2(G,k^*)$ such that $o(f)|n$. If we consider the generalized f-covering group $G_n(f)$ ([2, Section 2]), then it follows from [2, Theorem 6 and Corollary 3] that $G(\alpha) \cong G_n(f)$ and α can be regarded as a factor set arose from the sequence.

For a homomorphism $\chi \in \mathrm{Hom}(T,k^*)$ defined by $\chi(\varepsilon_n^i a_1) = \varepsilon_n^i$, let I be the augmentation ideal of kT which is the kernel of homomorphism $kT \to k$ induced from χ. Then $kG(\alpha)/(kG(\alpha)\cdot I)$ is isomorphic to $k^\alpha G$. Thus by taking composition of surjections $kG(\alpha) \to k^\alpha G$ and $k^\alpha G \to A$, it follows that A is a homomorphic image of the group algebra $kG(\alpha)$ of finite group $G(\alpha)$, which implies that A is a Schur algebra. □

3. Examples of Schur algebras

PROPOSITION 4. *Consider the following central simple algebras.*

(1) *Let* $R = Z(\sqrt{2})$. *Let* $A = R \oplus Ra \oplus Rb \oplus Rab$ *with* $a = (1+i)/\sqrt{2}$, $b = (1+j)/\sqrt{2}$ *and* $i^2 = j^2 = -1$, $ij = -ji$.

(2) *Let* $R = Z(\frac{1+\sqrt{5}}{2})$. *Let* $B = R \oplus Ra \oplus Rj \oplus Rb$ *with*

$$a = \frac{-1+\sqrt{5}+2i+(1+\sqrt{5})j}{4},$$

$$b = \frac{-1-\sqrt{5}+(-1+\sqrt{5})j+2ij}{4}$$

and i, j *are as above.*

Then A *and* B *are projective Schur algebras represented by nontrivial cocycles.*

Proof. The algebra A is an Azumaya algebra which is nontrivial in $B(R)$ ([4, p.148]). Let H be a group generated by a and b. It was erroneously claimed in [5] that $a^8 = b^8 = 1$, $ab = b^4a$ and A is a homomorphic image of RH with $|H| = 64$. However, $ab \neq b^4a$. Moreover $a^4 = b^4 = -1$ and $a^2 = i$, $b^2 = j$ and by tedious calculations we have

$$H = \{\pm 1,\ \pm i,\ \pm j,\ \pm ij,\ \frac{(\pm 1 \pm i \pm j \pm ij)}{2},\ \frac{(\pm 1 \pm i)}{\sqrt{2}},$$
$$\frac{(\pm 1 \pm j)}{\sqrt{2}}, \frac{(\pm 1 \pm ij)}{\sqrt{2}}, \frac{(\pm i \pm j)}{\sqrt{2}}, \frac{(\pm i \pm ij)}{\sqrt{2}}, \frac{(\pm j \pm ij)}{\sqrt{2}}\}$$

so that the order of H is 48. Let

$$u = (-1+i+j+ij)/2, \quad v = (1-i)/\sqrt{2}, \quad x = i \quad \text{and} \quad y = j.$$

Then we have relations that $x^2 = y^2 = (xy)^2$, $x^4 = u^3 = 1$, $y^u = xy$, $v^y = v^{-1}$, $u^v = u^{-1}xy^{-1}$, $v^2 = x^{-1}$ and $x^u = y$. Thus H equals $\langle u, v\rangle$ with the defining relations $o(u) = 3$ and $o(v) = 8$, hence $H \cong E_{48}$ ([6, p.389]). Define

$$\psi: \; RH \to A \;\text{ by } \psi(v) = (1-i)/\sqrt{2},\; \psi(u) = (-1+i+j+ij)/2.$$

Then ψ is a homomorphism and $\psi(v^{-1}) = a$, $\psi(v^2uv) = b$. This shows that A is a Schur algebra determined by the group algebra RH.

