International Journal of Reliability and Applications
Vol. 3, No. 1, pp. 1-16, 2002

Invited Paper

A Vtub-Shaped Hazard Rate Function with Applications to System Safety

Hoang Pham*
Department of Industrial Engineering
Rutgers University, Piscataway, New Jersey 08854, USA

Abstract. In reliability engineering, the bathtub-shaped hazard rates play an important role in survival analysis and many other applications as well. For the bathtub-shaped, initially the hazard rate decreases from a relatively high value due to manufacturing defects or infant mortality to a relatively stable middle useful life value and then slowly increases with the onset of old age or wear out.

In this paper, we present a new two-parameter lifetime distribution function, called the Loglog distribution, with Vtub-shaped hazard rate function. We illustrate the usefulness of the new Vtub-shaped hazard rate function by evaluating the reliability of several helicopter parts based on the data obtained in the maintenance malfunction information reporting system database collected from October 1995 to September 1999. We develop the S-Plus add-in software tool, called Reliability and Safety Assessment (RSA), to calculate reliability measures include mean time to failure, mean residual function, and confidence intervals of the two helicopter critical parts. We use the mean squared error to compare relative goodness of fit test of the distribution models include normal, lognormal, and Weibull within the two data sets. This research indicates that the result of the new Vtub-shaped hazard rate function is worth the extra function-complexity for a better relative fit. More application in broader validation of this conclusion is needed using other data sets for reliability modeling in a general industrial setting.

Key Words : *hazard rate, Vtub-shaped failure rate, component reliability function, loglog distribution, bathtub-shaped failure rate.*

1. INTRODUCTION

In today's technological world nearly everyone depends upon the continued functioning of a wide array of complex machinery and equipment for their everyday safety, mobility and economic welfare. We expect our electric appliances, lights, safety-critical parts, hospital monitoring control room, etc. to function whenever we need them. When they fail the results can be catastrophic; injury or loss of life. Reliability is the probability that a

* *E-mail address : hopham@rci.rutgers.edu*

product or part will operate properly for a specified period of time under the design operating conditions such as temperature, volt, humidity, stress level, etc., without failure.

In reliability engineering, the bathtub-shaped hazard rates play an important role in survival analysis and many other biological applications. For the bathtub-shaped, initially the hazard rate decreases from a relatively high value due to manufacturing defects or infant mortality to a relatively stable middle useful life value and then slowly increases with the onset of old age or wear out. This behavior can be observed in human life, electronic system products, and many others.

In the last two decades, many authors [Mann, Schafer and Singpurwalla,1974; Smith and Bain,1975; Gaver and Acar,1979; Mudholkar and Srivastava,1993; Block & Joe,1997; Wang, Muller & Capra,1998; Aalen & Gjessing,2001] have been proposed lifetime distributions with various bathtub-shaped failure rates. For example, Mann, Schafer and Singpurwalla [1974] proposed mixtures of Weibull distributions; Smith and Bain [1975] proposed the exponential power distribution; Gaver and Acar [1979] presented a four-parameter bathtub-shaped failure rate function; Mudholkar and Srivastava [1993] proposed a simple generalization of the Weibull family called the exponentiated-Weibull family with three parameters. Such exponentiated-Weibull distribution is well suited for modeling bathtub shaped failure rate lifetime data. Aalen & Gjessing [2001] recently discussed the shaped of various hazard rates including bathtub-shaped hazard rate.

In this paper, we present a new lifetime distribution function with two parameters, called the Loglog distribution. The corresponding hazard rate of the new distribution, called the Vtub-shaped hazard rate, not only includes distributions with bathtub, increasing and decreasing failure rates, but also provides a broader class of monotone failure rates.
The Vtub-shaped hazard rate is defined as: If there exists a change point t_0 such that the hazard rate h(t) is decreasing in $[0, t_0]$ and slowly increasing, as a Vtub-shaped, in $[t_0, \infty)$.

We illustrate the usefulness of the new Vtub-shaped hazard rate function by evaluating the reliability of several helicopter parts based on the data obtained in the Maintenance Malfunction Information Reporting (MMIR) system database collected from October 1995 to September 1999. We also develop the S-Plus add-in software tool, called Reliability and Safety Assessment (RSA), to calculate reliability, mean time to failure, failure rate function, mean residual life, and confidence intervals of the two helicopter main rotor blade and rotor blade assembly parts.

2. RELIABILITY MEASURES

Acronyms

MTTF	Mean time to failure
MMIR	Maintenance Malfunction Information Reporting
FAA	Federal Aviation Administration

2.1 Vtub-Shaped Hazard Rate

We present a new two-parameter lifetime distribution, called the Loglog distribution, with parameters a and α. Let t_1, t_2 ,..., t_n be failure times of a random variable having a density function as follows:

$$f(t) = \alpha \cdot \ln a \cdot t^{\alpha-1} \cdot a^{t^\alpha} \cdot e^{1-a^{t^\alpha}} \quad \text{for } t > 0,\ \alpha > 0,\ a > 1 \tag{1}$$

Then the loglog distribution and reliability functions are

$$F(t) = \int_0^t f(x)dx = 1 - e^{1-a^{t^\alpha}}$$

and

$$R(t) = e^{1-a^{t^\alpha}} \tag{2}$$

respectively. The corresponding hazard rate of the new distribution, called the Vtub-shaped hazard rate, is

$$h(t) = \alpha \cdot \ln a \cdot t^{\alpha-1} \cdot a^{t^\alpha} \tag{3}$$

Figures 1 and 2 (both a and b) describe the density function and hazard rate function for various values of a and α.

