A History of Researches of a Nonlinear Wave Equation with Jumping Nonlinearity*

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Abstract

We investigate the history of the research of the existence of periodic solutions of a nonlinear wave equation with jumping nonlinearity, suggested by McKenna and Lazer (cf. [15]). We also investigate the recent research of it; a relation between multiplicity of solutions and source terms of the equation when the nonlinearity $-(bu^+ - au^-)$ crosses eigenvalues and the source term f is generated by eigenfuntions.

0. Introduction

In this article we investigate the history of the research of the existence of periodic solutions of a nonlinear wave equation with jumping nonlinearity, under Dirichlet boundary condition

$$u_{tt} - u_{xx} + bu^+ - au^- = f(x, t) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \tag{0.1}$$

$$u\left(\pm\frac{\pi}{2}, t\right) = 0,$$

u is π-periodic in t and even in x, where $u^+ = \max\{0, u\}$, f is a forcing term, and t time variable. We also investigate the recent research of it; a relation between multiplicity of solutions and source terms of the equation when the nonlinearity

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* This work was supported by Inha University Research Grant.
2000 Mathematics Subject Classification: 35B10, 35L20

$-(bu^+ - au^-)$ crosses eigenvalues and the source term f.

Choi and Jung [12] proved that if $-1 < a < 3 < b < 7$ and $f(x, t) = s\phi_{00}$ (ϕ_{00} is the positive eigenfunction and $s \in R$), then for the case $s > 0$, (0.1) has at least three solutions, one of which is positive, and for the case $s < 0$, (0.1) has at least one solution, one of which is negative. The authors proved this result by the variational reduction method. In this paper we improve this result. To state our result, let us consider the set

$$\{j^2 - k^2 \mid j,\ k \in N,\ j \text{ is odd and } k \text{ is even}\}$$

whose role will be explained in the third section. Since this set is unbounded from above and from below, and has no finite accumulation point, we can denote by $\{\mu_k \mid k \in Z\}$ a strictly increasing enumeration of it. Let $\{\mu_k^+ \mid k \in N\}$ and $\{\mu_k^- \mid k \in N\}$ the sequence of positive and negative eigenvalues in $\{\mu_k \mid k \in N\}$, respectively. That is,

$$\cdots < \mu_3^- < \mu_2^- < \mu_1^- < 0 < \mu_1^+ < \mu_2^+ < \mu_3^+ \cdots$$

Let X and $|||\cdot|||$ be the Hilbert space and norm in X introduced in section 1. Suppose that the forcing term is supposed to be a multiple $s\phi_{00}(s \neq 0,\ s \in R)$ of the positive eigenfunction. That is, we consider the following problem

$$u_{tt} - u_{xx} + bu^+ - au^- = s\phi_{00} \text{in} \quad \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right) \times R, \tag{0.2}$$

$$u\left(\pm\frac{\pi}{2},\ t\right) = 0,$$

u is π-periodic in t and even in x.

Now we state our main results:

In section 1 we introduce an invariant space X spanned by eigenfunctions and investigate some properties. In section 3 we recall a critical point theory which is a crucial role to prove the multiplicity results for the periodic solutions of (0.2). In section 3 we prove Theorem 0.1 and 0.2. In section 3 and 4 we investigate a relation between multiplicity of solutions and source terms of the equation when the nonlinearity $-(bu^+ - au^-)$ crosses an eigenvalue λ_{10} and the source term f is generated by three eigenfunctions.

1. Invariant space spanned by eigenfunctions

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In $L^1(Q)$ we can consider the Hilbert basis given by

$$\phi_{j0}=\frac{\sqrt{2}}{\pi}\cos jx \quad (j\in N),\quad \phi_{jk}=\frac{2}{\pi}\cos jx\cos kt,\ \ \psi_{jk}=\frac{2}{\pi}\cos jx\sin kt\ (j,\ k\in N).$$

Let $u\in L^2(Q)$. u can be written by

$$u=\sum h_{jk}\phi_{jk}+\tilde{h}_{jk}\psi_{jk},\ j,\ k\in N,\quad \sum[|h_{jk}|^2+|\tilde{h}_{jk}|^2]<\infty.$$

It is easily seen that the eigenvalues of A ($Au=u_{tt}-u_{xx}$) are $\lambda_{jk}=j^2-k^2$. Let

$$E=\{u\in L^2(Q)|\sum_{jk}|\lambda_{jk}|(h_{jk}^2+\tilde{h}_{jk}^2)<\infty\}.$$

The space E, endowed with the norm

$$|||u|||=[\sum_{j,k}|\lambda_{jk}|(h_{jk}^2+\tilde{h}_{jk}^2)]^{\frac{1}{2}}$$

is a real Hilbert space continuously embedded in $L^2(Q)$. We define a subspace X of E as follows:

$$X=\{u\in E|h_{jk},\ \tilde{h}_{jk}=0 \text{ whenever } j \text{ is even or } k \text{ is odd}\}.$$

Then X is a closed invariant linear subspace of E compactly embedded in $L^2(Q)$. Moreover X is invariant under shifts: let $u\in X$ and τ be a real number, if $v(x,\ t)=u(x,\ t+\tau)$, then $v\in X$. X is invariant under a map from u to u^+ and a mapfrom u to u^-; if $u\in X$, then u^+, $u^-\in X$. Moreover $A(X)\subseteq X(Au=u_{tt}-u_{xx})$), $A:X\rightarrow X$ is an isomorphism. As stated before, we denote by $\{\mu_k^+|k\geq 1\}$ and $\{\mu_k^-|k\geq 1\}$ the sequence of positive and negative eigenvalues, respectively.

We need the following some properties:

Proposition 1.1. (i) $|||u|||\geq\|u\|$, where $\|u\|$ denote the L^2 norm of u.

(ii) $\|u\|=0$ if and only if $|||u|||=0$.