Since the center of H equals $\langle v^4\rangle \cong \langle -1\rangle$, we have a central extension

$$1 \to \langle -1\rangle \cong \langle v^4\rangle \to H \xrightarrow{\pi} H/\langle v^4\rangle \to 1,$$

where $H/\langle v^4\rangle$ is of order 24 generated by $\bar{u}$, $\bar{v}$ with $\bar{u}^3 = \bar{v}^4 = (\bar{v}\bar{u})^2 = 1$. Then $H/\langle v^4\rangle$ is isomorphic to the symmetric group S_4 by considering $\bar{u} \leftrightarrow (1,2,3)$ and $\bar{v} \leftrightarrow (1,3,2,4)$.

Moreover the sequence yields a factor set β of order 2, for $vu(v^5u)^{-1} \in \mathrm{Ker}\pi$, $v^5u = \beta(\bar{v},\bar{u})vu$, hence $\beta(\bar{v},\bar{u}) \in \mathrm{Ker}\pi \cong \langle -1\rangle$. Therefore $R^\beta S_4$ having basis $\{d_g |\; g \in S_4\}$ represents A with relations $d_{\bar{v}} \mapsto a$, $d_{\bar{u}} \mapsto b$, thus A is a projective Schur algebra with respect to $\beta \neq 1$. This proves (1).

For the algebra B in (2), it is an Azumaya algebra due to [8], and by some straightforward computations we have the following relations:
$b = aj = \omega ja$, where $\omega = (2 + (-1+\sqrt{5})i + (-1-\sqrt{5})ij)/4$;
$ab = \zeta ba$, where $\zeta = (1-i+j+ij)/2$;
$a^5 = b^5 = 1$; $\omega^3 = \zeta^3 = -1$; $(a\omega)^2 = (a\zeta)^2 = -1$;
$(a\omega^2)^5 = (a\zeta^2)^5 = -1$.
Now let $P = a\zeta$ and $W = \zeta$. Then $P^2 = W^3 = (PW)^5 = -1$. Moreover if $S = \zeta^2$ and $T = a$ then it follows that $TS^{-1} = a\zeta^4 = -a\zeta$, $S^3 = T^5 = 1$ and $(TS^{-1})^2 = -1$.

Let G be a group generated by P and W. Then G is also generated by -1, S and T, thus $|G|$ =120 ([11, p.176]). If we define a map ψ

$$\psi: \; RG \to B \quad \text{by} \quad P \mapsto a\zeta \;\text{ and } W \mapsto \zeta$$

then it is a surjection, because $\psi(S) = \zeta^2$, $\psi(T) = \psi(PW^{-1}) = a$ and $\psi(T^4S) = a^4\zeta^2 = i$, hence this shows that B is a Schur R-algebra.

Following Theorem 1, we consider a central group extension

$$1 \to \langle -1\rangle \cong \langle W^3\rangle \to G \xrightarrow{\pi} G/\langle W^3\rangle \to 1.$$

Then $G/\langle W^3\rangle$ is a group of order 60 and is isomorphic to the alternating group A_5 (refer to [1]). Thus with $\alpha \in Z^2(G/\langle W^3\rangle, \langle -1\rangle)$ which corresponds to the above sequence, B can be represented by a twisted group ring $R^{\alpha}(G/\langle W^3\rangle)$, so that B is a projective Schur R-algebra. This completes the proposition. □

An algebra is called a generalized Clifford R-algebra of rank m [9, Section 3] if there exist m generators $u_1, \cdots, u_m$ such that $u_i^n = a_i \in U(R)$ for $u_i u_j = \omega u_j u_i$ for all $1 \le i, j \le m$ and for a fixed n-th root of unity ω. In case of $m = n = 2$, this is only the generalized quaternion algebra $(a, b)/R = R \oplus Ri \oplus Rj \oplus Rij$ with relations $i^2 = a,\ j^2 = b$ and $ij = -ji$ for $a, b \in U(R)$, which is a known example of projective Schur algebra. We recall the following lemma.