The Vtub-shaped and bathtub-shaped hazard rates are not the same. As for the bathtub-shaped, for example, after the infant mortality period, the useful life of the system begins. During its useful life, the system fails as a constant rate. This period is then followed by a wear out period during which the system starts slowly increases with the on set of wear out. But for the Vtub-shaped, after the infant mortality period, the system starts to experience at a relatively low increasing rate, but not constant, and then increasingly more failures due to aging.

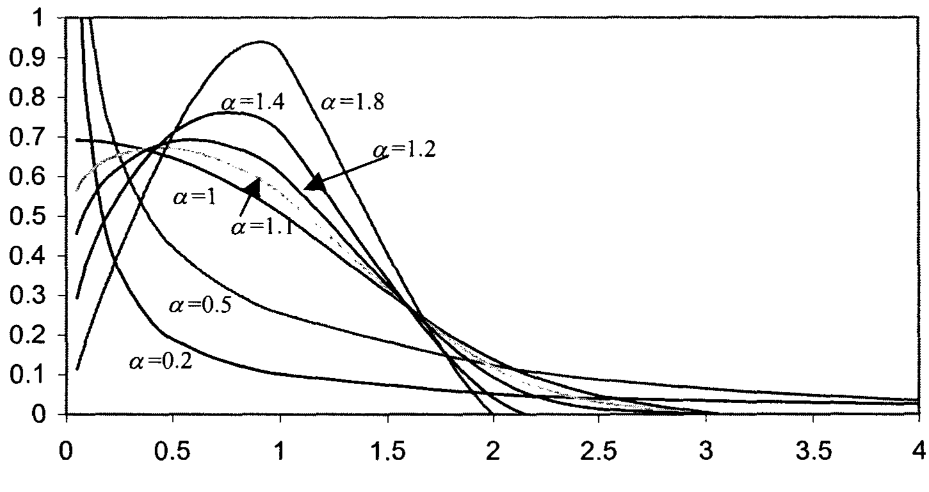

Figure 1. (a) Probability density function with a=2

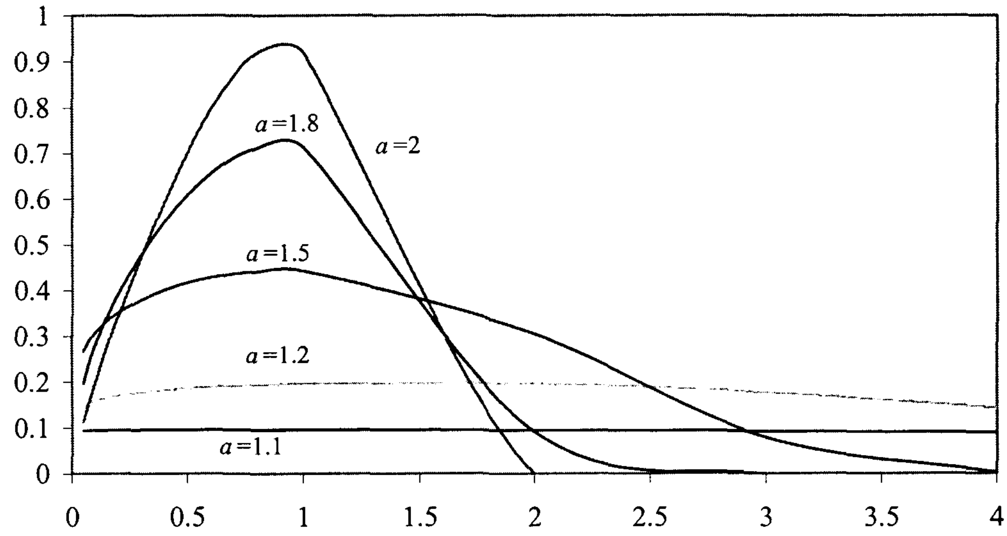

Figure 1. (b) Probability density function with $\alpha = 1.5$

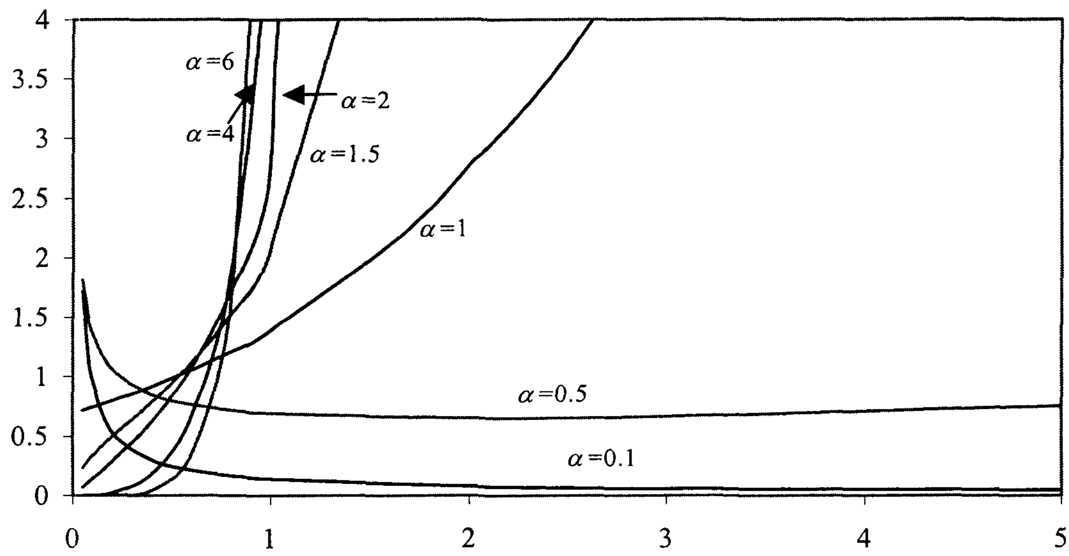

Figure 2. (a) Hazard rate function with a=2

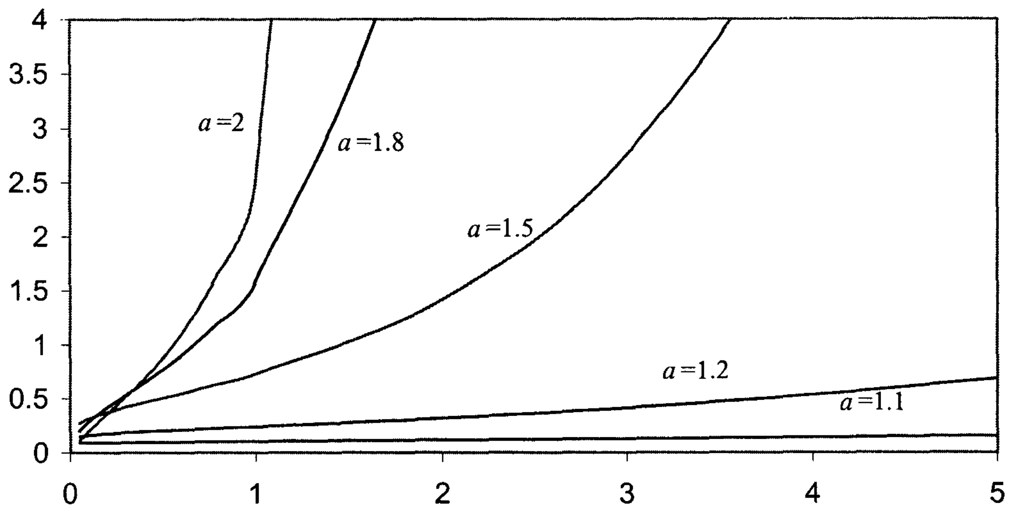

Figure 2. (b) Hazard rate function with $\alpha = 1.5$

Note that the distribution F is an increasing failure rate (IFR) if the hazard rate *h*(*t*) is an increasing function of t. Similarly, F is a decreasing failure rate (DFR) if *h*(*t*) is a decreasing function of t.

Definition: The distribution F has a Vtub-shaped failure rate if there exists a change point t_0 such that the distribution F is DFR for $t \le t_0$ and IFR for $t \ge t_0$.

Result: For any given a and α, the loglog distribution F is DFR for $t \le t_0$ and IFR for $t \ge t_0$, where

$$t_0 = \left(\frac{1-\alpha}{\alpha \ln a} \right)^{1/\alpha} \tag{4}$$

Proof: From equation (3), upon differentiation, we obtain

$$h'(t) = (\alpha \ln a)\left(a^{t^\alpha} t^{\alpha-2}\right)\left[(\alpha - 1) + \alpha \ln a \cdot t^\alpha\right]$$

Set h'(t)=0 and after simplifications, we obtain

$$t = \left(\frac{1-\alpha}{\alpha \ln a} \right)^{1/\alpha} \equiv t_0$$

It easily follows that the sign of *h*'(*t*) is determined by $(\alpha - 1) + \alpha \ln a \cdot t^\alpha$ which is negative for all $t \le t_0$ and positive for all $t \ge t_0$. Therefore, *h*(*t*) is initially decreasing and then increasing in t.

It is easy to see that when $\alpha \geq 1$ the distribution F is an IFR.

