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Balanced Accuracy and Confidence Probability of Interval Estimates

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Abstract. Simultaneous estimation of accuracy and probability corresponding to a prediction interval is considered in this study. Traditional application of confidence interval forecasting consists in evaluation of interval limits for a given significance level. The wider is this interval, the higher is probability and the lower is the forecast precision. In this paper a measure of stochastic forecast accuracy is introduced, and a procedure for balanced estimation of both the predicting accuracy and confidence probability is elaborated. Solution can be obtained in an optimizing approach. Suggested method is applied to constructing confidence intervals for parameters estimated by normal and t distributions

Key Words : *prediction interval, confidence probability, accuracy index, Chebyshev's Inequality*

1. INTRODUCTION

Interval estimates of statistical parameters are usually constructed for big confidence probability values, say, 95%, 99%, or 99.9%. However, a higher level of this probability makes the interval itself very wide, which translates into a low accuracy of the parameter evaluation. It is a natural feature of statistical measurements and estimations, very well esxpressed in Chebyshev's Inequality and other theorems concerning probability and a range of estimates (see, for example, [5]). High probability corresponds to a large span of a parameter's possible values, that means a poor precision in this parameter evaluation. In a larger sense, it is a natural phenomenon

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expressed in Heisenberg's Uncertainty Principle: the better is known a particle localization, the less is probability to find it in this place. A similar theorem is known in the Fourier analysis of signals and in wavelets analysis: a signal does not have simultaneously a precise location in time and a precise frequency. The Gabor theorem states that a brief signal has a Fourier transform of a broad range of frequencies ([2], [6]). Thus we know much about nothing (high probability, poor identification) or we know nothing about everything (low probability, high identification).

However, as Abraham and Ledolter state ([1], pp 1-2), "the objective of forecasting is to reduce the forecast error: to produce forecasts that are seldom incorrect and have small forecast errors." They continue on page 5, stating that "the most important criterion for choosing a forecast method is its accuracy, or how closely the forecast predicts the actual event." In this work we elaborate a methodology and optimizing procedure of a compromise solution, i.e., the obtaining a high accuracy of the interval estimates together with high confidence probability. This paper is structured as follows. Section II sets up the notation. Section III introduces the measures of accuracy for an interval estimate. Feasible solutions for both probability and accuracy are considered in Section IV for normal and in Section V for t distributions. Section VI presents regression models for estimation of optimal percentiles of these distributions. Section VII gives a rough estimation of accuracy-probability probem via Chebyshev's Inequallity. Some further extensions of the suggested approach are discussed in Section VIII, and Section IX summarizes.

2. NOTATION

Consider confidence interval estimate of the following structure:

$$\hat{y}_{\pm} = \hat{y} \pm \Delta, \tag{2.1}$$

where $\hat{y}$ is a point estimate of a random parameter y, and $\hat{y_{\pm}}$ are confidence limits. Various kinds of interval estimators with normal and with t-distributions correspond to this relation with the deviation length Δ defined as

$$\Delta = u_{\alpha/2} \cdot \sigma \tag{2.2}$$

where σ is the standard deviation of the parameter y. If σ is known, we use a normal distribution for $100\%(1-\alpha)$ probability confidence interval for y,

$$P[\hat{y} - z_{\alpha/2}\sigma \leq y \leq \hat{y} + z_{\alpha/2}\sigma] = 1 - \alpha \tag{2.3}$$

where α is a the significance level. If σ is a sample estimator, we have a $t_{\alpha/2}$ distribution with known degrees of freedom m, i.e., there is a t-distribution for $100\%(1-\alpha)$ confidence interval

$$P[\hat{y} - t_{\alpha/2}(m)\sigma \leq y \leq \hat{y} + t_{\alpha/2}(m)\sigma] = 1 - \alpha. \tag{2.4}$$

Both cases (2.3) and (2.4) we can describe by the relations (2.1) and (2.2) where $u_{\alpha/2}$ could correspond to the normal ($z_{\alpha/2}$) distribution with a known σ, or to a t-distribution ($t_{\alpha/2}(m)$) with estimated σ in the general representation (2.2). Thus (2.3) and (2.4) can be written as

$$P[\hat{y} - u_{\alpha/2}\sigma \le y \le \hat{y} + u_{\alpha/2}\sigma] = P[\hat{y} - \Delta \le y \le \hat{y} + \Delta] = 1 - \alpha. \tag{2.5}$$

For two-sided estimation (2.5), a confidence probability $p(u)$ is connected with cumulative distribution function $F(u)$ as follows:

$$p(u) = 2F(u) - 1, \tag{2.6}$$

thus, symetricized probability density function (pdf) value is

$$f(u) =\equiv p'(u) = 2F'(u) = 2\psi(u), \tag{2.7}$$

where $\psi(u)$ is pdf of the random variable u, and $F(u)$ is defined by

$$F(u) = \int_{-\infty}^{u} f(x)dx, \tag{2.8}$$

where normal or t pdf could be used as $f(x)$.

