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A History of Researches of Jumping Problems in Elliptic Equations*

Department of Mathematics, Inha University **Q-Heung Choi**
Department of Mathematics, Kunsan National University **Tacksun Jung**

Abstract

We investigate a history of reseahches of a nonlinear elliptic equation with jumping nonlinearity, under Dirichlet boundary condition. The investigation will be focussed on the researches by topological methods. We also add recent researches, relations between multiplicity of solutions and source terms of the equation when the nonlinearity crosses two eigenvalues and the source term is generated by three eigenfunctions.

0. Introduction

We investigate a history of researches of a nonlinear elliptic equation with jumping nonlinearity, under Dirichlet boundary condition. The investigation will be focussed on the researches by topological methods. We also add a recent research.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L=\sum_{1\le i,j\le n}\frac{\partial}{\partial}x_i\left(a_{ij}\frac{\partial}{\partial x_j}\right),$$

where $a_{ij}=a_{ji}\in C^\infty(\overline{\Omega})$. We investigate multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition

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$$Lu + g(u) = f(x) \text{ in } \Omega, \tag{0.1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{0.2}$$

where the semilinear term $g(u) = bu^{+} - au^{-}$ and we assume that L is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_i \leq \cdots \to \infty$$

Here the source term f is generated by the eigenfunctions of the second order elliptic operator with Dirichlet boundary condition.

In [4, 5, 6, 7], the authors have investigated multiplicity of solutions of (1.1) when the forcing term f is supposed to be a multiple of the first eigenfunction and the nonlinearity $-(bu^{+} - au^{-})$ croses eigenvalues. In [3], the authors investigated a relation between multiplicity of solutions and source terms of (0.1) when the forcing term f is supposed to be spanned two eigenfunction ϕ_1, ϕ_2 and the nonlinearity $-(bu^{+} - au^{-})$ croses two eigenvalues λ_1, λ_2 .

Let ϕ_i be the eigenfunction corresponding to λ_i. Then the set of eigenfunctions $\{\phi_i\}$ is an orthogonal in $L^2(\Omega)$.

Let us denote an element u, in H_0, as $u = \sum h_n \phi_n$ and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_n| h_n^2 < \infty\}.$$

Then this is a complete normed space with a norm $\|u\| = (\sum |\lambda_n| h_n^2)^{\frac{1}{2}}$. If $f \in H_0$ and a, b are not eigenvalues of L, then every solution in H_0 of $Lu + bu^{+} - au^{-} = f$ belongs to H (cf. [3]). Hence equation (0.1) with (0.2) and (0.3) is equivalent to

$$Lu + bu^{+} - au^{-} = f \text{ in } H. \tag{0.4}$$

In Section 1, we state the beging researches of a nonlinear elliptic equation with jumping nonlinearity, under Dirichlet boundary condition. In Section 2, we suppose that the nonlinearity crosses two eigenvalues and the source term is generated by two eigenfunctions and we investigate the properties of the reduced map ϕ(see equation (2.6)). In Section 3, we reveal a relation between multiplicity of solutions and source terms in equation (0.4) when the source term belongs to the three dimensional space spanned by ϕ_1, ϕ_2, and ϕ_3.

1. Beging Researches

In this section, we state the beging researches (cf. [4, 9]) of a nonlinear elliptic equation with jumping nonlinearity, under Dirichlet boundary condition. In [4, 9], the authors suppose that the forcing term is a multiple of the first eigenfunction.

Theorem 1.1. Let $a < \lambda_1 < b < \lambda_2$. Let $f = s\phi_1$. Then we have the followings.

(i) If $s > 0$, then equation (0.4) has a positive solution and a negative solution.

(ii) If $s = 0$, then equation (0.4) has only the trivial solution.

(iii) If $s < 0$, then equation (0.4) has no solution.

In [4], the authors showed multiplicity of solutions of the equation by the cotraction mapping principle and basic topological methods when the nonlinearity crosses two eigenvalues.

Theorem 1.2. Let $a < \lambda_1 < \lambda_2 < b < \lambda_3$. Let $f = s\phi_1$. Then we have the followings.

(i) If $s > 0$, then equation (0.4) has a positive solution, a negative solution, and at least two sign changing solutions.

(ii) If $s = 0$ then equation (0.4) has only the trivial solution

(iii) If $s < 0$, then equation (0.4) has no solution.