2.2 The Mean Residual Life

The mean residual life function is the expected remaining life, *T–t*, given that the item has survived to time *t*. Mathematically, the mean residual life function, MRL(t), is given by

$$MRL(t) = E[T - t \mid T \geq t], \qquad t \geq 0 \tag{5}$$

The above equation can be rewritten in terms of hazard rate as follows

$$MRL(t) = \frac{\int_t^\infty e^{-\int_0^x h(t)dt} dx}{e^{-\int_0^t h(x)dx}}, \qquad t \geq 0$$

From equation (3),

$$MRL(t) = \frac{\int_t^\infty e^{-\int_0^x \alpha \cdot \ln a \cdot t^{\alpha-1} \cdot a^{t^\alpha} dt} dx}{e^{-\int_0^t \alpha \cdot \ln a \cdot x^{\alpha-1} \cdot a^{x^\alpha} dx}}, \qquad t \geq 0$$

After simplifications, we obtain

$$MRL(t) = e^{\left(a^{t^\alpha} - 1\right)} \int_t^\infty e^{\left(1 - a^{x^\alpha}\right)} dx, \qquad t \geq 0$$

When t=0, the mean residual life becomes the mean time to failure (MTTF), that is,

$$MRL(0) = \int_0^\infty e^{1 - a^{t^\alpha}} dt \equiv MTTF$$

3. PARAMETER ESTIMATION

We now wish to estimate the values of a and α using the maximum likelihood estimation (MLE) method. From equation (1), the likelihood function is

$$L(a,\alpha) = \prod_{i=1}^n \alpha \ln a \cdot t_i^{\alpha-1} e^{1-a^{t_i^\alpha}} a^{t_i^\alpha}$$

$$= \alpha^n (\ln a)^n \left(\prod_{i=1}^n t_i \right)^{\alpha-1} a^{\sum_{i=1}^n t_i^\alpha} e^{n - \sum_{i=1}^n a^{t_i^\alpha}}$$

The log likelihood function is

$$\log L(a,\alpha) = n \log \alpha + n \ln(\ln a) + (\alpha - 1)\left(\sum_{i=1}^n \ln t_i \right) + \ln a \cdot \sum_{i=1}^n t_i^\alpha + n - \sum_{i=1}^n a^{t_i^\alpha} \tag{6}$$

The first derivatives of the log likelihood function with respect to a and α are, respectively,

$$\frac{\partial}{\partial a}\log L(a,\alpha)=\frac{n}{a\ln a}+\frac{1}{a}\cdot\sum_{i=1}^{n}t_i^{\alpha}-\sum_{i=1}^{n}t_i^{\alpha}a^{t_i^{\alpha}-1} \tag{7}$$

and

$$\frac{\partial}{\partial \alpha}\log L(a,\alpha)=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln t_i+\ln a\cdot\sum_{i=1}^{n}\ln t_i\cdot t_i^{\alpha}-\sum_{i=1}^{n}t_i^{\alpha}\cdot a^{t_i^{\alpha}}\cdot\ln a\cdot\ln t_i \tag{8}$$

Set the equations (7) and (8) equate to zero, we can obtain the MLE of a and α by solving the following simultaneous equations:

$$\frac{n}{\ln a}+\sum_{i=1}^{n}t_i^{\alpha}-\sum_{i=1}^{n}t_i^{\alpha}a^{t_i^{\alpha}}=0$$

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\ln t_i+\ln a\cdot\sum_{i=1}^{n}\ln t_i\cdot t_i^{\alpha}\left(1-a^{t_i^{\alpha}}\right)=0$$

After rearrangements, we obtain

$$\ln a\sum_{i=1}^{n}t_i^{\alpha}\left(a^{t_i^{\alpha}}-1\right)=n$$

$$\ln a\cdot\sum_{i=1}^{n}\ln t_i\cdot t_i^{\alpha}\cdot\left(a^{t_i^{\alpha}}-1\right)-\frac{n}{\alpha}=\sum_{i=1}^{n}\ln t_i$$

We next determine the confidence intervals for parameter estimates a and α. For the log-likelihood function in (6), we can obtain the Fisher information matrix H as

$$H=\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}$$

where

$$h_{11}=E\left[-\frac{\partial^2\log L}{\partial a^2}\right],\ h_{12}=h_{21}=E\left[-\frac{\partial^2\log L}{\partial a\partial\alpha}\right],\ h_{22}=E\left[-\frac{\partial^2\log L}{\partial\alpha^2}\right].$$

The variance matrix, V, can be obtained as follows

$$V=[H]^{-1}=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix} \tag{9}$$

The variances of a and α are

$$Var(a)=v_{11} \qquad Var(\alpha)=v_{22}$$

One can approximately obtain the (1-β)100% confidence intervals for a and α based on the normal distribution as $[\hat{a}-z_{\beta}\sqrt{v_{11}}\ ,\hat{a}+z_{\beta}\sqrt{v_{11}}\ ]$ and $[\hat{\alpha}-z_{\beta}\sqrt{v_{22}}\ ,\ \hat{\alpha}+z_{\beta}\sqrt{v_{22}}\ ]$, respectively, where v_{ij} is given in (9) and z_β is $(1-\beta/2)100\%$ of the standard normal distribution.