Formulae (2.1) and (2.2) are widely applied for estimation of mean value, of sums and differences, of two means differences, of proportions parameters, in forecasting by regression and time-series models, for coefficients of regression interval estimates, etc. For each case a standard deviation σ is calculated by corresponding formulae. See, for example, [3], [5].

Together with standard deviation σ (square root of the variance), other measures of variability are used too, for example, mean error, mean percent error, mean absolute error, mean absolute percent error (see [1]). However, in confidence interval estimations, the mean square error, or standard deviation (2.2), is typically used.

The value $p = 1-\alpha$ in (2.3) and (2.4), called the confidence coefficient, shows the probability that the random interval (2.1) includes the parameter y. It is commonly interpreted that the repeated independent tryings of the random experiment produce the probability p that the random interval (2.5) includes the unknown fixed value of the parameter y.

3. ACCURACY OF CONFIDENCE INTERVAL ESTIMATES

In practice the interval estimation is usually conducted by the following scheme: taking one or two levels of significance, say $\alpha = 0.05$ and $\alpha = 0.01$, and the corresponding $u_{\alpha/2}$, a researcher applies normal (2.3) or t (2.4) distributions, i.e., calculates the length of deviation Δ (2.2) using known or estimated standard deviation σ for the case under construction, and combines it with the point estimate $\hat{y}$ (2.1) obtaining the upper $\hat{y}_+$ and the lower $\hat{y}_-$ confidence limits.

Here we encounter an interesting problem. The higher the confidence probability p, the wider the confidence interval itself, because $u_{\alpha/2}$ in (2.2) or (2.5) is bigger for a larger p (cumulative distribution (2.6) is a monotonically increasing function). In the limit case of probability reaching 1 the corresponding value of $u_{\alpha/2}$ goes to infinity, i.e., the interval (2.1) tends to be of infinite length. It means that the value of the parameter y is estimated with a lower precision when with a higher probability, and vice versa, a narrow interval (2.1) or (2.5) yields a better precision of the y value but it corresponds to a worse confidence probability p. In other words, we have either the higher precision evaluated with lower reliability or the lower precision evaluated with higer reliability (probability).

For practical aims usually the values $u_{\alpha/2} = 1$, $u_{\alpha/2} = 2$, and $u_{\alpha/2} = 3$ are used because they correspond to $\alpha = 0.1$, $\alpha = 0.05$, or $\alpha = 0.01$, i.e., to the high enough probability $p = 90\%$, $p = 95\%$ or 99%. These are reasonable conventional levels and for the given $u_{\alpha/2}$, the length of the confidence interval Δ in (2.1) is obtained automatically by (2.2). However, it could make sense to reduce a confidence coefficient if it will improve the precision of estimation. In practice the values less than one σ are not used, so in this paper we consider the values of u from 1 to 3.

Precision, or accuracy of interval estimation (or forecasting) depends both on $u_{\alpha/2}$ and σ values that in their product yield the length Δ in (2.2). Let us consider some measures of accuracy for interval estimates, taking into account the above described features of confidence probability intervals.

As a measure of accuracty with which the interval identifies the value of estimated parameter y, we can introduce the ratio of the lower and upper confidence limits assuming that the point estimate $\hat{y}$ has a positive value (otherwise we use its absolute value):

$$\frac{\hat{y}_-}{\hat{y}_+} = \frac{\hat{y} - \Delta}{\hat{y} + \Delta} = \frac{\hat{y} - \sigma u}{\hat{y} + \sigma u} = \frac{1 - \epsilon u}{1 + \epsilon u}, \tag{3.1}$$

where here and henceforth u is used to denote $u_\alpha/2$, and ϵ denotes a parameter of precision

$$\epsilon = \frac{\sigma}{|\hat{y}|}, \tag{3.2}$$

i.e., ϵ is a positive value of the relative deviation σ in units of $\hat{y}$. If $\Delta \to 0$ ($\hat{y} \gg \sigma$) then the index (3.1) goes to $+1$; if $\Delta \gg \hat{y}$ ($\sigma \gg \hat{y}$) then this index goes to -1, that corresponds to highest and the lowest accuracy levels in the interval estimates, respectively.