(iii) $Au\in X$ implies $u\in X$.

Proposition 1.2. Let c not be an eigenvalue of A and $u\in X$. Then we have $(A-c)^{-1}u\in X$.

From Proposition 1.2, we have:

Proposition 1.3. Let $f(x, t) \in X$. Let a and b not be eigenvalues of A. Then all the solutions of

$$Au + bu^+ - au^- = f(x, t)$$

belong to X.

For simplicity of notations, a weak solution of (0.2) is characterized by

$$u_{tt} - u_{xx} + bu^+ - au^- = s\phi_{00} \quad \text{in} \quad X. \tag{1.1}$$

2. A recall of critical point theory

Let X be a real Hilbert space on which the compact Lie group S^1 acts by means of time translations, i.e. for $u \in X$ and $\theta \in [0, \pi]$, set:

$$s_\theta u(x, t) = u(x, t + \theta).$$

Let $Fix(S^1)$ be the set of fixed points of the action, i.e.

$$Fix(S^1) = \{u \in X \mid s_\theta u = u, \ \forall \theta \in [0, \pi]\}$$
$$= \{u \in X \mid u \text{ is independent of } t\}.$$

We call a subset B of X an invariant set if for all $u \in B$, $s_\theta u \in B$ for all $\theta \in [0, \pi]$. A function $f: X \to R^1$ is called S^1-invariant, if $f(s_\theta u) = f(u)$, $\forall u \in X$, for all $\theta \in [0, \pi]$. Let $C(B, X)$ be the set of continuous functions from B into X. If B is an invariant set we say $h \in C(B, X)$ is an equivariant map if $h(s_\theta u) = s_\theta h(u)$ for all $\theta \in [0, \pi]$ and $u \in B$. Let S_r be the sphere centered at the origin of radius r. Let $f: X \to R$ be a functional of the form

$$f(x) = \frac{1}{2}(Lx \mid x) - \psi(x)$$

where $L: X \to X$ is linear, continuous, symmetric and equivariant, $\psi: X \to X$ is of class C^1 and invariant and $D\psi: X \to X$ is compact. The following result follows from [9].

Theorem 2.1 Assume that $f \in C^1(X, R^1)$ is S^1-invariant and there exist two closed invariant linear subspaces V, W of X and $r > 0$ with the following properties:

(a) $V+W$ is closed and of finite codimension in X;

(b) $Fix(S^1) \subseteq V+W$;

(c) $L(W) \subseteq W$;

(d) $\sup_{S_r \cap V} f < +\infty$ and $\inf_W f > -\infty$;

(e) $u \notin Fix(S^1)$ whenever $Df(u)=0$ and $\inf_W f < f(u) < \sup_{S_r \cap V} f$;

(f) f satisfies $(PS)_c$ whenever $\inf_W f \le c \le \sup_{S_r \cap V} f$.

Then f possesses at least

$$\frac{1}{2}(\dim(V \cap W) - codim_X(V+W))$$

distinct critical orbits in $f^{-1}([\inf_W f, \sup_{S_r \cap V} f])$.

3. An application of critical point theory

Let $A: X \to X$ be the linear, continuous and symmetric operator defined by

$$(Au|v) = \int_Q -u_t \cdot v_t + u_x \cdot v_x, \tag{3.1}$$

and let $f: X \to R$ be the functional defined by

$$f(u) = \frac{1}{2}(Au|u) + \int_Q \frac{b}{2}|u^+|^2 + \frac{a}{2}|u^-|^2 - s\int_Q \phi_{00} u.$$

We note that f is well-defined. By the following proposition, f is of class C^1 and if u is a critical point of f, then u is a weak solution of (0.1) with (0.2) and (0.3). The functional $f(u)$ is continuous, Fréchet differentiable in X with

$$Df(u)v = \int_Q (u_{tt} - u_{xx})v + \int_Q bu^+ \cdot v - au^- \cdot v - s\int_Q \phi_{00} v.$$

Moreover $Df \in C$. That is $f \in C^1$.

We need the following two lemmas for Proposition 3.1.

Lemma 3.1. Let $u \in X$ be a critical point of f with u independent of t and $a < -\mu_1^-$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \ge 1$. Then the problem

$$u_{tt} - u_{xx} + bu^+ - au^- = 0 \quad \text{in } X \tag{3.2}$$

has only the trivial solution $u=0$(cf. [12]).

Lemma 3.1 also holds for the case

$$-\mu_{k+1}^{+} < b < -\mu_{k}^{+},\ \cdots,\ -\mu_{1}^{+} < a,\ k \geq 1.$$

Lemma 3.2. Let $u \in X$ be a critical point of f with independent of t, $a < -\mu_1^-$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \geq 1$, $s > 0$, and $\alpha > 0$ be given. Then there exists $C > 0$ such that for all b and a with $a < -\mu_1^- - \alpha$, $-\mu_k^- + \alpha < b < -\mu_{k+1}^- - \alpha$ the solutions of (1.1) satisfy $|||u||| \leq C$.

Lemma 3.2 also holds for the case $-\mu_{k+1}^{+} < b < -\mu_{k}^{+},\ \cdots,\ -\mu_{1}^{+} < a,\ k \geq 1$ and $s < 0$.

Proposition 3.1. Let $u \in X$ be a critical point of f with u independent of t, $a < -\mu_1^-$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \geq 1$ and $s > 0$. Then there exists $C' > 0$ such that

$$f(u) = -s\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi_{00}\, u\, dx > -C'.$$

The compact Lie group S^1 acts on X by means of time-translations, hence by orthogonal transformations. It is easily checked that A is equivariant and f is invariant. Moreover, $A(X) \subseteq X$, $A : X \to X$ is an isomorphism and $Df(X) \subseteq X$. Therefore constrained critical points on X are in fact free critical points on S. Moreover, distinct critical orbits give rise to geometrically distinct solutions.