After we obtain $\hat{a}$ and $\hat{\alpha}$, the MLE of reliability function can be computed as

$$\hat{R}(t) = e^{1-\hat{a}^{t^{\hat{\alpha}}}}$$

Let us define a partial derivative vector for reliability $R(t)$ as

$$v[R(t)] = \left[\frac{\partial R(t)}{\partial a}, \frac{\partial R(t)}{\partial \alpha}\right]$$

then the variance of $R(t)$ can be obtained as follows:

$$Var\left[R(t)\right] = v[R(t)] \cdot V \cdot \left(v[R(t)]\right)^T \qquad (10)$$

where V is given in (9).

One can approximately obtain the $(1-\beta)100\%$ confidence interval for $R(t)$ is $[\hat{R}(t) - z_\beta\sqrt{Var\left[R(t)\right]},\ \hat{R}(t) + z_\beta\sqrt{Var\left[R(t)\right]}\ ]$.

4. APPLICATIONS

Heliports continued to play a pivotal role in our society world. Just as persons are hard pressed to thrive without the benefits of a home, helicopters are hard pressed to thrive without the benefits of a place to land and re-fuel. We need to continue to do our best to support the construction and use of heliports in areas where vertical flight is more useful—namely urban centers [Rotor,2001].

One of the first uses for helicopters involved carrying individuals in need of medical care to medical facilities. Rotorcraft operators, of course, were able to do this with amazing speed. With this history of saving lives, today's rotorcraft can do the job even more quickly and with enhanced medical capabilities. After all, when it comes to saving lives, every second counts.

The helicopters community recently strives to improve communications regarding safety concerns within the air medical industry and solve problems that develop with governing and regulator agencies. Air medical flights are a highly favorable form of vertical flight in the public eye. Though some people insist on complaining when helicopters rush injured persons to hospitals at night, most people admire life saving rotorcraft operations.

Recently, the Federal Aviation Administration (FAA) and the Helicopter Association International (HAI) have jointly developed the Maintenance Malfunction Information Reporting (MMIR) system for the aviation industry [MMIR,1998]. MMIR is a web-based database tool that will provide crucial information for trend monitoring. Users of MMIR can search a database of more than 75,000 reported parts and generate a report of past experience with particular parts. The main thrust of the MMIR database program is to enhance aviation safety and reliability analysis. The MMIR program has the potential to provide the aviation industry with much needed data that would otherwise be unavailable, enhancing the overall progress in aviation safety and reliability studies.

Accurate prediction of reliability plays an important role in the profitability of a product and in the service industry. Service costs or repair costs for helicopter parts with the warranty period or under a service contract are a major expense and a significant pricing factor. Proper spare part stocking and support personnel training also depend upon good reliability fallout predictions. On the other hand, missing reliability targets may invoke contractual penalties and cost future business.

We illustrate the usefulness of the new Vtub-shaped hazard rate function by calculating reliability measures of two helicopter parts: main rotor blade and rotor blade assembly based on the MMIR system database collected from October 1995 to September 1999. We obtain the results by using our software RSA. The Main Rotor Blade data set and Rotor Brake Assembly data set are shown in Table 1 and Table 2, respectively.

4.1 Application 1: Main Rotor Blade

Following are some estimation results based on the data set in Table 1 using RSA tool.

Summary for Loglog Distribution Model (Vtub-Shaped Hazard Rate Function):

```
α = 1.1075035          Var[α] = 0.0162444
95% CI for α is [0.8576942, 1.3573128]

a = 1.0001629          Var[a] = 2.78202609e-008
95% CI for a is [0.9998360, 1.0004898]

MTTF = 1608.324131
MRL(t=MTTF) =  950.4751306
```

Figure 3 and 4 show the hazard rate function and reliability and its 95% confidence interval of main rotor blade, respectively. Figure 5 shows the reliability comparisons between the normal model, the lognormal model, the Weibull model and the loglog model for the main rotor blade data set.

4.2 Application 2: Rotor Brake Assembly

This example shows the reliability estimation results based on the data set given in Table 2 using RSA software tool.

Summary for Loglog Distribution Model

```
α = 0.8822884            Var[α] =  0.0369422
95% CI for α is [0.5055693, 1.2590075]

a =  1.0009176           Var[a] =  2.0577872e-006
95% CI for a is [0.9981060, 1.0037293]

MTTF = 1601.7326345
MRL(t=MTTF) =  1186.1544027
```

Figure 6 and 7 show the hazard rate function and reliability and its 95% confidence interval of rotor brake assembly, respectively. Figure 8 shows the reliability comparisons between the normal model, the lognormal model, the Weibull model and the loglog model for the main rotor brake assembly data set.

