THE SENSITIVITY OF STRUCTURAL RESPONSE USING FINITE ELEMENTS IN TIME

Sungho Park*, Seung Jo Kim**
Department of Aerospace Engineering
Seoul National University, Seoul 151-742 Korea

Abstract

The bilinear formulation proposed earlier by Peters and Izadpanah to develop finite elements in time to solve undamped linear systems, is extended (and found to be readily amenable) to develop time finite elements to obtain transient responses of both linear and nonlinear, and damped and undamped systems. The formulation is used in the *h*-, *p*- and *hp*-versions. The resulting linear and nonlinear algebraic equations are differentiated to obtain the first- and second-order sensitivities of the transient response with respect to various system parameters. The present developments were tested on a series of linear and nonlinear examples and were found to yield, when compared with results obtained using other methods, excellent results for both the transient response and its sensitivity to system parameters. Mostly, the results were obtained using the Legendre polynomials as basis functions, though, in some cases other orthogonal polynomials namely, the Hermite, the Chebyshev, and integrated Legendre polynomials were also employed (but to no great advantage). A key advantage of the time finite element method, and the one often overlooked in its past applications, is the ease in which the sensitivity of the transient response with respect to various system parameters can be obtained. The results of sensitivity analysis can be used for approximate schemes for efficient solution of design optimization problems. Also, the results can be applied to gradient-based parameter identification schemes.

Key words : Finite Elements in Time, Bilinear Formulation, First- and Second-Order Sensitivity, Orthogonal Polynomials

Introduction

The time finite element method, in which the time is discretized in a number of finite elements and the response history over each element is expressed in terms of basis functions in the time co-ordinate, has attracted a considerable attention in the past. In recent years, the method has found increasing popularity, especially with researchers involved with studying transient response and dynamic stability of periodic systems, such as the aeroelastic stability of helicopter rotor blades (Borri et al. 1985; Peters and Izadpanah 1988; Achar and Gaonkar 1993) and the multi-rigid body dynamics (Borri et al. 1990, 1991; Mello et al. 1990; Atluri and Cazzani 1995). Based on these and many other efforts (to name a few: Pian and O'Brien 1957; Gurtin 1964a, 1964b; Sandhu and Pister 1972; Tonti 1972; Atluri 1973; Herrera and Bielak 1974; Bailey 1975a, 1975b, 1976a, 1976b; Reddy 1975, 1976; Kim and Cho 1997) dedicated to develop appropriate variational framework for generating approximate solutions in space and time domain, it appears that the time finite element method has emerged as a viable approach for studying the transient response of systems (i.e. for solving initial value problems) also. A brief review of the key development leading to the current state of the finite element method as applied to initial value problems is given below. The time finite element method was first introduced by Argyris and Scharpf (1969) who employed Hermite cubic interpolation polynomials (akin to the beam finite element) to express the response over each

* : Postdoctoral Researcher
** : Professor

time finite element. The response and the velocity at any given instant were thus obtained in terms of the displacement and the velocity at two "pivotal points", namely the start and the end of the time element. The method, based on the Hamilton's principle, was applied to a single-degree-of-freedom system but no numerical examples were considered. A generalization of the formulation allowing for an arbitrary number (p) of pivotal points and each pivotal point having an arbitrary number of time derivatives (a) of the response, was also given (but, once again, without any numerical results). Fried (1969), using an 'Extended Hamilton's Principle', applied this approach to study the transient response of a damped system (nonconservative systems) and transient heat conduction in a slab whose surface was subjected to a harmonically varying temperature. Fried used a step by step approach in which the final conditions (for displacement and velocity) for one time element can be considered as initial conditions for the next time finite element, as in any time-marching technique. This was done to avoid storing and working with large matrices. Zienkiewicz and Parekh (1970), also working with problems in heat conduction, used a time finite element approach that was based on Galerkin procedure of obtaining weighted residual equations over a time interval. On integration, the resulting equation was given in terms of the initial and final states.