First we consider the case $a < -\mu_1^-$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \geq 1$ and $s > 0$. Let $c_\infty = \frac{1}{2}(\mu_k^- + \mu_{k+1}^-)$ and let $L : X \to X$ be the linear operator such that

$$(Lu|v) = (Au|v) - c_\infty \int_Q uv\, dx\, dt.$$

Then L is symmetric, bijective and equivariant. Let $X^-(L)$ be the negative space of L and $X^+(L)$ be the positive space of L. Then

$$X = X^-(L) \oplus X^+(L).$$

Moreover, we have

$$\forall u \in X^-(L) : (Lu|u) \leq (\mu_{k+1}^- - c_\infty)\int_Q u^2\, dx\, dt,$$

$$\forall u \in X^+(L) : (Lu|u) \geq (\mu_k^- - c_\infty) \int_Q u^2 dxdt.$$

Thus there exists $d > 0$ such that

$$(Lu|u) \leq -d|||u|||^2, \quad \forall u \in X^-(L),$$
$$(Lu|u) \geq d|||u|||^2, \quad \forall u \in X^+(L).$$

Then

$$f(u) = \frac{1}{2}(Lu|u) - \psi(u),$$

where

$$\psi(u) = \int_Q \left[-\frac{b}{2}|u^+|^2 - \frac{a}{2}|u^-|^2 + s\phi_{00}u - \frac{1}{2}c_\infty u^2 \right] dxdt.$$

Since X is compactly embedded in L^2, the map $D\psi$: $X \to X$ is compact.

Lemma 3.3. Under the same assumptions of Theorem 0.1, the functional f satisfies $(PS)_c$ condition for every $c \in R$: For every sequence (u_n) in X with $Df(u_n) \to 0$ and $f(u_n) \to c$, there exists a convergent subsequence of (u_n)(cf. [12]).

Lemma 3.4. Let $a < -\mu_1^-$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \geq 1$ and $s > 0$. Then the functional $f(u)$ is bounded from above on $X^-(L)$ and from below on $X^+(L)$. That is,

$$\sup_{u \in X^-(L)} f(u) < \infty \quad \text{and} \quad -\infty < \inf_{u \in X^+(L)} f(u).$$

Proof. For $u \in X^-(L)$,

$$\begin{aligned} f(u) &= \int_Q \left[\frac{1}{2}(u_{tt} - u_{xx}) \cdot u + \frac{b}{2}|u^+|^2 + \frac{a}{2}|u^-|^2 - s\phi_{00}u \right] dxdt \\ &\leq \frac{1}{2}\mu_{k+1}^- \int_Q u^2 + \frac{b}{2}\int_Q |u^+|^2 + \frac{a}{2}\int_Q |u^-|^2 - s\int_Q \phi_{00u} \\ &= \frac{1}{2}(\mu_{k+1}^- + b)\|u^+\|^2 + \frac{1}{2}(\mu_{k+1}^- + a)\|u^-\|^2 - s\int_Q \phi_{00}u < \infty, \end{aligned}$$

since $\mu_{k+1}^- + b < 0$ and mu$_{k+1}^- + a < 0$.

For $u \in X^+(L)$,

$$\begin{aligned} f(u) &= \int_Q \left[\frac{1}{2}(u_{tt} - u_{xx}) \cdot u + \frac{b}{2}|u^+|^2 + \frac{a}{2}|u^-|^2 - s\phi_{00}u \right] dxdt \\ &\geq \frac{1}{2}\mu_k^- \int_Q u^2 + \frac{b}{2}\int_Q |u^+|^2 + \frac{a}{2}\int_Q |u^-|^2 - s\int_Q \phi_{00}u \end{aligned} \tag{3.7}$$

$$\geq \frac{1}{2}(\mu_k^- + b)\|u^+\|^2 + \frac{1}{2}(\mu_k^- + a)\|u^-\|^2 - s\|u\| > -\infty,$$

since $u \in L^2(Q)$ and $\mu_{k+1}^- < -b < \mu_k^- < -a$.

Let $X^-(A)$ be the negative space of A and $X^+(L)$ be the positive space of A.

Lemma 3.5. Under the same assumptions of Theorem 0.1, there exists a neighborhood S_r of 0 with radius $r > 0$ such that

$$\sup_{u \in S_r \cap X^-(A)} f(u) < 0.$$

Proof. For $u \in X^-(A)$.

$$f(u) = \int_Q \left[\frac{1}{2}(u_{tt} - u_{xx}) \cdot u + \frac{b}{2}|u^+|^2 + \frac{a}{2}|u^-|^2 - s\phi_{00}u\right]dx\,dt$$

$$\leq \frac{1}{2}\mu_1^- \int_Q u^2 + \frac{b}{2}\int_Q |u^+|^2 + \frac{a}{2}\int_Q |u^-|^2 - s\int_Q \phi_{00}u$$

$$= \frac{1}{2}(\mu_1^- + b)\int_Q |u^+|^2 + \frac{1}{2}(\mu_1^- + a)\int_Q |u^-|^2 - s\int_Q \phi_{00}u$$

$$\leq \frac{1}{2}\frac{\mu_1^- + b}{|\mu_1^-|}|||u^+|||^2 + \frac{1}{2}\frac{\mu_1^- + a}{|\mu_1^-|}|||u^-|||^2 - s\int_Q \phi_{00}u.$$

Since $\mu_1^- + a < 0$, there exists a neighborhood S_r of 0 with radius $r > 0$ such that

$$\sup_{u \in S_r \cap X^-(A)} f(u) < 0,$$

where $|||u||| < r$ and $|||u|||^2 \geq |\mu_1^-|||u||^2$.