Table 1. Main Rotor Blade Data

Part code	Part(hours)
xxx-015-001-107	1634.3
xxx-015-001-107	1100.5
xxx-015-001-107	1100.5
xxx-015-001-107	819.9
xxx-015-001-105	1398.3
xxx-015-001-107	1181
xxx-015-001-107	128.7
xxx-015-001-107	1193.6
xxx-015-001-107	254.1
xxx-015-001-107	3078.5
xxx-015-001-107	3078.5
xxx-015-001-107	3078.5
xxx-015-001-107	26.5
xxx-015-001-107	26.5
xxx-015-001-107	3265.9
xxx-015-001-107	254.1
xxx-015-001-107	2888.3
xxx-015-001-107	2080.2
xxx-015-001-107	2094.3
xxx-015-001-107	2166.2
xxx-015-001-107	2956.2
xxx-015-001-107	795.5
xxx-015-001-107	795.5
xxx-015-001-107	204.5
xxx-015-001-107	204.5
xxx-015-001-107	1723.2
xxx-015-001-107	403.2
xxx-015-001-107	2898.5

xxx-015-001-107	2869.1
xxx-015-001-107	26.5
xxx-015-001-107	26.5
xxx-015-001-107	3180.6
xxx-015-001-107	644.1
xxx-015-001-107	1898.5
xxx-015-001-107	3318.2
xxx-015-001-107	1940.1
xxx-015-001-107	3318.2
xxx-015-001-107	2317.3
xxx-015-001-107	1081.3
xxx-015-001-107	1953.5
xxx-015-001-107	2418.5
xxx-015-001-107	1485.1
xxx-015-001-107	2663.7
xxx-015-001-107	1778.3
xxx-015-001-107	1778.3
xxx-015-001-107	2943.6
xxx-015-001-107	2260
xxx-015-001-107	2299.2
xxx-015-001-107	1655
xxx-015-001-107	1683.1
xxx-015-001-107	1683.1
xxx-015-001-107	2751.4

Table 2. Rotor Brake Assembly data

Part.code	Part(hours)
xxx-301-103	716.5
xxx-301-103	221.3
xxx-301-103	1076.6
xxx-301-103	1144.1
xxx-301-103	502.4
xxx-301-103	1557.2
xxx-301-103	552.1
xxx-301-103	1394.3
xxx-301-103	677.3
xxx-301-103	552.1
xxx-301-103	1622.4
xxx-301-103	3077
xxx-301-103	3077
xxx-301-101	4980.2