An important development in the evolution of time finite element method was a series of papers by Bailey (1975a, 1975b, 1976a, 1976b) in which the author pointed out the need for applying Hamilton's law of varying action, briefly, Hamilton's law, not Hamilton's principle, in solving problems in elastodynamics. Using Hamilton's law, Bailey used the classical Ritz method, with simple polynomials as the basis functions, to study elastodynamic response of beams. Baruch and Riff (1982, 1984a, 1984b), and Borri et al. (1985) developed time finite element using Hamilton's law. Baruch and Riff (1982) noted that one can use six different formulations (each having different combinations of initial and final constraints) of the Hamilton's law for each degree-of-freedom. Borri et al. (1990, 1991), Mello et al. (1990) and Atluri and Cazzani (1995) applied the primal and mixed form of Hamilton's law to develop piecewise Lagrange-type time finite elements to solve nonlinear equations of multi-rigid body dynamics.

Wu (1977) recognizing the limitations of constrained variational principles like the extended Hamilton's principle to nonconservative forces, presented an unconstrained variational principle in which the constraints were applied using the well known technique of Lagrange multipliers. Simkins (1978) presented a procedure, consisting of introducing all boundary and essential conditions into the 'variational statement' as natural boundary conditions, making the variational statement suitable for obtaining approximate solutions for initial and boundary value problems. It was pointed out that the proposed procedure when applied to the Euler equations leads to a modified Hamilton's principle as was given by Tiersten (1968). In a subsequent paper, Simkins (1981) employed this variational statement to develop finite element in time.

Commenting on the paper by Simkins (1978), Smith (1979) noted that what the former calls a variational statement for solving initial value problems is really an application of the weighted residual method to those problems. Building on this observation, Peters and Izadpanah (1988) offered a bilinear formulation of elastodynamics as an alternative source to develop approximate methods to solve these problems. As was ealier done by Wu (1977), Peters and Izadpanah (1988) employed Lagrange multiplier method to account for various end constraints. More specific choice and use of Lagrange multiplier to satisfy the various constraint conditions can be seen in the papers by Atluri (1973), Borri et al. (1990, 1991), Mello et al. (1990) and Atluri and Cazzani (1995). A distinct advantage of this augmented, bilinear formulation is the natural convergence of the end conditions. Also, the method can be applied as a step by step procedure, thereby avoiding the need for dealing with large matrices. To achieve this, the natural convergence of the end constraints is very important as the end conditions of one segment are used as initial conditions for the next segment. Using the proposed augmented bilinear formulation and the h-, p-, and the hp-versions, Peters and Izadpanah (1988) solved a number of examples related to dynamic response of linear systems.

One of the objectives of the present paper is to evaluate the performance of the augmented bilinear formulation of elastodynamics in determining the transient response of damped (nonconservative) linear and nonlinear systems. Whenever possible, the numerical results are compared with existing results or those obtained using Runge-Kutta fourth-order method. For all the cases studied, the results are obtained using Legendre polynomials as basis functions. The results were also obtained using other polynomials, namely: Hermite, integrated Legendre, and Chebyshev polynomials as basis functions, but without any significant improvement in either accuracy or efficiency. The results obtained by using these polynomials are thus not being presented.

In addition to the transient response calculations, the time finite element approach is natural for obtaining the sensitivity of the transient response of linear and nonlinear and damped and undamped systems, as the sensitivities can be easily obtained by performing direct differentiation of the nonlinear algebraic equations resulting from the application of the proposed finite element method. The study of the response sensitivities of the dynamic systems are important for performing system identification and evaluating the effect of design changes on the dynamic response of the system. Additionally it is easier to determine the character of the motion of the dynamic system from the critical stability viewpoint by observing the sensitivity coefficients. A number of approaches are currently being used for obtaining sensitivity of the transient response; namely: direct differentiation of the governing differential equation, adjoint variable method, finite difference method, Green's function method, and Fourier amplitude sensitivity method. A review of the existing methods to obtain sensitivity of the transient response is given by Adelman and Haftka (1986) and Haftka and Adelman (1989). In a recent paper, Wang and Lu (1993) have presented a procedure that uses discrete Fourier transform to obtain sensitivity of the transient response of the linear systems. Among the current methods the direct differentiation method is straight forward and is quite efficient when the number of design variables is small. The proposed time finite element method based approach can be considered to be similar to the direct differentiation approach. The key difference being that in the proposed approach differentiation of the algebraic equations and not that of the original differential equations is performed. The present approach is thus simpler than the direct differentiation approach. For some applications, expressions for second-order sensitivity using the proposed method are also obtained for nonlinear structural systems and the results are compared with the ones obtained using the finite-difference method. Although a number of approaches are currently being used for obtaining first-order sensitivity of the transient response, little research progress has been made towards obtaining higher order sensitivity. Second-order sensitivity derivatives were obtained by Haug (1981) and by Haftka (1982), using the adjoint variable method for the structural systems, and by Bindolino and Mantegazza (1987), using a direct differentiation method for an aeroelastic response problem. Also, Van Belle (1982) derived expressions for second-order sensitivities using flexibility approach rather than the stiffness approach. Part of the motivation for second-order sensitivity derivatives is that they estimate nonlinear sensitivity effects including interactions between variables.