Dancer showed the following by degree theory and critical point theory.

Theorem 1.3. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Let $f = s\phi_1$, Then there exists $\varepsilon > 0$ such that if $\lambda_1 < a < \lambda_2 < b < \lambda_2 - \varepsilon$ then the followings hold.

(i) If $s > 0$, then equation (0.4) has a positive solution and at least two sign changing solutions.

(ii) If $s = 0$ then equation (0.4) has only the trivial solution

(iii) If $s < 0$, then equation (0.4) has a negative solution.

2. Multiplicity for Source Terms in Two Dimensional Space

We suppose that the nonlinearity $-(bu^{+}-au^{-})$ crossing two eigenvalues λ_1 and λ_2 i.e., $a<\lambda_1<\lambda_2<b<\lambda_3$. We have a concern with a relation between multiplicity of solutions and source terms of a nonlinear elliptic equation

$$Lu+bu^{+}-au^{-}=f \text{ in } L^2(\Omega). \tag{2.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_1 and ϕ_2.

Let V be the two dimensional subspace of $L^2(\Omega)$ spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in $L^2(\Omega)$. Let P be an orthogonal projection $L^2(\Omega)$ onto V. Then every element $u\in H$ is expressed by

$$u=v+w,$$

where $v=Pu$, $w=(I-P)u$. Hence equation (2.1) is equivalent to a system

$$Lw+(I-P)(b(v+w)^{+}-a(v+w)^{-})=0, \tag{2.2}$$

$$Lv+P(b(v+w)^{+}-a(v+w)^{-})=s_1\phi_1+s_2\phi_2. \tag{2.3}$$

Lemma 2.1. For fixed $v\in V$, (2.2) has a unique solution $w=\theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v.

The proof of the lemma is similar to that of Lemma 2.1 of [3].

By Lemma 2.1, the study of multiplicity of solutions of (2.1) is reduced to that of an equivalent problem

$$Lv+P(b(v+\theta(v))^{+}-a(v+\theta(v))^{-})=s_1\phi_1+s_2\phi_2 \tag{2.4}$$

defined on the two dimensional subspace V spanned by $\{\phi_1, \phi_2\}$.

We note that if $v\geq 0$ or $v\leq 0$ then $\theta(v)\equiv 0$.

Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and $\phi_1(x)>0$ in Ω, there exists a cone C_1 defined by

$$C_1=\{v=c_1\phi_1+c_2\phi_2 : c_1\geq 0, |c_2|\leq kc_1\}$$

for some $k>0$ so that $v\geq 0$ for all $v\in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 : c_1 \le 0,\ |c_2| \le k|c_1|\}$$

so that $v \le 0$ for all $v \in C_3$.

We define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-),\ v \in V. \tag{2.5}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma(cf. Lemma 2.2 of [3]).

Lemma 2.2. $\Phi(cv) = c\Phi(v)$ for $c \ge 0$ and $v \in V$.

Lemma 2.2 implies that Φ maps a cone with vertex 0 onto a cone with vertex 0. We set he cones C_2, C_4 as follows

$$C_2 = \{c_1\phi_1 + c_2\phi_2 : c_2 \ge 0,\ c_2 \ge k|c_1|\},$$
$$C_4 = \{c_1\phi_1 + c_2\phi_2 : c_2 \le 0,\ c_2 \le -k|c_1|\}.$$

Then the union of four cones C_i $(1 \le i \le 4)$ is the space V.

The map Φ maps C_1 onto the cone

$$R_1 = \{d_1\phi_1 + d_2\phi_2 : d_1 \ge 0,\ |d_2| \le k\left(\frac{b-\lambda_2}{b-\lambda_1}\right)d_1\}.$$

The cone R_1 is in the right half-plane of V and the restriction $\Phi|_{C_1} : C_1 \to R_1$ is bijective.

The map Φ maps the cone C_3 onto the cone

$$R_3 = \{d_1\phi_1 + d_2\phi_2 : d_1 \ge 0,\ d_2 \le k\left|\frac{\lambda_2 - a}{\lambda_1 - a}\right||d_1|\}.$$

The cone R_3 is in the right half-plane of V and the restriction $\Phi|_{C_3} : C_3 \to R_3$ is bijective. We note that $R_1 \subset R_3$ since $a < \lambda_1 < \lambda_2 < b < \lambda_3$.