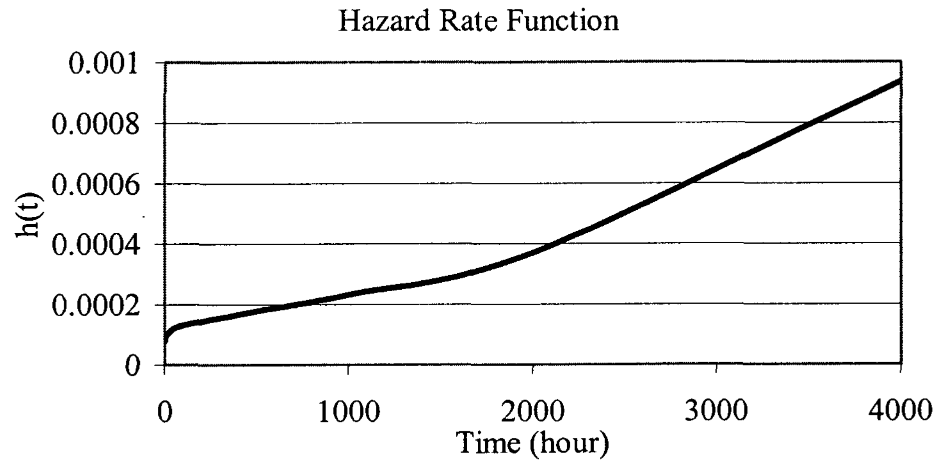

Figure 3. Hazard rate function for a main rotor blade data set

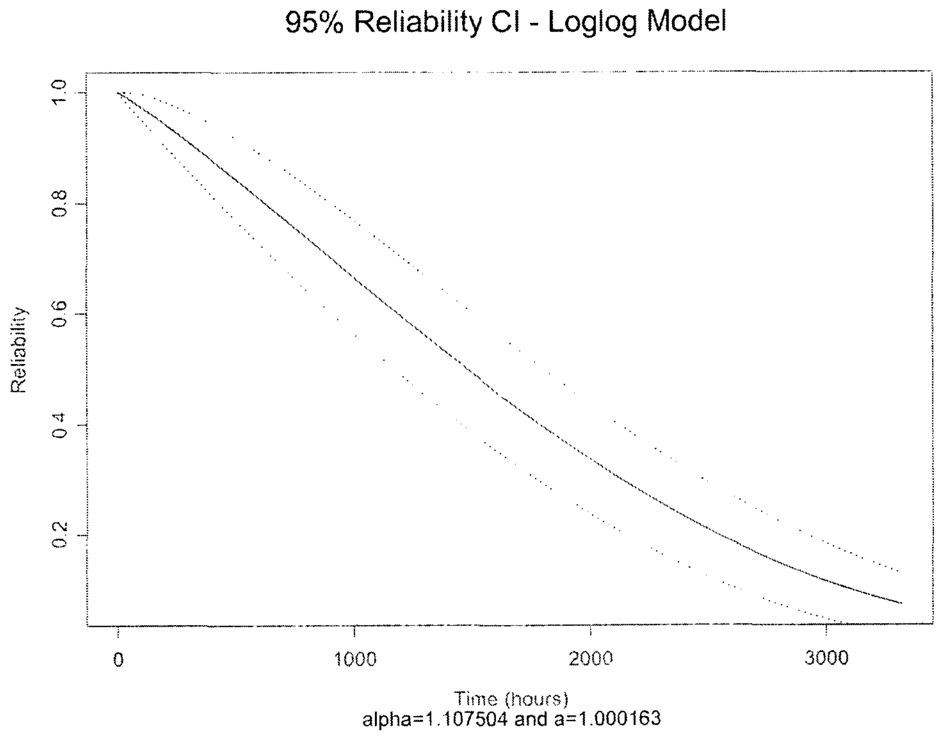

Figure 4. Estimated reliability and its confidence interval for a main rotor blade data set