The second objective of this paper is to study the performance of the time finite element in obtaining the sensitivity of the transient response of various linear and nonlinear and damped and undamped systems. The results obtained from the present approach are compared with those obtained with the central finite difference approach, using step sizes obtained from a convergence study. No such convergence study is needed when the present approach, employing direct differentiation of the algebraic equations resulting from the time finite element method, is employed.

Mathematical formulation

Transient response of linear systems

Consider a simple one-degree-of-freedom spring-mass-damper system given as

$$M\ddot{u}(t)+C\dot{u}(t)+Ku(t)=F(t) \qquad T_0 < t \leq T_f \tag{1}$$

with initial conditions $u(0)=u_0 \quad \dot{u}(0)=u_1$

Here M, C and K are respectively the mass, damping and stiffness coefficients, and $F(t)$

is the externally applied dynamic load. Multiplying Eq. (1) with a test function (or weight function) $v(t)$ and integrating with respect to time gives

$$\int_{T_0}^{T_f} v \cdot (M\ddot{u} + C\dot{u} + Ku - F)dt = 0 \tag{2}$$

Integrating the first term in Eq. (2) by parts, we get

$$[Mv\dot{u}]_{T_0}^{T_f} + \int_{T_0}^{T_f}(Kvu + Cv\dot{u} - M\dot{v}\dot{u} - vF)dt = 0 \tag{3}$$

For the correct formulation of Eq. (3), we need to investigate the possible constraints on the initial and end conditions of the system. Many previous works (Baruch and Riff (1982)) offered various ideas. Peters and Izadpanah (1988) suggested that $v(T_f)$ in Eq. (3) has to be zero to eliminate the unknown final momentum $M\dot{u}(T_f)$ but $v(t)$ must not be allowed to vanish at T_0 to insure a natural convergence of the initial momentum $M\dot{u}(T_0)$. The variational equation (Bailey 1975a,1975b,1976a, 1976b; Hitzl and Levinson 1979; Simkins 1981; Riff and Baruch 1984a,1984b; Borri et al. 1985, 1990, 1991; Mello et al. 1990) based on Hamilton's law gives the following form

$$\delta\int_{T_0}^{T_f}(T - V)dt + \int_{T_0}^{T_f} Q_i\delta s_i dt - \frac{\partial T}{\partial \dot{s}_i}\delta s_i \,|\, _{T_0}^{T_f} = 0 \tag{4}$$

and included the displacement variations at T_0 and T_f. Baruch and Riff (1982) demonstrated that Hamilton's law with a constraint of $\delta s_i(T_f) = 0$ showed the best convergence among the 6" correct formulations. The trial function $u(t)$ can be expressed as

$$u(t) = \sum_{j=1}^{N} q_j\phi_j(t) \tag{5}$$

where $\phi_j(t)$ are the basis functions in the form of Legendre polynomials, though same results were also obtained using Hermite, Chebyshev, and integrated Legendre polynomials. Substituting Eq. (5) into Eq. (3), we get a equation of the form (Zienkiewicz and wood 1987)

$$\sum_{j=1}^{N} q_j\left(\int_{T_0}^{T_f}(Kv\phi_j + Cv\dot{\phi} - M\dot{v}\dot{\phi}_j)dt\right) = \int_{T_0}^{T_f} Fvdt - Mv(T_f)\dot{u}(T_f) + Mv(T_0)\dot{u}(T_0) \tag{6}$$