Theorem 2.1. If f belongs to R_1, then equation (2.1) has a positive solution and a negative solution.

Lemma 2.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R_2=\{d_1\phi_1+d_2\phi_2 : d_1\geq 0,\ -k\left|\frac{\lambda_2-a}{\lambda_1-a}\right|d_1\leq d_2\leq k\left|\frac{\lambda_2-b}{\lambda_1-b}\right|d_1\},$$

$$R_4=\{d_1\phi_1+d_2\phi_2 : d_1\geq 0,\ -k\left(\frac{\lambda_2-b}{\lambda_1-b}\right)d_1\leq d_2\leq k\left(\frac{\lambda_2-a}{\lambda_1-a}\right)d_1\}.$$

We note that all the cones R_2, R_3, R_4 contain R_1. R_3 contain R_1, R_2, R_4.

To investigate a relation between multiplicity of solutions and source terms in the nonlinear equation

$$Lu+bu^+-au^-=f \text{ in } H, \tag{2.6}$$

we consider the restrictions $\Phi|_{C_i}(1\leq i\leq 4)$ of Φ to the cones C_i. Let $\Phi_i=\Phi|_{C_i}$, i.e., $\Phi_i: C_i\rightarrow V$.

For $i=1,3$, the image of Φ_i is R_i and $\Phi_i: C_i\rightarrow R_i$ is bijective.

Lemma 2.3. For every $v=c_1\phi_1+c_2\phi_2$, there exists a constant $d>0$ such that

$$(\Phi(v),\ \phi_1)\geq d|c_2|.$$

For the proof see [3].

Let us find the image of C_i under Φ_i for $i=2,4$. Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ is on the ray $c_1(b-\lambda_1)\phi_1+kc_1(b-\lambda_2)\phi_2$, $c_1\geq 0$(a boundary ray of R_1) and the image of the other end point of γ under Φ is on the ray $-c_1(-\lambda_1+a)\phi_1+kc_1(-\lambda_2+a)\phi_2$, $c_1\geq 0$(a boundary ray of R_3). Since Φ is continuous, $\Phi(\gamma)$ is a path in V. By Lemma 2.2, $\Phi(\gamma)$ does not meet the origin. Hence the path $\Phi(\gamma)$ meets all rays (starting from the origin) in R_2.

Therefore it follows from Lemma 2.3 that the image $\Phi(C_2)$ of C_2 contains R_2. Similarly, we have that the image $\Phi(C_4)$ of C_4 contains R_4.

If a solution of (2.1) is in $\mathrm{Int}C_1$, then it is positive. If a solution of (2.1) is in $\mathrm{Int}C_3$, then it is negative. If it is in $\mathrm{Int}(C_2 \cup C_4)$, then it has both signs. Therefore we have the main theorem of this section.

Theorem 2.2. Let $a < \lambda_1 < \lambda_2 < b < \lambda_3$. Let $v = c_1\phi_1 + c_2\phi_2$. Then we have the followings.

(i) If $f \in \mathrm{Int}\, R_1$, then equation (2.1) has a positive solution, a negative solution, and at least two solutions changing sign.

(ii) If $f \in \partial R_1$, then equation (2.1) has a positive solution, a negative solution, and at least one solution changing sign.

(iii) If $f \in \mathrm{Int}(R_3 \setminus R_1)$, then equation (3.1) has a negative solution and at least one sign changing solution.

(iv) If $f \in \partial R_3$, then equation (2.1) has a negative solution.

3. Multiplicity for Source Terms in Three Dimensional Space

In this section, we suppose that $\lambda_2 < \lambda_3 - 1$ and $a < \lambda_1 < \lambda_2 < b < \lambda_3 - 1$. We investigate a relation between multiplicity of solutions and the source terms of a nonlinear equation when the source terms belong to the three dimensional space spanned by ϕ_1, ϕ_2, and ϕ_3.

First we consider

$$Lu + bu^+ - au^- = s_1\phi_1 + s_2\phi_2 + \varepsilon\phi_3. \tag{3.1}$$

Let V, W and P be the same as in section 2. Then equation (3.1) is equivalent to a system

$$Lw + (I-P)(b(v+w)^+ - a(v+w)^-) = \varepsilon\phi_3, \tag{3.2}$$

$$Lv + P(b(v+w)^+ - a(v+w)^-) = s_1\phi_1 + s_2\phi_2. \tag{3.3}$$

Lemma 3.1 For fixed $v \in V$, (3.2) has a unique solution $w = \theta_\varepsilon(v)$. Furthermore, $\theta_\varepsilon(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v.