From Lemma 3.5 we have the following:

Lemma 3.6. Under the same assumptions of Theorem 3.1, there exists a neighborhood S_r of 0 with radius $r > 0$ such that

$$\sup_{S_r \cap X^-(A)} f(u) - f(u)|_{u \in Fix(S^1)} < 0.$$

Proof. From Proposition 3.2, there exists $C' > 0$ such that $f(u) = -\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi_{00}u > -C'$.

From Lemma 4.5, there exists a neighborhood S_r of 0 with radius $r > 0$ such that $\sup_{S_r \cap X^-(A)} f(u) - f(u)|_{u \in Fix(S^1)} < 0$.

From Lemma 3.4 and Lemma 3.6, we have

Lemma 3.7. Under the same assumptions of Theorem1.1, we have
$$\inf_{u\in X^+(L)} f(u) < \sup_{u\in S_r\cap X^-(A)} f(u) < f(u)|_{u\in Fix(S^1)}.$$

Theorem 3.1. Let $a < -\mu_1^1$, $-\mu_k^- < b < -\mu_{k+1}^-$, $k \ge 1$, and $s > 0$. Then (0.2) has at least k solutions.

Proof. If we set $V = X^-(A)$ and $W = X^+(L)$, then V and W are closed invariant subspaces of X with $L(W) \subseteq W$. We note that $Fix(S^1) \subseteq W$. Since $c_\infty < 0$, we have $V + W = X$. By Proposition 3.1, f is $C^1(X, R^1)$. By Lemma 3.4 and Lemma 3.5, assumption (d) of Theorem 1.1 is satisfied. By Lemma 3.3, assumption (f) of Theorem 1.1 is satisfied. By Lemma 3.7, assumption (e) of Theorem 1.1 is satisfied. Thus by Theorem 1.1, the problem (0.1) has at least $\frac{1}{2}\dim(V\cap W) = k$ nontrivial solutions. In case $-\mu_{k+1}^+ < b < -\mu_k^+, \cdots, -\mu_1^+ < a$, $k \ge 1$ and $s < 0$, denote by $c_\infty = \frac{1}{2}(\mu_k^+ + \mu_{k+1}^+)$. We introduce L as in the previous case and set $V = X^+(A)$ and $W = X^-(L)$.

We can apply the same argument of the functional $-f$ as in the proof of Theorem 3.1 and we have the following.

Theorem 4.2. Let $-\mu_{k+1}^+ < b < -\mu_k^+, \cdots, -\mu_1^+ < a$, $k \ge 1$ and $s < 0$. Then (0.2) has at least k solutions.

4. Multiplicity for source terms in two dimensional space

Let H_0 be the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x \text{ and } t\}.$$

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H.
Let us denote an element u, in H_0, as $u = \sum h_{mn}\phi_{mn}$ and we define a subspace H of H_0 as

$$H=\{u\in H_0: \sum|\lambda_{mn}|h_{mn}^2<\infty\}.$$

Then this is a complete normed space with a norm

$$\|u\|=(\sum|\lambda_{mn}|h_{mn}^2)^{\frac{1}{2}}.$$

In this section, we investigate multiplicity of solutions $u(x, t)$ for a piecewise linear perturbation $-(bu^+-au^-)$ of the one-dimensional wave operator $u_{tt}-u_{xx}$ with the nonlinearity $-(bu^+-au^-)$ crossing the eigenvalue λ_{10}. We suppose that $-1<a<3$ and $3<b<7$

Under this assumption, we have a concern with a relation between multiplicity of solutions and source terms of a nonlinear wave equation

$$Lu+bu^+-au^-=f \text{ in } H. \quad (4.1)$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}.

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u\in H$ is expressed by $u=v+w$, where $v=Pu$, $w=(I-P)u$. Hence equation (4.1) is equivalent to a system

$$Lw+(I-P)(b(v+w)^+-a(v+w)^-)=0, \quad (4.2)$$

$$Lv+P(b(v+w)^+-a(v+w)^-)=s_1\phi_{00}+s_2\phi_{10}. \quad (4.3)$$

Lemma 4.1. For fixed $v\in V$, (4.2) has a unique solution $w=\theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v (*cf.* [12]).

By Lemma 4.1, the study of multiplicity of solutions of (4.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$Lv+P(b(v+\theta(v))^+-a(v+\theta(v))^-)=s_1\phi_{00}+s_2\phi_{10} \quad (4.4)$$

defined on the two dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}\}$.

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$ and $\phi_{00}(x, t)>0$ in Q there exists a cone C_1 defined by

$$C_1=\left\{v=c_1\phi_{00}+c_2\phi_{10}: c_1\geq 0,\ |c_2|\leq\frac{c_1}{\sqrt{2}}\right\}$$

so that $v\geq 0$ for all $v\in C_1$ and a cone C_3 defined by

$$C_3 = \left\{ v = c_1\phi_{00} + c_2\phi_{10} : c_1 \le 0,\ |c_2| \le \frac{|c_1|}{\sqrt{2}} \right\}$$

so that $v \le 0$ for all $v \in C_3$.

We define a map $\Phi: V \to V$ given by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V. \tag{4.5}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma.

Lemma 4.2. $\Phi(cv) = c\Phi(v)$ for $c \ge 0$ and $v \in V$.

Lemma 4.2 implies that Φ maps a cone with vertex 0 onto a cone with vertex 0. Let $C_i (1 \le i \le 4)$ be the same cones of V as in Section 1. We investigate the images of the cones C_1 and C_3 under Φ. First we consider the image of the cone C_1. If $v = c_1\phi_{00} + c_2\phi_{10} \ge 0$, we have

$$\Phi(v) = c_1(b+\lambda_{00})\phi_{00} + c_2(b+\lambda_{10})\phi_{10}.$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} : d_1 \ge 0,\ |d_2| \le \frac{1}{\sqrt{2}} \left(\frac{b+\lambda_{10}}{b+\lambda_{00}} \right) d_1 \right\}.$$

The cone R_1 is in the right half-plane of V and the restriction $\Phi|_{C_1}: C_1 \to R_1$ is bijective.

