Stability for Nonlinear Systems with Slowly Varying and Small Jump Inputs

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Abstract: Stability theorem for nonlinear systems with slowly varying inputs is extended to systems with slowly varying and small jump inputs.

Keywords: *stability, nonlinear systems, slowly baring input, and small jump input*

I. Introduction

Consider a nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0); \text{given}, \quad t \geq 0 \qquad (1)$$

Where x(t) and u(t) are the n state and m input vectors, respectively. If the inputs vary slowly (that is, the Euclidean norm of $\dot{u}(t)$ is small) and the system has an isolated equilibrium defined by $x = h(u)$ for each $u \in \Gamma$ (a compact subset of R^m), we can analyze the system (1) by treating u(t) as constant parameters. It is reasonable to expect that the slowly varying system (1) will posses a property similar to the frozen system of each fixed input u.

One example of such system is a two time-scale system, in which a fast subsystem is perturbed by the slowly varying states of a slow subsystem [6]. We can design the fast subsystem under the assumption that the states of the slow subsystem are constant parameters. Another example is a control system designed by the extended linearization method [7] or by the pseudo linearization method [8]. In this case the command signals and load disturbances are the slowly varying inputs. Gain-scheduled control systems are also included in this class. Stability of the system (1) with slowly varying inputs is studied by some authors via the Gronwall-Bellman inequality and the Lyapunov method [1]-[4]. Bounds on the norm magnitude of $\dot{u}(t)$ under which states of the system remain bounded are obtained. Since the derivative of u(t) is used in the analysis, step changes in the inputs can not be handled. Here, we are to show that stability theorem via the Lyapunov method [4] can be easily adapted to permit step changes in the inputs. Since the command signals or load disturbances in control systems designed with the extended linearization method can often be step changes [9], it would be important in practice.

II. Result

Stability theorem which can handle small jump changes in the inputs is investigated. In the sequel, as a magnitude of a time varying vector (pointwise in time), the Euclidean norm is used and denoted by $\|\bullet\|$.

Assume that ([3] and [4])

[H1]: f(x,u) is twice continuously differentiable.

[H2]: For every $u \in \Gamma$. the equation f(x,u)=0 has a twice continuoucly differentiable isolated root x=h(u). That is, f(h(u),u)=0 and h(u) is twice continuously differentiable.

[H3]: For each fixed $u \in \Gamma$ and $z = x - h(u) \in B_r = \{z \in R^n \mid \|z\| < r\}$,the frozen system;

$$\dot{z}(t) = f(z(t) + h(u), u) \qquad (2)$$

has a Lyapunov function V(z,u) such that

$$c_1\|z\|^2 \leq V(z,u) \leq c_2\|z\|^2 \qquad (3)$$

$$\frac{\partial V(z,u)}{\partial z} f(z + h(u), u) \leq -c_3\|z\|^2 \qquad (4)$$

$$\left\|\frac{\partial V(z,u)}{\partial z}\right\| \leq c_4\|z\| \qquad (5)$$

$$\left\|\frac{\partial V(z,u)}{\partial u}\right\| \leq c_5\|z\|^2 \qquad (6)$$

Assumptions [H1] and [H2] are about smoothness of the function f(x,u) and the steady-state solution x=h(u). Assumption [H3] is about the quadratic stability at each steady-state. Instead of the assumption [H3], we may use more explicit eigenvalue condition ([2] and [3]) such that real parts of all the eigenvalues of $\partial f(x,u)/\partial x$ at any $u \in \Gamma$ and $x = h(u)$ are less than $-\sigma(\sigma > 0)$. Lemma due to Hoppendteadt [1] shows that the assumption [H3] is implied by the above eigenvalue condition with sufficiently small B_r [4]. Here, we use [H3] for simplicity.

From the smoothness conditions of [H1] and [H2], we have for $u \in \Gamma$ and $z = x - h(u) \in B_r$:

$$\left\|\frac{\partial f(x,u)}{\partial x}\right\| \leq L_1 \qquad (7)$$

$$\left\|\frac{\partial f(x,u)}{\partial u}\right\| \leq L_2 \qquad (8)$$

$$\left\|\frac{\partial h(u)}{\partial u}\right\| \leq L_3 \qquad (9)$$

Transforming the state variables to $z(t) = x(t) - h(u(t))$, the system (1) becomes

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$$\dot{z}(t) = f(z(t) + h(u(t)), u(t)) - \frac{\partial h(u(t))}{\partial u}\dot{u}(t) \quad (10)$$

