한국산업정보학회논문지 제7권 제3호 (2002. 9)

Canonical foliations of almost f-cosymplectic structures*

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요 약 본 논문은 주로 개 f-코심플렉틱 다양체를 다룬다. 이 개념은 개 코심플렉틱 다양체와 개 겐모츠 다양체를 포함한다. 개 코심플렉틱 다양체는 [1]에서 도입된 이래 [2], [3], [4] 등 여러 학자들에 의해 연구되어져 왔으며 개 겐모츠 다양체는 [5]에서 도입된 이래 [6], [7] 등에서 연구되어져 왔다. 본 논문에서는 개 f-코심플렉틱 다양체의 접촉 초함수에 의해 정의되는 정규 엽층구조의 기하학적 성질을 연구한다. 본 논문의 목적은 [8], [9]에서 얻은 성과를 확장하는 것이다.

Abstract The present paper mainly treats with almost f-cosymplectic manifolds. This notion contains almost cosymplectic and almost Kenmotsu manifolds. Almost cosymplectic manifolds introduced in [1] have been studied by many schalors, say [2], [3], [4], and almost Kenmotsu manifolds introduced in [5] have been studied in [6], [7]. The present paper studies some geometrical and topological properties of the canonical foliation defined by the contact distribution of an almost f-cosymplectic manifold. The purpose of the present paper is to extend the results obtained in [8], [9].

1. Introduction

Let M be a $(2n+1)$-dimensional manifold endowed with *an almost contact structure* (ϕ, ξ, η), that is, ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form satisfying the conditions

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1.$$

If M admits a Riemannian metric g compatible with (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$ (:= the Lie algebra of all vector fields on M), M is called *an almost contact metric manifold.* On such a manifold the fundamental form Φ is defined by

$$\Phi(X, Y) := g(\phi X, Y), \quad X, Y \in \Gamma(TM).$$

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *almost cosymplectic* (resp. *almost α-Kenmotsu*) if $d\eta = 0$ and $d\Phi = 0$ (resp. $d\Phi = 2\alpha\eta\wedge\Phi$, α being a non-zero real constant), where d is the exterior differential operator. Geometrical properties and examples of almost cosymplectic manifolds are found in [1], [2], [3], [4], [7]. The case of almost α-Kenmotsu manifolds are found in [5], [6], [7].

Recently a new notion of an almost α-cosymplectic manifold was introduced in [9], which is defined by

(1.1) $\quad d\eta = 0, \quad d\Phi = 2\alpha\eta\wedge\Phi$

*The present work was supported by the Research Fund of Kyungsan University, 2001.

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for $\alpha \in \boldsymbol{R}$. This notion corresponds to an almost cosymplectic or almost α-Kenmotsu manifold according to $\alpha = 0$ or not.

In the present paper we extend this notion to that of *almost f-cosymplectic manifold* which is defined by

$$(1.2) \quad d\eta = 0, \ d\Phi = 2f\eta \wedge \Phi,$$

where f is a function on M satisfying

$$(1.3) \quad df \wedge \eta = 0.$$

It is obvious that an almost f-cosymplectic manifold with $f = \alpha$ constant is just almost α-cosymplectic. We will see an example of an almost f-cosymplectic manifold which f is not constant.

The notion of an almost f-cosymplectic manifold is closely related to the locally conformal geometry of almost cosymplectic manifolds. It is known in [10], [7] that an almost contact metric manifold M is locally conformal almost cosymplectic if and only if there exists a 1-form ω on M such that

$$(1.4) \quad d\eta = \omega \wedge \eta, \ d\Phi = 2\omega \wedge \Phi, \ d\omega = 0.$$

ω is called *the characteristic form* of such a manifold in the sense that if the form ω satisfying (1.4) exists, then it is unique. We easily notice that an almost f-cosymplectic manifold is a special case of an locally conformal almost cosymplectic manifold (take ω as $f\eta$).

The present paper studies the canonical foliations of an almost f-cosymplectic manifold. The purpose of the present paper is to extend the results obtained in [8], [9].

2. Almost f-cosymplectic manifolds

To begin with, it is useful to give a characterization of almost f-cosymplectic manifolds from the viewpoint of locally conformal geometry of almost cosymplectic manifolds. The proof is an immediate consequence of (1.4).

Proposition 2.1 *A locally conformal almost cosymplectic manifold M is almost f-cosymplectic if and only if the characteristic form of M is given by $\omega = f\eta$.*

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost f-cosymplectic manifold. By virtue of Propositoin 2.1, [7] provides some formulas.