LEMMA 5 ([9]). *Under the same notations above, a generalized Clifford R-algebra Γ of rank m is isomorphic to a twisted group ring R^cG for a 2-cocycle c determined by $a_1, \cdots, a_m$.*

Indeed if $u_1, \ldots, u_m$ are m-generators of Γ, let $G = Z_n^m = Z_n \times \cdots \times Z_n$ (m-times) and let $\{x_1, \cdots, x_m\}$ be generators of $Z_n^m = G$. Then by defining $c : G \times G \to U(R)$ as below, it follows that R^cG is isomorphic to Γ:

$$c(x_i^s, x_j^t) = \omega^{st} \text{ if } i < j\ ; \qquad c(x_i^s, x_j^t) = 1 \text{ if } i > j$$
$$c(x_i^s, x_j^t) = a_i \text{ if } i = j,\ s + t \ge n\ ; \quad c(x_i^s, x_j^t) = 1 \text{ if } i = j,\ s + t < n.$$

Furthermore if R contains n as a unit then R^cG is an Azumaya algebra thus $R^cG \cong \Gamma$ implies that Γ is a projective Schur algebra.

EXAMPLE. Due to Lemma 5, the quaternion algebra D in Section 2 and the algebra A in Proposition 4 are projective Schur algebras with nontrivial cocycles whose orders are finite, thus they are also Schur algebras.

In fact for $D = (-1, -1)/K$, consider $G = Z_2 \times Z_2$ with generators $\{x_1, x_2\}$, and 2-cocycle $c : G \times G \to U(K)$ defined by $c(x_1, x_2) = -1$, $c(x_2, x_1) = 1$, $c(x_1, x_1) = -1$ and $c(x_2, x_2) = -1$. Then D is an image of K^cG by Lemma 5.

Similarly, A has generators a, b satisfying $a^4 = b^4 = -1$ and $ab = \omega ba$ where $\omega^6 = 1$, indeed $\omega = (1 + i - j + ij)/2$. Let $G = Z_{12} \times Z_{12}$ and $d : G \times G \to U(R)$ be a cocycle defined in the way of Lemma 5. Then $A \cong R^dG$.

Moreover, since the cocycles c and d are of finite order, Theorem 3 shows that the algebras are Schur algebras.

REMARK. Theorem 3 shows that the order of 2-cocycle α plays an important role to determine whether a projective Schur algebra with respect to α is a Schur algebra. For the converse of the theorem, we ask that if A is a projective Schur algebra represented by $R^\alpha G$ and if A itself is a Schur algebra, then the order of α is finite. But this is not necessarily true as in the following example.

A generalized quaternion algebra $A = (\sqrt{2}, \sqrt{2})/Q(\sqrt{2})$ is a projective Schur algebra due to Lemma 5, and moreover A is a Schur algebra since A is isomorphic to a matrix algebra $M_{2\times 2}(Q(\sqrt{2}))$. However the 2-cocycle c defined in Lemma 5 which represents A is not of finite order.

Thus it would be interesting to show that if A is a projective Schur algebra represented by $R^\alpha G$ and if A is a Schur algebra, then there is a 2-cocycle $\beta \in Z^2(G, U(R))$ whose order is finite.

4. Schur algebras and group character

We discussed (projective) Schur algebras as homomorphic images of certain (twisted) group algebras. However, as a simple component of kG, a Schur k-algebra A can be expressed precisely as $A = kGe$ with a block idempotent e of kG. The block idempotent is uniquely determined by the group character χ of G which is afforded by A.

Let ρ be a projective α-representation of G over k and χ_α be an irreducible α-character afforded by ρ. In case that k is a splitting field for $k^\alpha G$, the block idempotent $e(\chi_\alpha)$ of $k^\alpha G$ corresponding to χ_α forms

$$e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g\in G} \alpha^{-1}(g, g^{-1})\chi_\alpha(g^{-1})a_g$$

where $\{a_g | g \in G\}$ is a k-basis of $k^\alpha G$ ([7, 1.11.1]).

Generally, for nonsplitting field k we have the following theorem.

THEOREM 6. *Let $\alpha \in Z^2(G, k^*)$. Let E be an extension of k which is a splitting field for $E^\alpha G$ and U be a simple $E^\alpha G$-module. Let ρ be an irreducible α-representation afforded by U and χ_α be the α-character corresponding to ρ. Then for the block idempotent $e(\chi_\alpha)$ of $E^\alpha G$ and for $\mathcal{G} = Gal(k(\chi_\alpha)/k)$,*

$$v(\chi_\alpha) = \sum_{\tau\in\mathcal{G}} e(\chi_\alpha^\tau) \quad \textit{with} \quad \chi_\alpha^\tau(g) = \tau(\chi_\alpha(g))$$

is a block idempotent of $k^\alpha G$, where $k(\chi_\alpha)$ is the field obtained by adjoining to k the values $\chi_\alpha(g)$ for $g \in G$. Moreover, $k^\alpha G v(\chi_\alpha) \cong \rho(k^\alpha G)$ and the center $Z(\rho(k^\alpha G)) \cong k(\chi_\alpha)$.