Stability bound on $\|\dot{u}(t)\|$ can be obtained from that the function V(z,u) of [H3] become a Lyapunov function still for the system(10). Since the time derivative of u(t) is used, step changes are not permitted. For the differentiation free condition, we introduce another transformation as

$$z(t) = x(t) - h(u_m(t)) \quad (11)$$

$$u(t) = u_m(t) + u_j(t) \quad (12)$$

Where $u_m(t)$ and $u_j(t)$ represent differentiable slowly varying components and small jump components of u(t), respectively. Then the system (1) becomes

$$\dot{z}(t) = f(z(t) + h(u_m(t)), u_m(t) + u_j(t)) - \frac{\partial h(u_m(t))}{\partial u_m}\dot{u}_m(t) \quad (13)$$

Stability bounds on $u_m(t)$ and $u_j(t)$ are investigated through the method very similar to [4] and [5].

Theorem 1: Consider the system which satisfied the assumptions [H1]-[H3] and

$$\|z(0) = x(0) - h(u_m(0))\| \le r\sqrt{c_1 / c_2} \quad (14)$$

$$\|\dot{u}_m(t)\| \le \varepsilon_1 \quad (15)$$

$$\|u_j(t)\| \le \varepsilon_2, \qquad t \ge 0 \quad (16)$$

If, for some positive $\theta < 1$,

$$(c_4 L_3 + \theta r c_5 \sqrt{c_1 / c_2})\varepsilon_1 + c_4 L_2 \varepsilon_2 < \theta r c_3 \sqrt{c_1 / c_2} \quad (17)$$

Then the solutions of system (1) are uniformly bounded for $t \ge 0$ and uniformly ultimately bounded with an ultimate bound

$$b = \frac{c_4(L_3\varepsilon_1 + L_2\varepsilon_2)}{\theta(c_3 - c_5\varepsilon_1)} \quad (18)$$

Proof: We show that $V(z, u_m)$ of [H3] is a Lyapunov function for the system (13). The time derivative of $V(x, u_m)$ is

$$\begin{aligned}
\dot{V}(z,u_m) &= \frac{\partial V(z,u_m)}{\partial z}\dot{z} + \frac{\partial V(z,u_m)}{\partial u_m}\dot{u}_m \\
&= \frac{\partial V(z,u_m)}{\partial z}\left[f(z+h(u_m), u_m + u_j) - \frac{\partial h(u_m)}{\partial u_m}\dot{u}_m\right] + \frac{\partial V(z,u_m)}{\partial u_m}\dot{u}_m \\
&= \frac{\partial V(z,u_m)}{\partial z}\left[f(z+h(u_m), u_m) + \left.\frac{\partial f(z+h(u_m),u)}{\partial u}\right|_{u\in[u_m, u_m+u_j]} u_j - \frac{\partial h(u_m)}{\partial u_m}\dot{u}_m\right] \\
&\quad + \frac{\partial V(z,u_m)}{\partial u_m}\dot{u}_m \\
&\le -c_3\|z\|^2 + (c_4 L_2\varepsilon_2 + c_4 L_3 \varepsilon_1)\|z\| + c_5\varepsilon_1\|z\|^2
\end{aligned} \quad (19)$$

For $\|z\| \ge b$,

$$\begin{aligned}
\dot{V}(z,u_m) &\le (-c_3 + c_4(L_3\varepsilon_1 + L_2\varepsilon_2)/b + c_5\varepsilon_1)\|z\|^2 \\
&= -(1-\theta)(c_3 - c_5\varepsilon_1)\|z\|^2
\end{aligned} \quad (20)$$

Since $\varepsilon_1 < c_3 / c_5$ from (17), $\dot{V}(z, u_m)$ is negative definite for $\|z\| \ge b$. From (17) and (18) we have $b < r\sqrt{c_1 / c_2}$ and hence the theorem follows. q.e.d.

We may choose $u_m(t)$ and θ which will provide better stability bounds. In the case of no jump changes in the inputs u(t), ε_2 can be chosen as zero and the theorem is reduced to that of [5]. Sometimes it is also beneficial to introduce $u_m(t)$ and $u_j(t)$ for no jump changes in the inputs. For example, small fast triangular wave inputs produce very large $\|\dot{u}(t)\|$ and previous theorems may fail to provide stability. Instead, the proposed method can be applied with choosing $u_m(t)$ as the average values of the triangular wave inputs. As in Lawrence and Rugh [3], the stabillity bouned of (15) and (16) may be improved to the time averaging ones.

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