Define a (1,1)-tensor field h on M by

$$(2.1) \quad hX := \nabla_X \xi + f\Phi^2 X \quad X \in \Gamma(TM),$$

where ∇ denotes the Levi-Civita connection of (M, g).

Lemma 2.2 *Let (M, ϕ, ξ, η, g) be an almost f-cosymplectic manifold. Then*

(1) the linear operator h is symmetric and satisfies $h\Phi + \Phi h = 0, \ h\xi = 0,$

(2) $h = -\frac{1}{2}\Phi(L_\xi \Phi)$, *where L denotes the Lie derivative operator,*

(3)

$$(\nabla_X \Phi)Y + (\nabla_{\Phi X}\Phi)\Phi Y = f\{2g(\Phi X, Y)\xi - \eta(Y)\Phi X\} + \eta(Y)h\Phi X.$$

Proof. (1) follows from Proposition 2.1 and a result established in [7] (Lemma 3.1).

(2) The condition $d\eta = 0$ implies

$$(2.2) \quad \nabla_\xi \xi = 0.$$

On the other hand, a general formula for $\nabla\Phi$ in an almost contact metric manifold ([11]) yields that an almost f-cosymplectic manifold satisfies

(2.3) $\nabla_\xi \Phi = 0.$

Then (1), combined with (2.2) and (2.3) gives rise to

$$\begin{aligned}\Phi(L_\xi \Phi)X &= \Phi[\xi, \Phi X] - \Phi^2[\xi, X] \\ &= -\Phi h \Phi X - hX = -2hX\end{aligned}$$

for any $X \in \Gamma(TM)$.

(3) is an immediate consequence of a result established in [7] (Lemma 3.2) with $\omega = f\eta$.

Remark. Lemma 2.2. (3) induces the formula (2.1). Furthermore, we also have

(2.4) $\delta\Phi = 0,$

where δ is the codifferential operator.

3. Canonical foliations of almost f-cosymplectic manifolds

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost f-cosymplectic manifold. Since $d\eta = 0$, the contact distribution $D = \ker \eta$ defines a foliation F on M of codimension 1.

Theorem 3.1 *Let (M, ϕ, ξ, η, g) be an almost f-cosymplectic manifold. Then*

(1) F is Riemannian,

(2) F is tangentially almost Kähler,

(3) when $f = 0$ (resp. $f \neq 0$), F is totally geodesic (resp. totally umbilic) if and only if $h = 0$.

Proof. (1) follows from (2.2).

(2) We note that $\Phi\xi = 0$ and $\Phi(\xi, X) = 0$ for each $X \in \Gamma(TM)$. Hence the restriction (g_D, ϕ_D, Φ_D) of (g, ϕ, Φ) to the contact dustribution D induces an almost Kähler structure for F.

(3) A direct computation using (2.1) yields for any $X, Y \in \Gamma(D)$

(3.1) $$\begin{aligned} g(\nabla_X Y, \xi) &= -g(hX - f\Phi^2 X, Y) \\ &= -g(hX, Y) - fg(X, Y),\end{aligned}$$

which means that $h = 0$ if and only if $\nabla_X Y = -fg(X, Y)\xi$. Therefore, if $f = 0$ then F is totally geodesic. Otherwise, F is totally umbilic with mean curvature $-f\xi$.

Recall that an almost contact manifold (M, Φ, ξ, η) is said to be *normal* if

$$\begin{aligned} N_\Phi(X, Y) := &[\Phi X, \Phi Y] - \Phi[\Phi X, Y] \\ &- \Phi[X, \Phi Y] + \Phi^2[X, Y] \\ &+ 2d\eta(X, Y)\xi = 0\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. A normal almost f-cosymplectic manifold is called *an f-cosymplectic manifold*. The class of f-cosymplectic manifolds contains cosymplectic manifolds and α-Kenmotsu manifolds. The following result provides a characterization of f-cosymplectic manifolds in terms of the canonical foliation F and the operator h.

Proposition 3.2 *An almost f-cosymplectic manifold M is f-cosymplectic if and only if the canonical foliation F is tangentially Kähler and $h = 0$.*

Proof. It can be easily seen from Lemma 2.2 (2) that for each $X \in \Gamma(D)$

(3.2) $N_\Phi(\xi, X) = -\Phi[\xi, \Phi X] - [\xi, X] = 2hX.$

On the other hand, for each $X, Y \in \Gamma(D)$

(3.3) $N_\Phi(X, Y) = N_{\Phi_D}(X, Y),$

where Φ_D is an almost complex structure for F defined in Theorem 3.1. Therefore the proof is completed by (3.2) and (3.3).