We determine the image of the cone C_3. If $v = -c_1\phi_{00} + c_2\phi_{10} \le 0$, we have

$$\Phi(v) = -c_1(\lambda_{00}+a)\phi_{00} + c_2(\lambda_{10}+a)\phi_{10}.$$

Thus Φ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} : d_1 \le 0,\ d_2 \le \frac{1}{\sqrt{2}} \left| \frac{\lambda_{10}+a}{\lambda_{00}+a} \right| |d_1| \right\}.$$

The cone R_3 is in the left half-plane of V and the restriction $\Phi|_{C_3}: C_3 \to R_3$ is bijective.

We note that R_1 is in the right half plane and R_3 is in the left half plane.

Theorem 4.1. (i) If f belongs to R_1, then equation (4.1) has a positive solution and no negative solution. (ii) If f belongs to R_3, then equation (4.1) has a negative solution and no positive solution.

The cones C_2, C_4 are as follows

$$C_2=\left\{c_1\phi_{00}+c_2\phi_{10}: c_2\geq 0,\ c_2\geq \frac{1}{\sqrt{2}}|c_1|\right\},$$

$$C_4=\left\{c_1\phi_{00}+c_2\phi_{10}: c_2\leq 0,\ c_2\leq -\frac{1}{\sqrt{2}}|c_1|\right\},$$

Then the union of four cones $C_i(1\leq i\leq 4)$ is the space V.

Lemma 4.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R'_2=\left\{d_1\phi_{00}+d_2\phi_{10}: d_2\geq 0,\ -\sqrt{2}\left|\frac{\lambda_{00}+a}{\lambda_{10}+a}\right|d_2\leq d_1\leq\sqrt{2}\left|\frac{b+\lambda_{00}}{b+\lambda_{10}}\right|d_2\right\},$$

$$R'_4=\left\{d_1\phi_{00}+d_2\phi_{10}: d_2\leq 0,\ \sqrt{2}\left(\frac{\lambda_{00}+a}{\lambda_{10}+a}\right)d_2\leq d_1\leq\sqrt{2}\left(\frac{b+\lambda_{00}}{b+\lambda_{10}}\right)|d_2|\right\}.$$

Then the union of four cones R_1, R'_2, R_3, R'_4 is also the space V.

To investigate a relation between multiplicity of solutions and source terms in the nonlinear wave equation

$$Lu+bu^+-au^-=f \text{ in } H, \tag{4.6}$$

we consider the restrictions $\Phi|_{C_i}(1\leq i\leq 4)$ of Φ to the cones C_i. Let $\Phi_i=\Phi|_{C_i}$, i.e.,

$$\Phi_i: C_i\rightarrow V.$$

For $i=1$, 3, the image of Φ_i is R_i and $\Phi_i: C_i\rightarrow R_i$ is bijective.

From now on, our goal is to find the image of C_i under Φ_i for $i=2,4$. Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ is on the ray $c_1(b+\lambda_{00})\phi_{00}+\frac{1}{\sqrt{2}}c_1(b+\lambda_{10})\phi_{10},\ c_1\geq 0$ (a boundary ray of R_1) and the image of the other end point of γ under Φ is on the ray $c_1(\lambda_{00}+a)\phi_{00}+\frac{1}{\sqrt{2}}c_1(\lambda_{10}+a)\phi_{10}$,

$c_1 \ge 0$(a boundary ray of R_3). Since Φ is continuous, $\Phi(\gamma)$ is a path in V. By Lemma 1.2, $\Phi(\gamma)$ does not meet the origin. Hence the path $\Phi(\gamma)$ meets all rays (starting from the origin) in $R_1 \cup R'_4$ or all rays (starting from the origin) in $R'_2 \cup R_3$.

Therefore it follows from Lemma 4.2 that the image $\Phi(C_2)$ of C_2 contains one of sets $R_1 \cup R'_4$ and $R'_2 \cup R_3$.

Similarly, we have that the image $\Phi(C_4)$ of C_4 contains one of sets $R_1 \cup R'_2$ and $R'_4 \cup R_3$.

Theorem 4.2. Let $-1 < a < 3 < b < 7$ satisfy the condition

$$\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1. \tag{4.7}$$

Then we have the followings.

(i) If $f \in \mathrm{Int} R_1$, then equation (4.1) has a positive solution and at least two sign changing solutions.

(ii) If $f \in \partial R_1$, then equation (4.1) has a nonnegative solution and at least one sign changing solution.

(iii) If $f \in \mathrm{Int} R'_i (i=2, 4)$, then equation (4.1) has at least one sign changing solution.

(iv) If $f \in \mathrm{Int} R_3$, then equation (4.1) has only the negative solution.

(v) If $f \in \partial R_3$, then equation (4.1) has a nonpositive solution.

5. Multiplicity for source terms in three dimensional space

In this section, we investigate a relation between multiplicity of solutions and the source terms of a nonlinear wave equation when the source terms belong to the three dimensional space spanned by ϕ_{00}, ϕ_{10}, and $\phi_{mn} (\phi_{mn} \neq \phi_{00}, \phi_{10})$.

First we consider

$$Lu + bu^+ - au^- = s_1\phi_{00} + s_2\phi_{10} + \varepsilon\phi_{20}. \tag{5.1}$$

Let V, W and P be the same as in section 1. Then equation (5.1) is equivalent to a system

$$Lw+(I-P)(b(v+w)^{+}-a(v+w)^{-})=\varepsilon\phi_{20}, \tag{5.2}$$

$$Lv+P(b(v+w)^{+}-a(v+w)^{-})=s_1\phi_{00}+s_2\phi_{10}. \tag{5.3}$$

Lemma 5.1. For fixed $v\in V$, (5.2) has a unique solution $w=\theta_\varepsilon(v)$. Furthermore, $\theta_\varepsilon(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v.