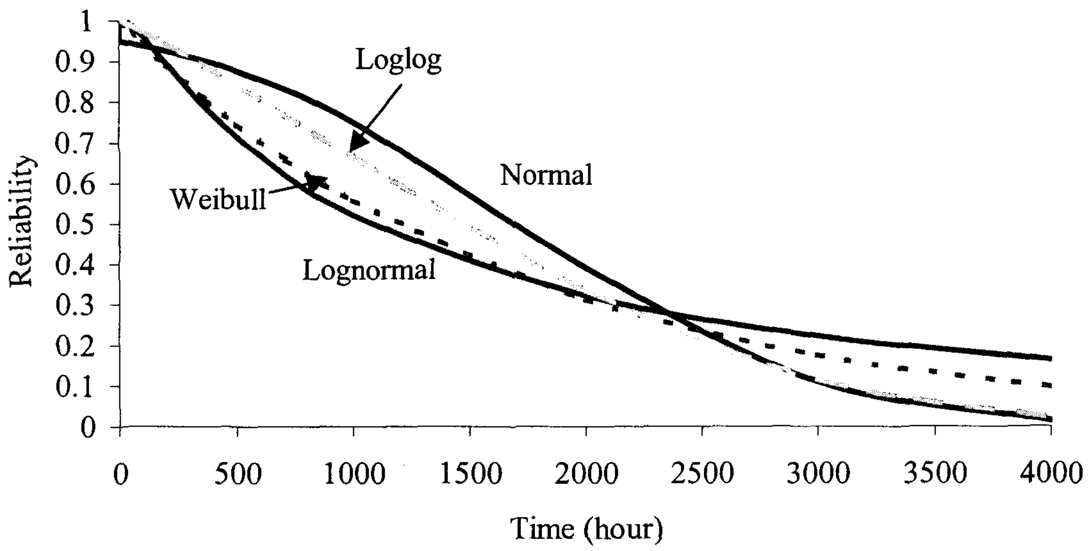

Figure 5. Reliability comparisons for a main rotor blade data set

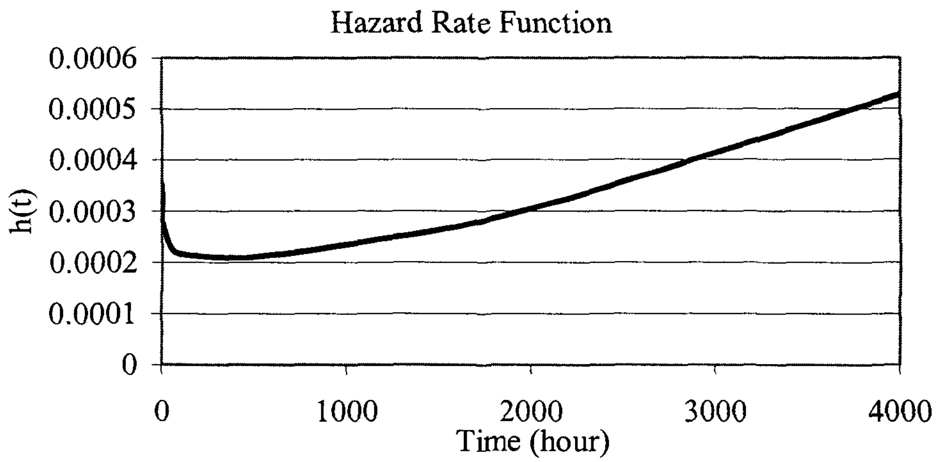

Figure 6. Hazard rate function for a rotor brake assembly data

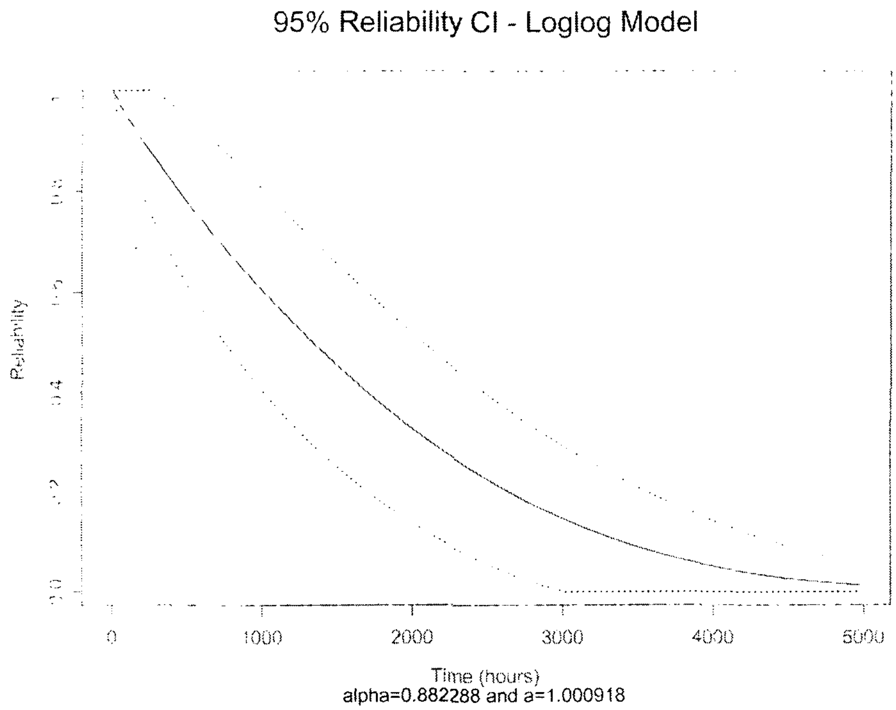

Figure 7. Estimated reliability and its confidence interval for a rotor brake assembly data set

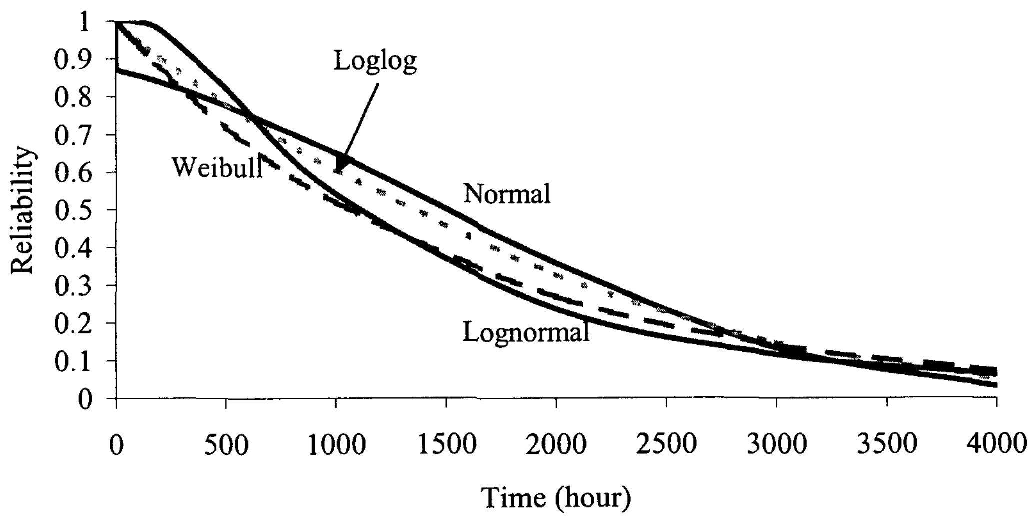

Figure 8. Reliability comparisons for a rotor brake assembly data set

ACKNOWLEDGEMENT

This research was supported in part by the *U.S. FAA William J. Hughes Technical Center* under Grant 98-G-006 and *U.S. National Science Foundation* under Grant INT-0107755

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BIOGRAPHY OF AUTHOR

Hoang Pham is an Associate Professor of the Department of Industrial Engineering at Rutgers University and a member of the Environmental and Occupational Health Sciences Institute of the University of Medicine and Dentistry New Jersey. Before joining Rutgers in July 1993, he was a senior engineering specialist at the Boeing Company, Seattle, and the Idaho National Engineering Laboratory, Idaho Falls. He is doing research in the areas of software reliability, reliability modeling, maintenance theory, environmental risk assessment, and optimization. He is the author of *Software Reliability* and the editor of fifteen volumes. He has published more than 65 journal articles and 15 book chapters. He is currently editing the *Handbook of Reliability Engineering* (Fall 2002).

He is the editor-in-chief of the *Int'l Journal of Reliability, Quality and Safety Engineering,* and an editorial board member of the *Int'l Journal of Systems Science, IIE Trans. on Quality and Reliability* and several others. He is also editor-in-chief of the book series on *Industrial and Systems Engineering,* Published by the World Scientific Publishers. He is the conference general chair of the *Eighth ISSAT Int'l Conference on Reliability and Quality in Design,* 2002.

He received his Ph.D. degree in industrial engineering from the State University of New York at Buffalo. He is listed in *Who's Who in the World, Who's Who in America, Who's Who in Science and Engineering*. He has received *The Outstanding Scientist of the 20^{th} Century Medal in honor of outstanding contribution in the field of Reliability Engineering.*