Let $v(t) = \delta q_i\psi_i(t)$, $1 \leq i \leq N$, where $\psi_i(t)$ are the admissible functions chosen from the same set for trial functions. The test functions may or may not be same as the trial functions. Then for each $\psi_i(t)$, Eq. (6) becomes

$$\boldsymbol{Bq} = \boldsymbol{a} \tag{7}$$

where

$$B_{ij} = \int_{T_0}^{T_f}(K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j)dt$$

$$a_i = M\psi_i(T_0)\dot{u}(T_0) - M\psi_i(T_f)\dot{u}(T_f) + \int_{T_0}^{T_f} F\psi_i dt$$

We impose the initial condition, $u(T_0) = u_0$ in the form of a constraint. This is done by augmenting Eq. (7) with an additional eq. (8).

$$u_0 = \sum_{j=1}^{N} q_j \phi_j(T_0) \tag{8}$$

$$v(T_f) = \delta q_i \psi_i(T_f) = 0 \tag{9}$$

The second constraint Eq. (9) can be included by using the method of Lagrange multiplier, by multiplying an arbitrary Lagrange multiplier λ to Eq. (9) and adding the product to the left hand side of Eq. (7).

$$\sum_{j=1}^{N} B_{ij} q_j + \lambda M \psi_i(T_f) = a_i^* \tag{10}$$

where

$$a_i^* = M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} F\psi_i dt$$

The $M\psi_i(T_f)\dot{u}(T_f)$ term in a_i has been eliminated due to the constraint Eq. (9). With this constraint on the test function the bilinear formulation can be proved to be convergent and the detail will be shown in appendix. Equations (7) and (10) can be written in the matrix form

$$\begin{bmatrix} B & \{M\psi_i(T_f)\} \\ \langle \phi_j(T_0) \rangle & 0 \end{bmatrix} \begin{pmatrix} q_j \\ \lambda \end{pmatrix} = \begin{pmatrix} a^* \\ u(T_0) \end{pmatrix} \tag{11}$$

where

$$\begin{aligned} B_{ij} &= \int_{T_0}^{T_f} (K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j) dt \\ a_i^* &= M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} F\psi_i dt \\ \lambda &= \dot{u}(T_f) \end{aligned}$$

and i, j are row and column index, respectively. In the case of multiple elements, $u(t)$ and λ at end point for a particular element should be used as the initial conditions for the following element.

Transient Response Sensitivity of Linear Systems

The transient sensitivity calculation is equivalent to the mathematical problem of obtaining the derivatives of the solutions with respect to the independent variables. The straightforward differentiation of Eq. (11) with respect to the design parameter p_k, allows us to write the following sensitivity equation.

$$\begin{bmatrix} B & \{M\psi_i(T_f)\} \\ \langle \phi_j(T_0) \rangle & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial q_i}{\partial p_k} \\ \frac{\partial \lambda}{\partial p_k} \end{pmatrix} + \begin{bmatrix} \frac{\partial B}{\partial p_k} & \{\frac{\partial M}{\partial p_k}\psi_i(T_f)\} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} q_j \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{\partial a^*}{\partial p_k} \\ \frac{\partial u(T_0)}{\partial p_k} \end{pmatrix} \tag{12}$$

where

$$\begin{aligned} \frac{\partial B_{ij}}{\partial p_k} &= \int_{T_0}^{T_f} \left(\frac{\partial K}{\partial p_k}\psi_i\phi_j + \frac{\partial C}{\partial p_k}\psi_i\dot{\phi}_j - \frac{\partial M}{\partial p_k}\dot{\psi}_i\dot{\phi}_j \right) dt \\ \frac{\partial a_i^*}{\partial p_k} &= \frac{\partial M}{\partial p_k}\psi_i(T_0)\dot{u}(T_0) + M\psi_i(T_0)\frac{\partial \dot{u}(t)}{\partial p_k} \end{aligned}$$