More convenient for using is the range $[0, 1]$ of an accuracy index constructed from (3.1) as follows:

$$A \equiv \frac{1}{2}\Big(\frac{\hat{y}_-}{\hat{y}_+} + 1\Big) = \frac{\hat{y}}{\hat{y}_+} = \frac{\hat{y}}{\hat{y} + \sigma u} = \frac{1}{1 + \epsilon u}, \tag{3.3}$$

with ϵ defined in (3.2). If $\epsilon \to 0$ ($\hat{y} \gg \sigma$), then $A \to 1$ and if $\epsilon \to \infty$ ($\hat{y} \ll \sigma$), then $A \to 0$, i.e., we have the highest or the lowest accuracy level, respectively. For arbitrary positive or negative values of $\hat{y}$, it is more correct to define accuracy index (3.3) as

$$A \equiv \frac{|\hat{y}|}{|\hat{y}| + \Delta} = \frac{1}{1 + \epsilon u} \tag{3.4}$$

with ϵ defined in (3.2). Measuring the accuracy of an interval estimate by the index (3.3) or (3.4), we compare the mean interval value $\hat{y}$ with the maximal interval value $\hat{y}_+$, thus we do not overestimate the accuracy. Rather we evaluate the bottom level of accuracty (when all the realization of a random experiment would occurr near the upper confidence limit $\hat{y}_+$).

4. NORMAL DISTRIBUTION: FEASIBLE SOLUTIONS FOR PROBABILITY AND ACCURACY

Now we have cumulative probability function $p(u)$ (2.6) and accuracy function $A(u)$ (3.4). The former is growing monotonically from 0 to 1, and the latter is decreasing monotonically (as a hyperbolic curve) from 1 to 0, when the argument u changes from 0 to ∞.

Mutual simultaneous changing of $p(u)$ and $A(u)$ and choosing their "optimal" in some sense values could be considered as a cooperative game of a duopolistic kind (see [4], [7], [8]). In a probability-accuracty duopoly, each point of their values occupies a particular position in the two-dimensional space of these two attributes. Maximizing utility corresponds to elaborating a feasible balanced solution of a trade-off equilibrium.

Both $p(u)$ and $A(u)$ change from 0 to 100% although in opposite directions, but these percentages are of different gauges: percents of probability do not coincide with percents of relative deviations. Thus we can use an expression $kA(u)$ where k transforms the units of A into the same units as p so that the two can be compared.

An additive utility function is

$$v(u) = \alpha k A(u) + (1 - \alpha) p(u), \tag{4.1}$$

with $\alpha = 0.5$ due to bound conditions: if $A(u) = 0$, then $p(u) = 1$ and if $A(u) = 1$, then $p(u) = 0$. However, it is not convenient to consider the linear from (4.1) with unknown coefficient of transformation k. Thus we will use a multiplicative utility function that describes a general quality of a prediction:

$$v(u) = kA(u)p(u) \tag{4.2}$$

with A and p defined in (3.4) and (2.6) respectively.

The product $v(u)$ of the increasing $p(u)$ and the decreasing $A(u)$ functions has a maximum, the area around which localizes a range of the feasible solutions u^* that

yield both high probability $p(u^*)$ and accuracy $A(u^*)$ levels, i.e., the best general quality of the prediction.

Numerical values of these functions $p(u)$, $A(u)$, and $v(u)$ for different values of u and for several values of the precision parameter ϵ (3.2) are presented in Table 1.

The columns in Table 1 correspond to ϵ from 2% to 200% (when standard deviation equals to 0.02 part of the point estimate $\hat{y}$, or is twice more than $\hat{y}$, respectively). Each column with utility function $v(u)$ (4.2) shows that its value contains a point of a maximum $v_{max} = v(u^*)$, and this point u^* defines the corresponding values of probability $p(u^*)$ and accuracy $A(u^*)$ functions for each given value of the parameter ϵ (in Table 1 all these connected values are denoted by asterics in a row).