Proof. We use the cotraction mapping principle. Let delta= $\frac{1}{2}(a+b)$. We rewrite (5.2) as

$$(-L-\delta)w=(I-P)(b(v+w)^{+}-a(v+w)^{-}\delta(v+w)-\varepsilon\phi_{20}),$$

or equivalently,

$$w=(-L-\delta)^{-1}(I-P)(g_v(w)-\varepsilon\phi_{20}), \tag{5.4}$$

where

$$g_v(w)=b(v+w)^{+}-a(v+w)^{-}-\delta(v+w).$$

Since $|g_v(w_1)-g_v(w_2)|\le|b-\delta||w_1-w_2|$, we have

$$|g_v(w_1)-g_v(w_2)|\le|b-\delta||w_1-w_2|,$$

where $|\ |$ is the L^2 norm in H. The operator $(-L-\delta)^{-1}(I-P)$ is a selfadjoint compact map from $(I-P)H$ into itself. The eigenvalues of $(-L-\delta)^{-1}(I-P)$ in W are $(\lambda_{mn}-\delta)^{-1}$, where $\lambda_{mn}\ge 7$ or $\lambda_{mn}\le -1$. Therefore its L^2 norm is $\max\left\{\frac{1}{7-\delta}, \frac{1}{1+\delta}\right\}$.

Since $|b-\delta|<\min\{7-\delta, 1+\delta\}$, it follows that for fixed $v\in V$, the right hand side of (5.4) defines a Lipschitz mapping W into itself with Lipschitz constant $\gamma<1$. Hence, by the cotraction mapping principle, for given $v\in V$, there is a unique win W which satisfies (5.2).

Also, it follows, by the standard arguement principle, that $\theta_\varepsilon(v)$ is Lipschitz continuous in terms of v.

By Lemma 5.1, the study of multiplicity of solutions of (5.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$Lv+P(b(v+\theta_\varepsilon(v))^{+}-a(v+\theta_\varepsilon(v))^{-})=s_1\phi_{00}+s_2\phi_{10} \tag{5.5}$$

defined on the two dimensional subspace V.

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$ and $v>0$ in Q for all $v\in C_1$, there exists a convex subset $C_{1\varepsilon}$ of C_1 defined by

$$C_{1\varepsilon}=\{v=c_1\phi_{00}+c_2\phi_{10}: v+\varepsilon\phi_{20}>0 \text{ in } Q\}$$

and a convex subset $C_{3\varepsilon}$ of C_3 defined by

$$C_{3\varepsilon}=\{v=c_1\phi_{00}+c_2\phi_{10}: v+\varepsilon\phi_{20}<0 \text{ in } Q\}.$$

We define a map $J: R\times V\to V$ given by

$$J(\varepsilon, v)=Lv+P(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-), \quad v\in V. \tag{5.6}$$

Then for fixed $\{\varepsilon\}$, J is continuous on V, since $\theta_\varepsilon(v)$ is continuous on V. Also, it is easily proved that for fixed v, J is continuous on $\dot{V}$.

Lemma 5.2. For fixed $v\in V$, J is continuous on R.

Proof. It is enough to show that for fixed v, $\theta_\varepsilon(v)$ is a continuous function of ε. We use the cotraction mapping principle. Let $\delta=\frac{1}{2}(a+b)$. Let $w_i(i=1, 2)$ be the unique solution of

$$(-L-\delta)w=(I-P)(b(v+w)^+-a(v+w)^-\delta(v+w)-\varepsilon_i\phi_{20}),$$

or equivalently,

$$w=(-L-\delta)^{-1}(I-P)(g_v(w)-\varepsilon_i\phi_{20}), \tag{5.7}$$

where

$$g_v(w)=b(v+w)^+-a(v+w)^--\delta(v+w).$$

Then we have, by Lemma 5.2,

$$|w_1-w_2|\le\gamma|w_1-w_2|+\frac{|\phi_{20}|}{|\lambda_{20}+\delta|}|\varepsilon_1-\varepsilon_2|,$$

or equivalently,

$$(1-\gamma)|w_1-w_2|\le\frac{|\phi_{20}|}{|\lambda_{20}+\delta|}|\varepsilon_1-\varepsilon_2|,$$

which means that for fixed v, $\theta_\varepsilon(v)$ is a continuous function of ε, where $\gamma<1$.

We note that if v is in $C_{1\varepsilon}$, then $\theta_\varepsilon(v)=\frac{\varepsilon}{b-15}\phi_{20}$. In fact, if v is in $C_{1\varepsilon}$, then $v+\varepsilon\phi_{20}>0$ in Q and hence $v+\frac{\varepsilon}{b-15}\phi_{20}>0$ in Q. Hence $\theta_\varepsilon(v)=\frac{\varepsilon}{b-15}\phi_{20}$

satisfies $L\theta_\varepsilon(v)+(I-P)(b(v+\theta_\varepsilon(v))^+-a(v+\theta_\varepsilon(v))^-)=\varepsilon\phi_{20}$.

Also, if v is in $C_{3\varepsilon}$, then $v+\varepsilon\phi_{20}<0$ in Q and hence $v+\frac{\varepsilon}{a-15}\phi_{20}<0$ in Q

Hence $\theta_\varepsilon(v)=\frac{\varepsilon}{a-15}\phi_{20}$ satisfies the above equation.