For the first element, we have

$$\frac{\partial a_i^*}{\partial p_k} = \frac{\partial M}{\partial p_k} \phi_i(T_0)\dot{u}(T_0)$$

Transient Response Sensitivity of Nonlinear Systems

While the solution of a linear equation system can be accomplished without difficulty in a manner described above, this is not possible for nonlinear systems. Analytical procedures for the treatment of nonlinear differential equations are difficult and require extensive mathematical study. The differential equation describing a nonlinear oscillatory system over a given length of time, $T_0 < t \leq T_f$, may have a general form

$$g(u, \dot{u}, \ddot{u}, t, p) = 0 \tag{13}$$

where g may be nonlinear functions of $u(t)$ and $\dot{u}(t)$. The bilinear formulation of Eq. (13) gives us a general form

$$\tilde{g}(q, p) = 0 \tag{14}$$

where p_k, $k = 1, 2, \ldots, K$ are the K system parameters and the vector q denotes the generalized coordinates. Eq. (14), at times, may also be written as

$$\tilde{g} = a - Bq = 0 \tag{15}$$

where a is the load vector and B is the nonlinear "stiffness" matrix and a function of generalized coordinates q. The most obvious and direct way to solve Eq. (14) is by an iterative method (Burden and Faires 1993). The iteration is terminated when an 'error', i.e.

$$e = q^{(n)} - q^{(n-1)} \tag{16}$$

becomes sufficiently small. Usually some norm of the error is determined and iteration continues until this norm is sufficiently small. In this paper, the stopping criterion is to iterate until

$$\frac{\| e^{(k)} - e^{(k-1)} \|_\infty}{\| e^{(k)} \|_\infty} \leq \epsilon \; (= 5.0 \times 10^{-5}) \tag{17}$$

Transient Response Sensitivity of Nonlinear Systems: First- & Second-Order Sensitivity Derivatives

The sensitivity of the transient response of a nonlinear system can be obtained by taking the derivative of both sides of Eq. (14) with respect to p_k. This gives

$$\frac{\partial \tilde{g}_i}{\partial p_k} + \frac{\partial \tilde{g}_i}{\partial q_j}\frac{\partial q_j}{\partial p_k} = 0 \tag{18}$$

Note that from Eq. (18) it is clear that the design sensitivity equation is linear even though the analysis problem is nonlinear. $(\partial \tilde{g}_i / \partial q_j)$ is called Jacobian or the "tangent" stiffness matrix. Since the vector of generalized coordinates q_j is already available from the transient response analysis, the first derivatives of the generalized coordinates $(\partial q_j / \partial p_k)$ can be easily calculated by solving

Eq. (18).

$$\frac{\partial q_j}{\partial p_k} = -\left[\frac{\partial \tilde{g}_i}{\partial q_j}\right]^{-1} \frac{\partial \tilde{g}_i}{\partial p_k} \tag{19}$$

In matrix form, the sensitivity equation can be obtained by taking the derivatives of Eq. (15) with respect to p_k.

$$\frac{\partial a_i}{\partial p_k} - \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial p_k} q_j - \sum_{j=1}^{N}\left(\sum_{j=1}^{N} B_{ij}\frac{\partial q_j}{\partial q_m} + \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial q_m} q_j\right)\frac{\partial q_m}{\partial p_k} = 0 \tag{20}$$

This equation may be written symbolically as

$$[B_{ij}]\left\{\frac{\partial q_j}{\partial p_k}\right\} + \left[\frac{\partial B_{ij}}{\partial p_k}\right]\{q_j\} = \left\{\frac{\partial a_i}{\partial p_k}\right\} - \left\{\sum_{j=1}^{N}\left(\sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m\right)\frac{\partial q_j}{\partial p_k}\right\} \tag{21}$$