For small values of ϵ, e.g., 2%, 5%, or even 10%, the utility function reaches its maximum approximately at the points $u^* = 2.7$, 2.4, and 2.1, respectively, yielding both high probability confidence of $p(u^*) = 99.3\%, 98.4\%$, or 96.4%, and high levels of accuracy $A(u^*) = 94.9\%, 89.3\%$, and 82.6%, respectively. Intermediate values of the parameter $\epsilon = 25\%$ and 50% produce maximal $v(u^*)$ values at the points $u^* = 1.8$ and 1.5 with corresponding probability $p(u^*) = 92.8\%$ and 86.6%, and levels of accuracy $A(u^*) = 69\%$ and 57.1%. If for these cases we take higher probability values (i.e., higher u levels) it could influence the accuracy levels reducing them by several percents.

With big values of $\epsilon = 100\%$ or 200%, the maximal $v(u^*)$ are located at $u^* = 1.3$ or 1.1. That gives probability $p(u^*) = 80.6\%$ or 72.9%, and accuracy $A(u^*) = 43.5\%$ or 31.3%, respectively. In these cases, if confidence proobability is chosen at $u = 2$ (or 3), i.e., $p = 95.5\%$ (or 99.7%), it could drastically reduce the accuracy from 43.5% to 33.3% (or to 25%) in the case of $\epsilon = 100\%$, or from 31.3% to 20% (or to 14.3%) in the case of $\epsilon = 200\%$).

Thus if standard deviation σ is comparable with the point estimate $\hat{y}$ (i.e., for ϵ about or above 50%), the conventional choosing of confidence probability of about 95% or 99% produces too widely spreaded confidence interval limits, that corresponds to too low accuracy levels of the prediction. For these cases it is especially important to use reasonable lower values of probability compensated by higher levels of accuracy. Table 1 gives a possibility to choosing these balanced values of confidence probability and accuracy. A researcher could decide to choose not exactly "optimal" (asterics denoted) values, but some close to them and higher values of $p(u)$ if it corresponds to the accuracy levels in an admitable range.

5. t-DISTRIBUTION: FEASIBLE SOLUTIONS FOR PROBABILITY AND ACCURACY

The results presented in Figure 1 and Table 1 are obtained for normal distribution (when in (2.1) - (2.3) a standard deviation is known). The same kind of consideration could be performed for the t-distribution (when σ is estimated by sample data). However, another possibility to evaluate u^* and the functions $p(u^*)$ and $A(u^*)$ exists that makes all the evaluations much easier for normal and for t-distribution as well.

Table 1: Cumulative normal distribution $p(u)$, Accuracy $A(u)$, and Utility $v(u)$—all in %

		$\epsilon=2$		$\epsilon=5$		$\epsilon=10$		$\epsilon=25$		$\epsilon=50$		$\epsilon=100$		$\epsilon=200$	
u	$p(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$	$A(u)$	$v(u)$
1.0	68.3	98.0	66.9	95.2	65.0	90.9	62.1	80.0	54.6	66.7	45.5	50.0	34.1	33.3	22.7
*1.1	*72.9	97.8	71.3	94.8	69.1	90.1	65.6	78.4	57.2	64.5	47.0	47.6	34.7	*31.3	*22.8
1.2	77.0	97.7	75.2	94.3	72.6	89.3	68.7	76.9	59.2	62.5	48.1	45.5	35.0	29.4	22.6
*1.3	*80.6	97.5	78.6	93.9	75.7	88.5	71.4	75.5	60.9	60.6	48.9	*43.5	*35.1	27.8	22.4
1.4	83.8	97.3	81.6	93.5	78.4	87.7	73.6	74.1	62.1	58.8	49.3	41.7	34.9	26.3	22.1
*1.5	*86.6	97.1	84.1	93.0	80.6	87.0	75.3	72.7	63.0	*57.1	*49.5	40.0	34.7	25.0	21.7
1.6	89.0	96.9	86.3	92.6	82.4	86.2	76.8	71.4	63.6	55.6	49.4	38.5	34.2	23.8	21.2
1.7	91.1	96.7	88.1	92.2	84.0	85.5	77.9	70.2	63.9	54.1	49.2	37.0	33.7	22.7	20.7
*1.8	*92.8	96.5	89.6	91.7	85.2	84.7	78.7	*69.0	*64.0	52.6	48.8	35.7	33.1	21.7	20.2
1.9	94.3	96.3	90.8	91.3	86.1	84.0	79.2	67.8	63.9	51.3	48.3	34.5	32.5	20.8	19.6
2.0	95.5	96.2	91.8	90.9	86.8	83.3	79.5	66.7	63.6	50.0	47.7	33.3	31.8	20.0	19.1
*2.1	*96.4	96.0	92.5	90.5	87.3	*82.6	*79.7	65.6	63.2	48.8	47.0	32.3	31.1	19.2	18.5
2.2	97.2	95.8	93.1	90.1	87.6	82.0	79.7	64.5	62.7	47.6	46.3	31.3	30.4	18.5	18.0
2.3	97.9	95.6	93.6	89.7	87.8	81.3	79.6	63.5	62.1	46.5	45.5	30.3	29.7	17.9	17.5
*2.4	*98.4	95.4	93.9	*89.3	*87.8	80.6	79.3	62.5	61.5	45.5	44.7	29.4	28.9	17.2	17.0
2.5	98.8	95.2	94.1	88.9	87.8	80.0	79.0	61.5	60.8	44.4	43.9	28.6	28.2	16.7	16.5
2.6	99.1	95.1	94.2	88.5	87.7	79.4	78.6	60.6	60.0	43.5	43.1	27.8	27.5	16.1	16.0
*2.7	*99.3	*94.9	*94.2	88.1	87.5	78.7	78.2	59.7	59.3	42.6	42.3	27.0	26.8	15.6	15.5
2.8	99.5	94.7	94.2	87.7	87.3	78.1	77.7	58.8	58.5	41.7	41.5	26.3	26.2	15.2	15.1
2.9	99.6	94.5	94.2	87.3	87.0	77.5	77.2	58.0	57.8	40.8	40.7	25.6	25.5	14.7	14.7
3.0	99.7	94.3	94.1	87.0	86.7	76.9	76.7	57.1	57.0	40.0	39.9	25.0	24.9	14.3	14.2