Proof. For any $\tau \in \mathcal{G}$ and for $x = \sum_{g \in G} x_g a_g \in k(\chi_\alpha)^\alpha G$, where $x_g \in k(\chi_\alpha)$, if we define $\tau \cdot x$ by $\sum \tau(x_g) a_g$ then it is clear that $\mathcal{G}$ acts on $k(\chi_\alpha)^\alpha G$, thus χ_α^τ is also an α-character of G. Now since

$$\begin{aligned}\tau(e(\chi_\alpha)) &= \tau\Big(\frac{\chi_\alpha(1)}{|G|}\sum \alpha^{-1}(g,g^{-1})\chi_\alpha(g^{-1})a_g\Big)\\ &= \frac{\chi_\alpha(1)}{|G|}\sum \alpha^{-1}(g,g^{-1})\chi_\alpha^\tau(g^{-1})a_g = e(\chi_\alpha^\tau),\end{aligned}$$

we get that

$$v(\chi_\alpha) = \sum_{\tau \in \mathcal{G}} e(\chi_\alpha^\tau) = \frac{\chi_\alpha(1)}{|G|}\sum_{g \in G}\alpha^{-1}(g,g^{-1})\Big(\sum_{\tau \in \mathcal{G}}\chi_\alpha^\tau(g^{-1})\Big)a_g.$$

Furthermore for any $\sigma \in \mathcal{G}$, since $\sigma(v(\chi_\alpha)) = \sum_{\tau \in \mathcal{G}} \sigma e(\chi_\alpha^\tau) = \sum_{\tau \in \mathcal{G}} e(\chi_\alpha^{\sigma\tau}) = v(\chi_\alpha)$ it follows that $v(\chi_\alpha)$ is contained in $k^\alpha G$.

Now following similar arguments to [7, (14.1.14)], it is easy to see that $v(\chi_\alpha)$ is a primitive central idempotent of $k^\alpha G$. For the rest of theorem, we may refer to [7, (7.3.8)]. □

Therefore Theorem 6 provides a description for projective Schur k-algebra as a simple component of twisted group algebra.

THEOREM 7. *Let E be an algebraic closure of k. An algebra A is a projective Schur k-algebra if and only if there is a finite group G, a cocycle $\alpha \in Z^2(G, k^*)$ and an irreducible α-character χ_α of G over E such that $k = k(\chi_\alpha)$ and $A \cong k^\alpha G e(\chi_\alpha)$ as k-algebras.*

Proof. If A is a projective Schur algebra, i.e., if A is a simple component of $k^\alpha G$ which affords χ_α then A is central over k if and only if $k = Z(A) = Z(k^\alpha G v(\chi_\alpha)) = Z(\rho(kG^\alpha)) = k(\chi_\alpha)$ due to Theorem 6. Conversely if $A \cong k^\alpha G e(\chi_\alpha)$ then $k^\alpha G$ represents A and A is central since $k = k(\chi_\alpha)$. Thus A is a projective Schur algebra. □

Theorem 7 generalizes the statement in [7, (14.2.4)] about Schur algebras. By making use of group character, we prove Theorem 3 in more concrete form.

THEOREM 8. *Let A be a projective Schur algebra represented by $k^\alpha G$.*

(1) *If the order of $\alpha \in Z^2(G, k^*)$ is finite then A is a Schur algebra.*

(2) *Furthermore if A is a simple component $k^\alpha Ge$ with a block idempotent e of $k^\alpha G$ then A is a simple component of a group algebra with the same block idempotent e in the group algebra.*

Proof. As before, let ρ be an irreducible α-representation of G corresponding to the simple $k^\alpha G$-module A and χ_α be the irreducible α-character afforded by ρ. Let $\{a_g|\ g \in G\}$ denote a k-basis for $k^\alpha G$ and E be an algebraic extension of k which splits $E^\alpha G$. Theorem 7 implies that

$$k = k(\chi_\alpha) \quad \text{and } A \cong k^\alpha Ge(\chi_\alpha), \quad (e(\chi_\alpha): \quad \text{the block idempotent}).$$