This reduces to

$$[B^{*}_{ij}]\left\{\frac{\partial q_j}{\partial p_k}\right\} + \left[\frac{\partial B_{ij}}{\partial p_k}\right]\{q_j\} = \left\{\frac{\partial a_i}{\partial p_k}\right\} \tag{22}$$

where

$$B^{*}_{ij} = B_{ij} + \sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m \tag{23}$$

Again, straightforward differentiation of Eq. (20) with respect to p_l yields

$$\begin{aligned} &\frac{\partial^2 a_i}{\partial p_k \partial p_l} - \sum_{j=1}^{N} \frac{\partial^2 B_{ij}}{\partial p_k \partial p_l} q_j - \sum_{j=1}^{N}\left(\sum_{n=1}^{N} \frac{\partial^2 B_{ij}}{\partial p_k \partial q_n} \frac{\partial q_n}{\partial p_l}\right) q_j - \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial p_k} \frac{\partial q_j}{\partial p_l} - \\ &\sum_{j=1}^{N} \frac{\partial B^{*}_{ij}}{\partial p_l} \frac{\partial q_j}{\partial p_k} - \sum_{j=1}^{N}\left(\sum_{n=1}^{N} \frac{\partial B^{*}_{ij}}{\partial q_n} \frac{\partial q_n}{\partial p_l}\right)\frac{\partial q_j}{\partial p_k} - \sum_{j=1}^{N} B^{*}_{ij} \frac{\partial^2 q_j}{\partial p_k \partial p_l} = 0 \end{aligned} \tag{24}$$

Rearranging and simplifying Eq. (24) yields

$$\begin{aligned} &\frac{\partial^2 a_i}{\partial p_k \partial p_l} - \sum_{j=1}^{N} \frac{\partial^2 B_{ij}}{\partial p_k \partial p_l} q_j - \sum_{j=1}^{N}\left(\frac{\partial B_{ij}}{\partial p_k} + \sum_{n=1}^{N} \frac{\partial^2 B_{in}}{\partial p_k \partial q_j} q_n\right)\frac{\partial q_j}{\partial p_l} - \\ &\sum_{j=1}^{N}\left(\frac{\partial B^{*}_{ij}}{\partial p_l} + \sum_{n=1}^{N} \frac{\partial B^{*}_{ij}}{\partial q_n} \frac{\partial q_n}{\partial p_l}\right)\frac{\partial q_j}{\partial p_k} - \sum_{j=1}^{N} B^{*}_{ij} \frac{\partial^2 q_j}{\partial p_k \partial p_l} = 0 \end{aligned} \tag{25}$$

Presenting Eq. (25) in a matrix form yields

$$[B^{*}_{ij}]\left\{\frac{\partial^2 q_j}{\partial p_k \partial p_l}\right\} + [B^{**}_{ij}]\left\{\frac{\partial q_j}{\partial p_l}\right\} + [B^{***}_{ij}]\left\{\frac{\partial q_j}{\partial p_k}\right\} + \left[\frac{\partial^2 B_{ij}}{\partial p_k \partial p_l}\right]\{q_j\} = \left\{\frac{\partial^2 a_i}{\partial p_k \partial p_l}\right\} \tag{26}$$

where

$$B^{**}_{ij} = \frac{\partial B_{ij}}{\partial p_k} + \sum_{n=1}^{N} \frac{\partial^2 B_{in}}{\partial p_k \partial q_j} q_n$$

$$B^{***}_{ij} = \frac{\partial B^{*}_{ij}}{\partial p_l} + \sum_{n=1}^{N} \frac{\partial B^{*}_{ij}}{\partial q_n} \frac{\partial q_n}{\partial p_l}$$

Here q_j, $(\partial q_j/\partial p_k)$, and $(\partial q_j/\partial p_l)$ are known values since they were calculated in the previous analyses.

Numerical Examples

Nonlinear Softening System

Consider a softening spring-mass system (Chen *et al.* 1993) without viscous damping

$$\ddot{u}(t) + 100\tanh u(t) = 0 \tag{27}$$

with initial conditions $u(0) = 0.0$ $\dot{u}(0) = 25.0$

The domain with a range of time $0 < t \leq 5$ is divided into twenty five elements of equal time steps of $\Delta t = 0.2$ and, unless mentioned otherwise, Legendre polynomials of the sixth degree are used as basis functions in the calculation. Fig. 1 presents the responses of the undamped case.