Table 2. Normal Distribution: Estimation of $\epsilon(u^*)$, $p(u^*)$, and $A(u^*)$

u^*	$p(u^*)\%$	$\epsilon(u^*)\%$	$A(u^*)\%$
1.0	68.3	243.5	29.1
1.1	72.9	174.7	34.2
1.2	77.0	127.8	39.5
1.3	80.6	95.0	44.7
1.4	83.8	71.4	50.0
1.5	86.6	54.2	55.2
1.6	89.0	41.4	60.1
1.7	91.1	31.8	64.9
1.8	92.8	24.5	69.4
1.9	94.3	18.9	73.5
2.0	95.5	14.6	77.4
2.1	96.4	11.3	80.8
2.2	97.2	8.7	83.9
2.3	97.9	6.7	86.7
2.4	98.4	5.1	89.1
2.5	98.8	3.9	91.1
2.6	99.1	3.0	92.9
2.7	99.3	2.2	94.3
2.8	99.5	1.7	95.5
2.9	99.6	1.2	96.5
3.0	99.7	0.9	97.3

Consider the function (4.2) in explicit form:

$$v(u) = k\frac{1}{1+\epsilon u}p(u), \tag{5.1}$$

where the accuracy function is defined in (3.3) or (3.4), and $p(u)$ is defined in (2.6). For a given parameter ϵ the maxixmal value of $v(u)$ can be found from the condition:

$$v' \equiv \frac{dv(u)}{du} = k\frac{p'(u)(1+\epsilon u) - \epsilon p(u)}{(1+\epsilon u)^2} = 0. \tag{5.2}$$

The function $v(u)$ reaches its maximum at the point u^* that ccould be found from (5.2) as a numerical solution of the equation

$$\frac{p(u)}{p'(u)} - u = \frac{1}{\epsilon}, \tag{5.3}$$

where $p(u)$ is the function (2.6), and its derivative $p'(u)$ is defined in (2.7).

It is easier even not to solve (5.3) for u^* with a given value of ϵ, but to calculate $\epsilon = \epsilon(u^*)$ for different values of $u = u^*$ in the left-hand-side of (5.3). The results of

the evaluation for different values of u^* of normal distribution, together with $p(u^*)$ and $A(u^*)$ are presented in Table 2. They correspond to the data from Table 1.

Table 3 contains similar results obtained by solving (5.3) for ϵ when the t-distribution is used. Table 3 consists of sections for several values of degrees of freedom ($d.f.$) for the functions $p(u^*)$, $\epsilon(u^*)$, and $A(u^*)$. Table 2 could be considered a continuation of Table 3 for $d.f.$ more than 100. Using one of the values u^*, $p(u^*)$, $\epsilon(u^*)$, $A(u^*)$ as a chosen parameter, all three others values of optimal balanced solution could be obtained from Table 3 (with some interpolation for other values of these four variables and of $d.f.$.)