Remarks. (1) Theorem 3.1 and Proposition 3.2 extend results obtained in [8], [9].

(2) Let (N, J, G) be an almost Kähler manifold and $\boldsymbol{R}$ be the real line with coordinate t. Define an almost cosymplectic structure $(\tilde{\Phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the product manifold $M: =N\times \boldsymbol{R}$ by

$$\tilde{\Phi}X: =JX \quad X\in\Gamma(TN),$$
$$\tilde{\Phi}X: =0 \quad X=\frac{\partial}{\partial t}$$
$$\tilde{\xi}: =e^{\sigma}\frac{\partial}{\partial t}, \quad \tilde{g}: =G+e^{-2\sigma}dt\otimes dt$$
$$\tilde{\eta}: =e^{-\sigma}dt,$$

where σ is a function on $\boldsymbol{R}$.

Now consider the conformal change of the structure $(\tilde{\Phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ given by

(3.4)
$$\Phi: =\tilde{\Phi}, \quad \xi: =e^{-\sigma}\tilde{\xi}, \quad \eta: =e^{\sigma}\tilde{\eta}, \quad g: =e^{2\sigma}\tilde{g}$$

Then (Φ, ξ, η, g) is a (globally) conformal almost cosymplectic structure. Its characteristic form is $\omega=f\eta$ with $f=\sigma'$. Therefore, Proposition 2.1 says that (Φ, ξ, η, g) is almost f-cosymplectic on M. If the function σ is periodic, then the structure obtained can be projected to $N\times S^1$, where S^1 is a circle. Note that $N_\Phi(X, \xi)=0$ for $X\in\Gamma(TN)$, so that $h=0$ by means of (3.2).

The canonical foliation F consists of the leaves $N\times\{t\}$. It is tangentially almost Kähler and non-Kähler if N is non-Kähler. It follows from Proposition 3.2 that N is Kähler if and only if M is f-cosymplectic.

4. Semi-invariant submanifolds of almost f-cosymplectic manifolds

Let N be a submanifold of an almost contact metric manifold (M, Φ, ξ, η, g). N is called *a semi-invariant submanifold* if there exists a distribution D_N on N satisfying the following conditions

(4.1) $$\Phi D_N\subset D_N, \Phi D_N^{\perp}\subset TN^{\perp}, \xi\in\Gamma(TN^{\perp}),,$$

where $D_N^{\perp}$ is the orthogonal complementary distribution of D_N in TN. Recall the following result established in [9].

Lemma 4.1 *Let N be a semi-invariant submanifold of an almost contact metric manifold (M, Φ, ξ, η, g). The anti-invariant distribution $D_N^{\perp}$ is integrable if and only if for $X\in\Gamma(D_N)$ and $U, V\in\Gamma(D_{N^\perp})$,*

$$d\Phi(U, V, X)=0.$$

Proposition 4.2 *Let N be a semi-invariant submanifold of an almost f-cosymplectic manifold (M, Φ, ξ, η, g). Then*

(1) $D_N^{\perp}$ is integrable,

(2) the invariant distribution D_N is minimal.

Proof. (1) It is easy to see from (4.1) that (1.2) implies $d\Phi(U, V, X)=0$ for any $U, V\in\Gamma(D_N^{\perp})$, $X\in\Gamma(D_N)$. This proves (1) by Lemma 4.1.

(2) We need the following formulas (cf. [9]). For any $U\in\Gamma(D_N^{\perp})$, $X\in\Gamma(D_N)$

(4.2)
$$\begin{aligned} g(\nabla_X X, U) &= g(A_{\Phi U}X, \Phi X) - g((\nabla_X\Phi)X, \Phi U), \\ g(\nabla_{\Phi X}\Phi X, U) &= -g(A_{\Phi U}\Phi X, X) - g((\nabla_{\Phi X}\Phi)\Phi X, \Phi U), \end{aligned}$$

where A denotes the Weingarten map of $N\subset M$. (4.2), together with Lemma 2.2 (3), yields

(4.3) $$g(\nabla_X X+\nabla_{\Phi X}\Phi X, U)=0.$$

If we take an orthonormal frame field

$\{E_i, E_{p+i} = \Phi E_i\}$, $2p = \dim D_N$, of D_N, the mean curvature for D_N satisfies

$$x := \frac{1}{2p} \sum (\nabla_{E_i} E_i + \nabla_{\Phi E_i} \Phi E_i)^{\perp} = 0$$

by means of (4.3).