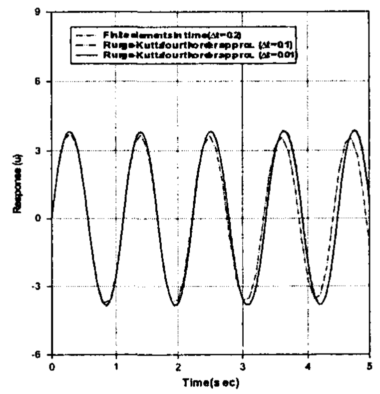

Fig. 1 Transient response of a nonlinear system without damping

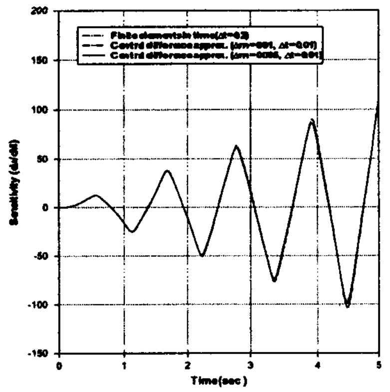

Fig. 2 Sensitivity of the response with respect to mass

For comparison, the Runge-Kutta fourth-order method (Burden and Faires 1993) for the second-order system is used to approximate the solutions using $\Delta t = 0.1$ and 0.01. To verify the results for the response sensitivities, the second-order central difference approximation is used to calculate the response sensitivities with respect to the design parameters. Given a function $u(p_k)$, the central difference approximations $(\Delta u / \Delta p_k)$ to the sensitivity (du/dp_k) of u with respect to a design parameter p_k is given as

$$\frac{\Delta u}{\Delta p_k} \cong \frac{u(p_k + \Delta p_k) - u(p_k - \Delta p_k)}{2\Delta p_k} \tag{28}$$

It is possible to employ higher-order finite difference approximations, but they are rarely used because of the high computational cost. Fig. 2 shows the sensitivities with respect to mass as obtained using the present approach and the central difference approach. The values of the central difference approximation with $\Delta t = 0.01$ and 0.005. are presented along with the result of the proposed method.

Fig. 3 presents the effect of the step size, Δp_k, on the response sensitivity. As can be seen from this Fig., as expected (Haftka and Gurdal 1992), step size plays an important role in the calculations of the response sensitivity. A step size of 0.01 appears to yield good results. The advantage of the proposed method is that there is no need to perform a convergence study as is the case in the finite difference method. Fig. 4 shows the sensitivity of the transient response with respect to the stiffness parameter for the undamped softening system. The approximate sensitivities as obtained from the central difference method using, $\Delta t = 0.1$ and 1.0, are also presented in that Figure. The step size appears to have limited effect on the sensitivity of the response with respect to the stiffness parameter.

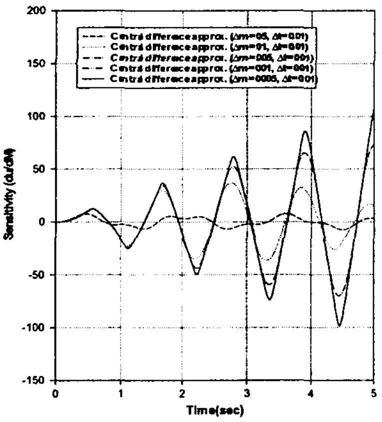

Fig. 3 Effect of step size on the response sensitivity with respect to mass

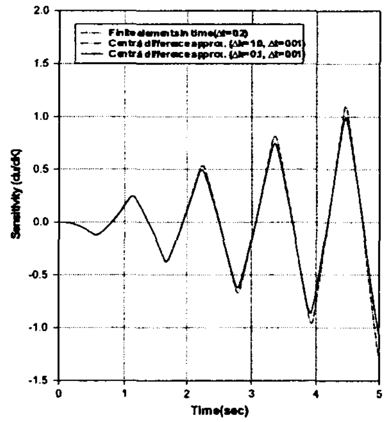

Fig. 4 Sensitivity of the response with respect to stiffness

Nonlinear Van der Pol Equation

As a second example, consider a nonlinear Van der Pol equation (Shampine 1994)