It is interesting to note that increasing $d.f.$ from 1 to ∞ shifts the 30% range of $p(u^*)$ from [50.0, 79.5]% to [68.3, 99.7]% and makes $\epsilon(u^*)$ and $A(u^*)$ more steeply changing functions of u^*. In general, $\epsilon(u^*)$ has values from more than 200% to less than 1% (all useful possible values of the parameter (3.2) are in this range), probability $p(u^*)$ is in the range from 50% to 99.7%, and level of accuracy is in the range from 30% to 97%. Desired values of both the confidence probability $p(u^*)$ and accuracy of the interval prediction $A(u^*)$ could be chosen simultaneously from the columns for $p(u^*)$ and $A(u^*)$ of Tables 2 and 3. Other parameters u^* and $\epsilon(u^*)$ will be defined from the same row (for the given $d.f.$ or with interpolation).

6. REGRESSION MODELING FOR OPTIMAL PERCENTILES

Results in Tables 2 and 3 reveal some interesting relations between the variables. If we consider the graphs that represent dependence of optimal values from Table 2 for u^*, $p^* = p(u^*)$ and $A^* = A(u^*)$ as functions of the parameter of precision ϵ (3.2) (ϵ values are given in the logarithm scale), then we realize that both p^* and A^* decrease for bigger ϵ, and A^* decreases more steeply. The u^* values are amazingly close to a straight decreasing line in dependence on $\ln(\epsilon)$, i.e.,

$$u^* = a + b \ln \epsilon \tag{6.1}$$

The same kind of a straight line corresponds to the data for u^* and logarithm of ϵ^* from Table 3 for the t-distribution for each number of degrees of freedom.

Regressions of u^* by $\ln \epsilon$ (considering ϵ as the independent parameter) are presented in Table 4. They are constructed by data for u^* and ϵ^* from Tables 2 and 3. The quality of these regressions is very high: t-statistics for coefficients are from dozens to hundreds, adjusted coefficients of determination are from 97% to 99%, F-statistics have values from hundreds to thousands. For a given value of ϵ we can estimate u^* by (6.1) and obtain probability $p(u^*)$ and accuracy $A(u^*)$ values by Tables 2 and 3 or by formulae (2.6) and (3.4).

Residual standard error $\hat{\sigma}$ (the last column in Table 4) is very small for all the regressions and being reduced by u^* values (they are in the range from 1 to 3 in Tables 2 and 3) it yields the precision parameter ϵ from 10% to 1% and less. It means that the accuracty of the forecasts by the model (6.1) from Table 4 are themselves very high.

Table 3: t-distribution: Estimation of $p(u^*)$, $\epsilon(u^*)$, and $A(u^*)$—all in %