We investigate the images of the convx sets $C_{1\varepsilon}$ and $C_{3\varepsilon}$ under J. First we consider the image of the cone $C_{1\varepsilon}$. If $v=c_1\phi_{00}+c_2\phi_{10}$ is in $C_{1\varepsilon}$, then $v+\theta_\varepsilon(v)>0$ in Q and hence we have

$$J(\varepsilon,\ v)=c_1(b+\lambda_{00})\phi_{00}+c_2(b+\lambda_{10})\phi_{10}.$$

Thus, for fixed epsilon, the image of $C_{1\varepsilon}$ under J, $J(\varepsilon,\ C_{1\varepsilon})$, is a convex subset of

$$R_1=\left\{d_1\phi_{00}+d_2\phi_{10}: d_1\geq 0,\ |d_2|\leq\frac{1}{\sqrt{2}}\leq\left(\frac{b+\lambda_{10}}{b+\lambda_{00}}\right)d_1\right\}.$$

For fixed ε, the restriction $J|_{C_{1\varepsilon}}: C_{1\varepsilon}\to J(\varepsilon,\ C_{1\varepsilon})$ is bijective.

We determine the image of the cone $C_{3\varepsilon}$. If $v=c_1\phi_{00}+c_2\phi_{10}$ is in $C_{3\varepsilon}$, then $v+\theta_\varepsilon(v)<0$ in Q and hence we have

$$J(\varepsilon,\ v)=-c_1(\lambda_{00}+a)\phi_{00}+c_2(\lambda_{10}+a)\phi_{10}.$$

Thus, for fixed epsilon, the image of $C_{3\varepsilon}$ under J, $J(\varepsilon, C_{3\varepsilon})$, is a convex subset of

$$R_3=\left\{d_1\phi_{00}+d_2\phi_{10}: d_1\leq 0,\ d_2\leq\frac{1}{\sqrt{2}}\left|\frac{\lambda_{10}+a}{\lambda_{00}+a}\right||d_1|\right\}.$$

For fixed ε, the restriction $J|_{C_{3\varepsilon}}: C_{3\varepsilon}\to J(\varepsilon,\ C_{3\varepsilon})$is bijective.

Let $\varepsilon>0$ be fixed. If v is in $C_{1\varepsilon}$, then $\theta_\varepsilon(v)=\frac{\varepsilon}{b-15}\phi_{20}$ and $\frac{\varepsilon}{b-15}(v+\phi_{20})>0$, $\frac{b-15-\varepsilon}{b-15}v>0$ in Q. Hence we have the lemma.

Lemma 5.3. Let $\varepsilon>0$ be fixed. Then there are open sets $C'_{1\varepsilon}, C'_{3\varepsilon}$ with $\overline{C_{1\varepsilon}}\subset C'_{1\varepsilon}\subset C_1$, $\overline{C_{3\varepsilon}}\subset C'_{3\varepsilon}\subset C_3$ such that $\theta_\varepsilon(v)=\frac{\varepsilon}{b-15}\phi_{20}$ for all $v\in C'_{1\varepsilon}$, $\theta_\varepsilon(v)=\frac{\varepsilon}{a-15}\phi_{20}$ for all $v\in C'_{3\varepsilon}$.

Theorem 5.1. Let $\varepsilon>0$ be fixed. Then we have:

(i) If f belongs to $J(\varepsilon,\ C_{1\varepsilon})$, then equation (5.1) has a positive solution and no negative solution.

(ii) If f belongs to $J(\varepsilon, C_{3\varepsilon})$, then equation (6.1) has a negative solution and no positive solution.

We define two sets $C_{2\varepsilon}$, $C_{4\varepsilon}$ as follows

$$C_{2\varepsilon}=\{v=c_1\phi_{00}+c_2\phi_{10}: c_2\geq 0,\ v\not\in \mathrm{Int}C_{1\varepsilon},\ v\not\in \mathrm{Int}C_{3\varepsilon}\},$$

$$C_{4\varepsilon}=\{v=c_1\phi_{00}+c_2\phi_{10}: c_2\leq 0,\ v\not\in \mathrm{Int}C_{1\varepsilon},\ v\not\in \mathrm{Int}C_{3\varepsilon}\}.$$

Then $C_2\subset C_{2\varepsilon}$, $C_4\subset C_{4\varepsilon}$ and the union of four sets $C_{i\varepsilon}(1\leq i\leq 4)$ is the space V.

For fixed $\varepsilon>0$, we set

$$R_{1\varepsilon}=J(\varepsilon, C_{1\varepsilon}),\quad R_{3\varepsilon}=J(\varepsilon, C_{3\varepsilon})$$

$$R'_{2\varepsilon}=\{v=d_1\phi_{00}+d_2\phi_{10}: d_2\geq 0,\ v\not\in \mathrm{Int}R_{1\varepsilon},\ v\not\in \mathrm{Int}R_{3\varepsilon}\},$$

$$R'_{4\varepsilon}=\{v=d_1\phi_{00}+d_2\phi_{10}: d_2\leq 0,\ v\not\in \mathrm{Int}R_{1\varepsilon},\ v\not\in \mathrm{Int}R_{3\varepsilon}\},$$

$$I_\varepsilon=\{v=c_1\phi_{00}: |c_1|\leq\varepsilon\},\quad I_{\varepsilon\delta}=\{v\in V: d(v, I_\varepsilon)<\delta\}\setminus(R_{1\varepsilon}\cup R_{3\varepsilon}),$$

where $d(v, I_\varepsilon)=\inf\{|v-w|: w\in I_\varepsilon\}$.

Then $R'_2\subset R'_{2\varepsilon}$, $R'_4\subset R'_{4\varepsilon}$ and the union of four sets $R_{1\varepsilon}$, $R'_{2\varepsilon}$, $R_{3\varepsilon}$, $R'_{2\varepsilon}$ is V.

We note that $C_i=C_{i0}(1\leq i\leq 4)$ and $R_i=R_{i0}(i=1, 3)$, $R'_j=R'_{j0}\,(j=2, 4)$.