From Proposition 4.2 we have the foliation F_N defined by $D_N^{\perp}$. Let $\nu := \omega_1 \wedge \cdots \wedge \omega_{2p}$ be the transversal volume form for F_N on N, where $\{\omega_i\}$ is the dual frame field of an orthonormal frame field $\{E_i, \Phi E_i\}$ of D_N. Then Proposition 4.2 says that $d\nu = 0$ ([8], [12]). By a similar argument as in [8] we have the following result. For the proof, it suffices to notice that the restriction Φ_N of Φ to N is a closed 2-form satisfying

(4.4) $\quad \Phi_N^p = (-1)^p p! \, \nu.$

Theorem 4.3 *Let N be a compact semi-invariant submanifold of an almost f-cosymplectic manifold (M, Φ, ξ, η, g). Then the transversal volume form ν defines a cohomology class $c(N) := [\nu] \in H^{2p}(N, \mathbf{R})$. Furthermore, if $D_N^{\perp}$ is minimal and D_N is integrable then for any $k \in \{1, \cdots, p\}$*

(4.5) $\quad H^{2k}(N, \mathbf{R}) \neq 0.$

For example, a leaf N of the canonical foliation F discussed in section 3 is a semi-invariant submanifold with trivial anti-invariant distribution. In this case, F_N is the point foliation and $D_N = D$ is integrable. Therefore, it holds (4.5) from Theorem 4.3. Indeed, $c(N)$ is explicitly represented by a $2n$-form $\frac{(-1)^n}{n!} \Phi_D^n$. It should be noted that Φ_D is harmonic by virtue of (2.4).

5. The transversal property for the canonical foliation F_N

It may be interesting to consider the problem when the canonical foliation F_N on a semi-invariant submanifold N of an almost f-cosymplectic manifold M is Riemannian. Related to this problem, we compute the Godbillon-Vey class for F_N.

In case that N is a leaf of the canonical foliation F discussed in section 3, F_N is the point foliation, so Riemannian. Thus all the secondary characteristic classes of F_N on a semi-invariant submanifold $N \subset M$ are zero ([13], [12]). In general, F_N may be non-Riemannian. Yet for its the Godbillon-Vey class we have

Theorem 5.1 *Let N be a semi-invariant submanifold of an almost f-cosymplectic manifold (M, Φ, ξ, η, g). Then the Godbillon-Vey class $GV(F_N)$ for the canonical foliation F_N is given by*

(5.1) $\quad GV(F_N) = (2fp)^{2p+1} [\eta_N \wedge (d\eta_N)^{2p}] = 0,$

where $2p := \dim D_N$ and η_N is the restriction of η to N.

Proof. Recall the definition of the Godbillon-Vey class for F_N given by

(5.2) $\quad GV(F_N) := [\psi \wedge (d\psi)^{2p}],$

where ψ is 1-form on N satisfying $d\nu = \psi \wedge \nu$. Now from (4.4) we get

$$(-1)^p p! \, d\nu = p2f(-1)^p p! \, \eta_N \wedge \nu.$$

Therefore, we can choose $\psi = 2fp\eta_N$. Therefore, (5.2) yields (5.1).

6. Conclusion

The main results of the present paper are as follows. One can derive several formulas in an almost f-cosymplectic manifold (section 2). These formulas enable to find the geometrical properties of the canonical foliation F defined by the contact distribution $D := \ker \eta$. One can prove that F is Riemannian and tangentially almost Kähler of codimension 1 and that F is tangentially Kähler if the manifold M is normal (section 3). Moreover, one can show that a semi-invariant submanifold N of such a manifold M admits a canonical foliation F_N defined by the anti-invariant distribution and a canonical cohomology class $c(N)$ generated by a transversal volume form for F_N. In addition, one can find the conditions when the even-dimensional cohomology classes $H^{2k}(N, R)$ of N are non-trivial. For example, when a leaf N of the canonical foliation F is a semi-invariant submanifold of M then $c(N)$ is explicitly represented by $\frac{(-1)^n}{n!} \Phi_D^n$ and $H^{2k}(N, R) \neq 0$ for $k = 1, \cdots, n$ ($2n = \dim D$) (section 4). Finally, one can consider the problem when F_N is Riemannian. Related to this problem, one can compute the Godbillon-Vey class for F_N (section 5).

These results extend those obtained in [8], [9] for the case of almost α-cosymplectic manifolds. Similar results in the locally conformal Kähler case is found in [14]. One may expect that these results will be applied to extend the recent researches such as [2], [3] on almost cosymplectic manifolds.

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