	$d.f. = 1$			$d.f. = 5$			$d.f. = 10$			$d.f. = 20$			$d.f. = 50$		
u^*	$p(u^*)$	$\epsilon(u^*)$	$A(u^*)$	$p(u^*)$	$\epsilon(u^*)$	$A(u^*)$	$p(u^*)$	$\epsilon(u^*)$	$A(u^*)$	$p(u^*)$	$\epsilon(u^*)$	$A(u^*)$	$p(u^*)$	$\epsilon(u^*)$	$A(u^*)$
1.0	50.0	175.2	36.3	63.7	222.5	31.0	65.9	232.2	30.1	67.1	237.6	29.6	67.8	241.1	29.3
1.1	53.0	135.0	40.2	67.9	163.3	35.8	70.3	168.7	35.0	71.6	171.6	34.6	72.3	173.4	34.4
1.2	55.8	106.7	43.9	71.6	122.6	40.6	74.2	125.2	40.0	75.6	126.5	39.7	76.4	127.3	39.6
1.3	58.3	86.1	47.2	75.0	93.9	45.0	77.7	94.6	44.9	79.2	94.8	44.8	80.0	94.9	44.8
1.4	60.5	70.7	50.2	78.0	73.0	49.5	80.8	72.5	49.6	82.3	72.0	49.8	83.2	71.7	49.9
1.5	62.6	59.0	53.0	80.6	57.6	53.7	83.5	56.2	54.2	85.1	55.3	54.6	86.0	54.7	54.9
1.6	64.4	49.9	55.6	83.0	45.9	57.6	85.9	44.1	58.6	87.5	42.9	59.3	88.4	42.0	59.8
1.7	66.1	42.7	57.9	85.0	37.0	61.4	88.0	34.9	62.8	89.5	33.5	63.7	90.5	32.5	64.4
1.8	67.7	36.9	60.1	86.8	30.1	64.8	89.8	27.8	66.7	91.3	26.3	67.9	92.2	25.6	68.7
1.9	69.2	32.2	62.1	88.4	24.7	68.0	91.3	22.3	70.3	92.8	20.7	71.8	93.7	19.7	72.8
2.0	70.5	28.3	63.9	89.8	20.4	71.0	92.7	17.9	73.6	94.1	16.4	75.3	94.9	15.4	76.5
2.1	71.7	25.0	65.5	91.0	17.0	73.7	93.8	14.5	76.6	95.1	13.0	78.5	95.9	12.0	79.9
2.2	72.8	22.3	67.1	92.1	14.2	76.2	94.8	11.8	79.4	96.0	10.4	81.5	96.8	9.4	82.9
2.3	73.9	20.0	68.5	93.0	11.9	78.5	95.6	9.6	81.9	96.8	8.2	84.1	97.4	7.3	85.6
2.4	74.9	18.0	69.8	93.8	10.1	80.5	96.3	7.9	84.1	97.4	6.6	86.4	98.0	5.7	87.9
2.5	75.8	16.3	71.0	94.6	8.6	82.4	96.9	6.5	86.1	97.9	5.2	88.4	98.4	4.4	90.0
2.6	76.6	14.8	72.2	95.2	7.3	84.1	97.4	5.3	87.9	98.3	4.2	90.2	98.8	3.5	91.8
2.7	77.4	13.5	73.2	95.7	6.2	85.6	97.8	4.4	89.4	98.6	3.3	91.7	99.1	2.7	93.3
2.8	78.2	12.4	74.2	96.2	5.4	87.0	98.1	3.6	90.8	98.9	2.7	93.1	99.3	2.1	94.5
2.9	78.9	11.4	75.1	96.6	4.6	88.2	98.4	3.0	92.0	99.1	2.1	94.2	99.4	1.6	95.6
3.0	79.4	10.5	76.0	97.0	4.0	89.3	98.7	2.5	93.1	99.3	1.7	95.2	99.6	1.2	96.5

Table 4. Regressions $u^* = a + b\ln(\epsilon)$ for t and Normal Distributions

$d.f.$	a	t_a	b	t_b	$R^2_{adj}\%$	F	σ
1	1.195	30.8	-0.721	25.8	97.1	666.8	0.106
5	1.265	55.3	-0.504	41.5	98.9	1724.6	0.066
10	1.274	81.4	-0.448	60.3	99.5	3640.3	0.046
20	1.281	126.0	-0.412	92.4	99.8	8540.9	0.030
50	1.287	203.6	-0.387	147.9	99.9	21863.6	0.020
∞	1.293	267.2	-0.368	192.3	99.9	36991.2	0.014

A very high quality of all the regressions in Table 4 could be understood by the following consideration. Cumulative functions of a normal or t-distribution behave very similarly with a cumulative logistic distribution function

$$F(u) = \frac{1}{1 + \alpha e^{-\beta u}}, \tag{6.2}$$

where α and β are some positive constants. Then our symetric function (2.6) is

$$p(u) = \frac{2}{1 + \alpha e^{-\beta u}} - 1, \tag{6.3}$$

and $A(u)$ is defined by (3.4). The first-order condition for maximum of a general quality of prediction (4.2) yields the following relation between ϵ and u:

$$\epsilon = \frac{2\alpha\beta}{e^{\beta u}(1 - \alpha^2 e^{-2\beta u} - 2\alpha\beta u e^{-\beta u})} \approx \frac{2\alpha\beta}{e^{\beta u}}. \tag{6.4}$$

Using this approximation we obtain the following after taking logarithms:

$$u = \frac{\ln(2\alpha\beta)}{\beta} - \frac{1}{\beta}\ln\epsilon, \tag{6.5}$$

i.e., exactly the model (6.1) for linear dependence of the optimal point $u = u^*$ on the logarithm of precision parameter ϵ.