By Lemma 3.1, the study of multiplicity of solutions of (3.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$Lv+P(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-)=s_1\phi_1+s_2\phi_2 \tag{3.4}$$

defined on the two dimensional subspace V.

Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and $v>0$ in Q for all $v\in C_1$, there exists a convex subset $C_{1\varepsilon}$ of C_1 defined by

$$C_{1\varepsilon}=\{v=c_1\phi_1+c_2\phi_2 : v+\varepsilon\phi_3>0 \text{ in } Q\}$$

and a convex subset $C_{3\varepsilon}$ of C_3 defined by

$$C_{3\varepsilon}=\{v=c_1\phi_1+c_2\phi_2 : v+\varepsilon\phi_3<0 \text{ in } Q\}.$$

We define a map $J: R\times V\to V$ given by

$$J(\varepsilon, v)=Lv+P(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-),\ v\in V. \tag{3.5}$$

Then for fixed ε, J is continuous on V since $\theta_\varepsilon(v)$ is continuous on V. Also, it is easily proved that for fixed v, J is continuous on V.

Lemma 3.2. For fixed $v\in V$, J is continuous on R.

Proof. It is enough to show that for fixed v, $\theta_\varepsilon(v)$ is a continuous function of ε. We use the cotraction mapping principle. Let $\delta=\frac{1}{2}(a+b)$. Let $w_i(i=1, 2)$ be the unique solution of

$$(-L-\delta)w=(I-P)(b(v+w)^+-a(v+w)^-\delta(v+w)-\varepsilon_i\phi_3)$$

or equivalently

$$w=(-L-\delta)^{-1}(I-P)(g_v(w)-\varepsilon_i\phi_3), \tag{3.7}$$

where

$$g_v(w)=b(v+w)^+-a(v+w)^--\delta(v+w).$$

Then we have, by Lemma 3.2,

$$\|w_1-w_2\|\le\gamma\|w_1-w_2\|+\frac{\|\phi_3\|}{|\lambda_3+\delta|}|\varepsilon_1-\varepsilon_2|,$$

or equivalently,

$$(1-\gamma)\|w_1-w_2\| \le \frac{\|\phi_3\|}{|\lambda_3+\delta|}|\varepsilon_1-\varepsilon_2|,$$

which means that for fixed v, $\theta_\varepsilon(v)$ is a continuous function of ε, where $\gamma<1$.

We note that if v is in $C_{1\varepsilon}$, then $\theta_\varepsilon(v)=\frac{\varepsilon}{b-\lambda_3}\phi_3$. In fact, if v is in $C_{1\varepsilon}$, then $v+\varepsilon\phi_3>0$ in Q and hence $v+\frac{\varepsilon}{b-\lambda_3}\phi_3>0$ in Q. Hence $\theta_\varepsilon(v)=\frac{\varepsilon}{b-\lambda_3}\phi_3$ satisfies $L\theta_\varepsilon(v)+(I-P)(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-)=\varepsilon\phi_3$. Also, if v is in $C_{3\varepsilon}$, then $v+\varepsilon\phi_3<0$ in Q and hence $v+\frac{\varepsilon}{a-\lambda_3}\phi_3<0$ in Q. Hence $\theta_\varepsilon(v)=\frac{\varepsilon}{a-\lambda_3}\phi_3$ satisfies the above equation.

We investigate the images of the convex sets $C_{1\varepsilon}$ and $C_{3\varepsilon}$ under J. First we consider the image of the cone $C_{1\varepsilon}$. If $v=c_1\phi_1+c_2\phi_2$ is in $C_{1\varepsilon}$, then $v+\theta_\varepsilon(v)>0$ in Q and hence we have

$$\begin{aligned}J(\varepsilon,v)&=L(v)+P(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-)\\&=c_1\lambda_1\phi_1+c_2\lambda_2\phi_2+b(c_1\phi_1+c_2\phi_2)\\&=c_1(b+\lambda_1)\phi_1+c_2(b+\lambda_2)\phi_2.\end{aligned}$$

Thus, for fixed ε, the image of $C_{1\varepsilon}$ under J, $J(\varepsilon,C_{1\varepsilon})$, is a convex subset of R_1. For fixed ε, the restriction $J|_{C_{1\varepsilon}}: C_{1\varepsilon}\to J(\varepsilon,C_{1\varepsilon})$ is bijective.