Since J is continuous, by Lemma 5.4 we have:

Lemma 5.4. Let $-1<a<3<b<7$ satisfy the condition (4.7). Then, for small $\varepsilon>0$, there is $\delta>0$ such that $J(\varepsilon, C_{2\varepsilon})$ contains $(R_{1\varepsilon}\cup R'_{4\varepsilon})\setminus I_{\varepsilon\delta}$ and $J(\varepsilon, C_{4\varepsilon})$ contains $(R_{1\varepsilon}\cup R'_{2\varepsilon})\setminus I_{\varepsilon\delta}$.

Lemma 5.5. Let $-1<a<3<b<7$ satisfy the condition (4.7). Then, for small $\varepsilon>0$, the equation $Lu+bu^+-au^-=s_1\phi_{00}+s_2\phi_{10}+\varepsilon\phi_{20}$ has at least one solution.

Proof $J(\varepsilon, \cdot)$ is continuous in v and homotopic to Φ. $J(\varepsilon, C_{3\varepsilon})=\Phi(C_{3\varepsilon})=R_{3\varepsilon}$ and $J(\varepsilon, \partial C_{3\varepsilon})=\Phi(\partial C_{3\varepsilon})$. Since $\Phi(V\setminus C_{3\varepsilon})$ contains $V\setminus R_{3\varepsilon}$, $J(\varepsilon, C_{3\varepsilon})$ contains it, which completes the proof.

With Lemma 5.3 , Lemma 5.4, and , we have the following.

Lemma 5.6. Let $-1 < a < 3 < b < 7$ satisfies the condition (4.7). Let $v = s_1\phi_{00} + s_2\phi_{10}$ and $f = v + \varepsilon\phi_{20}$. Then, for small $\varepsilon > 0$, we have the followings.

(i) If $v \in (\mathrm{Int} R_{1\varepsilon})$, then equation (5.1) has a positive solution and at least two sign changing solutions.

(ii) If $v \in (\mathrm{Int} R_{1\varepsilon})$, then equation (5.1) has a nonnegative solution and at least one sign changing solution.

(iii) If $v \in (\mathrm{Int}\ R'_{i\varepsilon})(i = 2, 4)$, then equation (5.1) has at least one sign changing solution.

(iv) If $v \in (\mathrm{Int}\ R_{3\varepsilon})$, then equation (5.1) has a negative solution.

(v) If $v \in (\partial R_{1\varepsilon})$, then equation (5.1) has a nonpositive solution.

If $Lu + bu^+ - au^- = s_1\phi_{00} + s_2\phi_{10} + \varepsilon\phi_{20}$ has m multiple solutions, then so is $Lu + bu^+ - au^- = k(s_1\phi_{00} + s_2\phi_{10} + \varepsilon\phi_{20})\ (k > 0)$.

If $\alpha = k\varepsilon\ (k > 0,\ \varepsilon > 0)$, then $C_{1\alpha} = kC_{1\varepsilon}$, $C_{3\alpha} = kC_{3\varepsilon}$.

We note that $C_{1(-\alpha)} = C_{1\alpha}$, $C_{3(-\alpha)} = C_{3\alpha}\}$.

If $v + \varepsilon\phi_{20} > 0$ in $Q(v \in V)$, then $v \in C_{1\varepsilon}$ and if $v + \varepsilon\phi_{20} < 0$ in $Q(v \in V)$, then $v \in C_{3\varepsilon}$.

We set :

$$U_1 = \{v + \varepsilon\phi_{20} : v \in R_{1\varepsilon},\ \varepsilon \in R\},\quad U_3 = \{v + \varepsilon\phi_{20} : v \in R_{3\varepsilon},\ \varepsilon \in R\},$$

$$U_2 = span\{\phi_{00},\ \phi_{10},\ \phi_{20}\} \setminus (\mathrm{Int}\, U_1 \cup \mathrm{Int}\, U_3)$$

With the above notations and facts, we have:

Theorem 5.2. Let $-1 < a < 3 < b < 7$ satisfy the condition (4.7). Let

$$f = s_1\phi_{00} + s_2\phi_{10} + \varepsilon\phi_{20}.$$

Then we have the followings.

(i) If $f \in \mathrm{Int}\, U_1$, then equation (5.1) has a positive solution and at least two sign changing solutions.

(ii) If $f \in \partial U_1$, then equation (5.1) has a nonnegative solution and at least one sign changing solution.

(iii) If $f \in \mathrm{Int}\, U_2$, then equation (5.1) has at least one sign changing solution.

(iv) If $f \in U_3$, then equation (5.1) has a negative solution.

Let ϕ_{mn} be an eigenfunction corresponding to $\lambda_{mn}(\lambda_{mn} \neq \lambda_{00}, \lambda_{10})$. We consider the equation

$$Lu + bu^{+} - au^{-} = s_1 \phi_{00} + s_2 \phi_{10} + s_3 \phi_{mn}. \qquad (5.8)$$

By the similar method of the proof of Theorem 6.2, we have the following.

Theorem 5.3. Let $-1 < a < 3 < b < 7$ satisfy the condition (4.7). Let

$$f = s_1 \phi_{00} + s_2 \phi_{10} + s_3 \phi_{mn} \ (\lambda_{mn} \neq \lambda_{00}, \lambda_{10}).$$

Then there are cones U_1, U_2, U_3 in $span\{\phi_{00}, \phi_{10}, \phi_{mn}\}$ which satisfy he followings.

(i) If $f \in \text{Int}\, U_1$, then equation (5.8) has a positive solution and at least two sign changing solutions.

(ii) If $f \in \partial U_1$, then equation (5.8) has a nonnegative solution and at least one sign changing solution.

(iii) If $f \in \text{Int}\, U_2$, then equation (5.8) has at least one sign changing solution.

(iv) If $f \in U_3$, then equation (5.8) has a negative solution.

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