7. A ROUGH ESTIMATION BY CHEBYSHEV'S INEQUALITY

Another way for estimation of balanced confidence probability and accuracy values independent on the kind of distribution of the random variable u could be found with the help of Chebyshev's Inequality (see [3] and [5]). In our notation (2.1) - (2.5) and for our purpose of an interval estimate this inequality implies that the fraction of measurements in the sample lying in the closed interval exceeds a particular value:

$$P[\hat{y} - u\sigma \le y \le \hat{y} + u\sigma] \ge 1 - \frac{1}{u^2} \tag{7.1}$$

where we would take u to be greater than 1 to have an inequality of interest. Using accurracy index (3.4) and cumulative probability of the right-had-side of (7.1), we can recalculate the above obtained results for an arbitrary probabilty function. However, these results would show a very rough estimate for the balanced probability and accuracy because Chebyshev's Inequality evaluates only the lower bound of a confidence probability.

We can use the inequality (7.1) to obtain simple estimates that are independent even of the parameter of precision ϵ. For this aim, we can use instead of (3.3) and (3.4) another index of the accuracy:

$$A = \frac{1}{u}. \tag{7.2}$$

Because of the range $u \geq 1$ for the inequality (7.1), the accuracy index (7.2) belongs to the interval $A \in [0, 1]$. The highest accuracty $A = 1$ corresponds to the minimal value $u = 1$, and $A \to 0$ when u increases.

The lower bound for the confidence probability, due to (7.1), is as follows:

$$p(u) = 1 - \frac{1}{u^2}. \tag{7.3}$$

The function of prediction quality (4.2) is now the product of (7.2) and (7.3), i.e.,

$$v(u) = \frac{1}{u} - \frac{1}{u^3} \tag{7.4}$$

Numerical values of these functions are given in Table 5, where the maximal $v(u)$ and all the values of variable in the same row are noted by asterics. More precisely, from the first order condition $\frac{dv}{du} = 0$ we can obtain the value $u^* = \sqrt{3} = 1.73$ for the numerical value of the function (7.4). The lower bound of the confidence probability is estimatedd as 65.4% (see Table 5 for an arbitrary distribution, and the corresponding accuracy value is 58.8%. However, in practical estimations the kind of distribution is usually known, thus, the more precise results of Tables 1, 2, and 3 could be used.

8. FURTHER EXTENSIONS OF PROBABILITY-ACCURACY APPROACH

The above described approach of simultaneous evaluation for probability confidence and accuracy of interval estimates could be extended in different other directions. For example, we considered two-sided estimates that correspond to two-tail tests. The same way we can consider a one-tail test, using α instead of $\alpha/2$ percentiles in (2.2) - (2.5), and working with original probability functions $F(u)$ and $f(u)$ without their transformation to the functions (2.6) - (2.7). A definition of the accuracy could be kept in the form of (3.4).

Table 5. Functions of $p(u^*)$, $A(u^*)$, and $v(u^*)$ for Chebyshev's Inequality

u^*	$p(u^*)\%$	$A(u^*)\%$	$v(u^*)\%$
1.0	0	100	0
1.1	17.4	90.9	15.8
1.2	30.6	83.3	25.5
1.3	40.8	76.9	31.4
1.4	49.0	71.4	35.0
1.5	55.6	66.7	37.0
1.6	60.9	62.5	38.1
*1.7	*65.4	*58.8	*38.5
1.8	69.1	55.6	38.4
1.9	72.3	52.6	38.1
2.0	75.0	50.0	37.5
2.1	77.3	47.6	36.8
2.2	79.3	45.5	36.1
2.3	81.1	43.5	35.3
2.4	82.6	41.7	34.4
2.5	84.0	40.0	33.6
2.6	85.2	38.5	32.8
2.7	86.3	37.0	32.0
2.8	87.2	35.7	31.2
2.9	88.1	34.5	30.4
3.0	88.9	33.3	29.6

Another interesting extension of our approach could be seen in its application to other statistical distributions used in interval estimations. For example, it could be applied for Chi-square and for F-statistics as well. Chi-square or F-statistics confidence intervals (for variance or for ratio of two variances, respectively) have known structure of multiplicative kind, and after logarithmic transformation, they could be represented in the linear form similar to (2.1) - (2.5). Then confidence probability and accuracy estimations can be elaborated for logarithms of the estimates.

9. SUMMARY

This paper describes the use of confidence probability together with accuracy of the forecast in interval estimates. The consideration was performed for normal and t-distributions, although it is possible to apply the suggested methodology to other statistics used for interval predictions. Balanced solutions for probability and accuracy are obtained in an optimizing approach of maximization of utility (quality of prediction) function of the multiplicative form. Tables of the optimal values for percentiles together with corresponding values of confidence probability, parameters of precision, and accuracy of interval estimates are constructed. Regressions are ob-

tained for more convenient estimation of optimal balanced values for charactersitcis of interval predictions.

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