We determine the image of the cone $C_{3\varepsilon}$. If $v=c_1\phi_1+c_2\phi_2$ is in $C_{3\varepsilon}$, then $v+\theta_\varepsilon(v)<0$ in Q and hence we have

$$\begin{aligned}J(\varepsilon,v)&=L(v)+P(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-)\\&=Lv+P(av)\\&=-c_1(\lambda_1+a)\phi_1+c_2(\lambda_2+a)\phi_2.\end{aligned}$$

Thus, for fixed ε, the image of $C_{3\varepsilon}$ under J, $J(\varepsilon,C_{1\varepsilon})$, is a convex subset of R_3. For fixed ε, the restriction $J|_{C_{3\varepsilon}}: C_{3\varepsilon}\to J(\varepsilon,C_{3\varepsilon})$ is bijective.

Let $\varepsilon>0$ be fixed. If v is in $C_{1\varepsilon}$, then $\theta_\varepsilon(v)=\frac{\varepsilon}{b-\lambda_3}\phi_3$ and $\frac{\varepsilon}{b-\lambda_3}(v+\phi_3)>0$,

$\frac{b-\lambda_3-\varepsilon}{b-\lambda_3} v>0$ in Q. Hence we have the lemma.

Lemma 3.3. Let $\varepsilon>0$ be fixed. Then there are open sets $C'_{1\varepsilon}$, $C'_{3\varepsilon}$ with $\overline{C}_{1\varepsilon}\subset C'_{1\varepsilon}\subset C_1$, $\overline{C}_{3\varepsilon}\subset C'_{3\varepsilon}\subset C_3$ such that $\theta_\varepsilon(v)=\frac{\varepsilon}{b-\lambda_3}\phi_3$ for all $v\in C'_{1\varepsilon}$, $\theta_\varepsilon(v)=\frac{\varepsilon}{a-\lambda_3}\phi_3$ for all $v\in C'_{3\varepsilon}$ Here the set $\overline{C}_{j\varepsilon}$ is the closure of $C_{j\varepsilon}$.

Theorem 3.1. Let $\varepsilon>0$ be fixed. Then we have:

(i) If f belongs to $J(\varepsilon, C_{1\varepsilon})$, then equation (3.1) has a positive solution and a negative solution.

(ii) If f belongs to $J(\varepsilon, C_{3\varepsilon})$, then equation (3.1) has a negative solution.

If $v+\varepsilon\phi_3>0$ in Ω $(v\in V)$, then $v\in C_{1\varepsilon}$ and if $v+\varepsilon\phi_3<0$ in Ω $(v\in V)$, then $v\in C_{3\varepsilon}$. We set

$$U_1=\{v+\varepsilon\phi_3: v\in J(\varepsilon, C_{1\varepsilon}),\ \varepsilon\in R\},\quad U_3=\{v+\varepsilon\phi_3: v\in J(\varepsilon, C_{3\varepsilon}),\ \varepsilon\in R\}$$

$$U_2=\text{span}\{\phi_1,\ \phi_2,\ \phi_3\}\setminus(U_1\cup U_3)$$

With the above notations and facts, we have the following.

Theorem 3.2. Suppose that $\lambda_2<\lambda_3-1$ and $a<\lambda_1<\lambda_2<b<\lambda_3-1$. Let $f=s_1\phi_1+s_2\phi_2+\varepsilon\phi_3$. Then we have the followings.

(i) If $f\in \text{Int}\,U_1$, then equation (3.1) has a positive solution, a negative solution, and at least two sign changing solutions.

(ii) If $f\in\partial U_1$, then equation (3.1) has a positive solution, a negative solution, and at least one sign changing solution.

(iii) If $f\in \text{Int}(U_3\setminus U_1)$, then equation (3.1) has a negative solution and at least one sign changing solution.

(iv) If $f\in\partial U_3$, then equation (3.1) has a negative solution.

(iv) If $f\in U_3$, then equation (3.1) has no solution.

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