It is known that the values of χ_α are sums of δ_g [7, (1.2.6)], where $\delta_g \in E^*$ is an $o(g)$-th root of

$$(a_g)^{o(g)} = \prod_{i=1}^{o(g)-1} \alpha(g, g^{-1}) a_{g^{o(g)}} = \prod_{i=1}^{o(g)-1} \alpha(g, g^{-1}) \ \in k^*.$$

If $o(\alpha) = n$ is finite then $(a_g)^{o(g)n} = 1$ for all $g \in G$. Thus we may choose δ_g as a root of unity in E and the values of χ_α are sums of roots of unity. Therefore $k(\chi_\alpha) = k(\varepsilon)$ for a primitive root of unity ε in E^*.

We denote the α-covering group $G(\alpha) = \langle \varepsilon_n^i a_g | g \in G, 1 \leq i \leq n \rangle$ by H. Then $|H| = n|G|$ and the map ξ on H defined by $\xi(\varepsilon_n^i a_g) = \varepsilon_n^i \rho(g)$ for $g \in G$, $i \in Z$ is an ordinary representation of H for

$$\xi(\varepsilon_n^i a_g \varepsilon_n^j a_x) = \varepsilon_n^{i+j} \alpha(g, x) \rho(gx) = \xi(\varepsilon_n^i a_g) \xi(\varepsilon_n^j a_x).$$

Let θ be the (ordinary) character of H afforded by ξ.

We claim that $k(\theta) = k(\chi_\alpha) = k$. Indeed since $\theta(\varepsilon_n^i a_g) = \mathrm{tr}(\xi(\varepsilon_n^i a_g)) = \varepsilon_n^i \mathrm{tr}\rho(g) = \varepsilon_n^i \chi_\alpha(g)$ and $\varepsilon_n \in k^*$ by Lemma 2, $\theta(\varepsilon_n^i a_g) \in k(\chi_\alpha)$. Thus all the values of θ are contained in $k(\chi_\alpha)$ and $k(\theta) \subseteq k(\chi_\alpha) = k$. It thus follows that $k(\theta) = k(\chi_\alpha) = k$.

Clearly, A is an EH-module and the block idempotent of EH which corresponds to the ordinary character θ forms

$$e(\theta) = \frac{\theta(1)}{|H|} \sum_{\varepsilon_n^i a_g \in H} \theta((\varepsilon_n^i a_g)^{-1}) \varepsilon_n^i a_g,$$

thus it belongs to $k(\theta)H = kH$.

Moreover we have that the idempotents $e(\theta)$ and $e(\chi_\alpha)$ are same because

$$
\begin{aligned}
e(\theta) &= \frac{\chi_\alpha(1)}{n|G|} \sum_{\varepsilon_n^i a_g \in H} \theta(\varepsilon_n^{-i}\alpha^{-1}(g,g^{-1})a_{g^{-1}})\varepsilon_n^i a_g \\
&= \frac{\chi_\alpha(1)}{n|G|} \sum_{\varepsilon_n^i a_g \in H} \varepsilon_n^{-i}\alpha^{-1}(g,g^{-1})\chi_\alpha(g^{-1})\varepsilon_n^i a_g \\
&= \frac{\chi_\alpha(1)}{n|G|} n \sum_{g\in G} \alpha^{-1}(g,g^{-1})\chi_\alpha(g^{-1})a_g = e(\chi_\alpha).
\end{aligned}
$$

We now consider the central simple component $kHe(\theta)$ of kH corresponding to an irreducible character θ of H. Since there is a surjective homomorphism $kH \to k^\alpha G$ induced from a surjection $H \to G$, $\varepsilon_n^i a_g \mapsto g$ for all $g \in G$, we have a surjective map $kHe(\theta) = kHe(\chi_\alpha) \to k^\alpha Ge(\chi_\alpha) \cong A$. Moreover this is an isomorphism because $kHe(\theta)$ is simple, hence we have $A \cong kHe(\theta)$. This finishes the proof. □

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