$$\ddot{u}(t) + \varepsilon(u(t)^2 - 1)\dot{u}(t) + u(t) = 0 \tag{29}$$

with initial conditions $u(0) = 2.0$ $\dot{u}(0) = 0.0$

For $\varepsilon > 0$, all non-trivial solutions converge to a limit cycle, a periodic solution. So the exact solution must oscillate between -2.0 and 2.0 in this example. Fig. 5 shows the approximate solution in the case of $\varepsilon = 5.0$ with time steps of 0.025, 0.0167 and 0.005 in the interval (0, 20). The Legendre polynomials of the fourth degree are used as basis functions. The accuracy of the solutions can be improved significantly by reducing the step size from $\Delta t = 0.025$ to 0.005. Results from the Runge-Kutta fourth-order method with a step size of 0.005 is also presented for comparison purpose. Fig. 6 shows the response sensitivity with respect to ε along with the system response.

In Fig. 7, the sensitivity results obtained from the present analytic approach are compared with those obtained using the central difference scheme. The result indicates that the peak values of the response sensitivity increases with time.

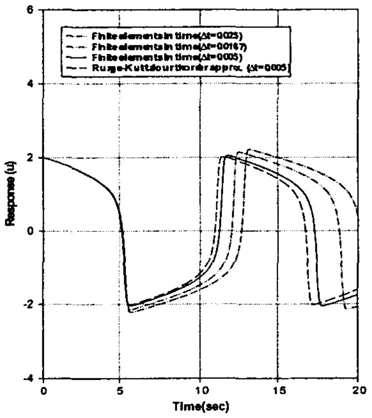

Fig. 5 Transient response of the Van der Pol equation ε=5.0

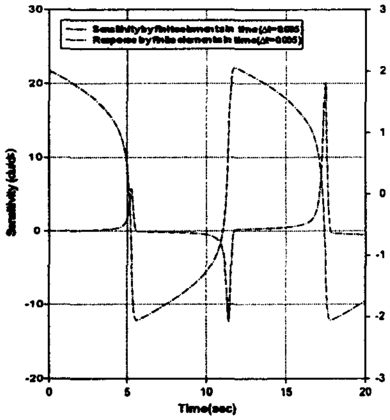

Fig. 6 Sensitivity of the response with respect to ε

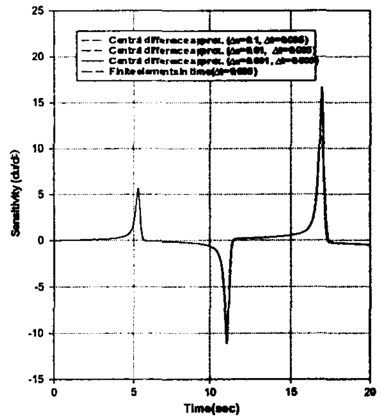

Fig.7 Comparison of the sensitivities with respect to c between TFM and CDM
$(\ddot{u}(t)+\varepsilon(u(t)^2-1)\dot{u}(t)+u(t)=0)$

System with Nonlinear Damping and Cubic Nonlinearity

The proposed method is first applied to a single-degree-of-freedom system, presented in Kapania and Park (1997) of the form

$$\ddot{u}(t)+a_1u(t)+a_2u(t)^3+a_3\dot{u}(t)+a_4\dot{u}(t)^3=0,\quad 0< t\le 30 \tag{30}$$

where

$$u(0^+)=0.0 \qquad \dot{u}(0^+)=5.0$$

and the following numerical values are given for system parameters

$$a_1 = 25.0, \quad a_2 = 2.5, \quad a_3 = 1.0, \quad a_4 = 0.1$$

Here an impulse, as the forcing function, is applied. The impulse is simulated using the initial conditions of zero displacement and an initial velocity of $\dot{u}(0)$.

Using the time finite element method based on the bilinear formulation, the second-order sensitivity equations are derived for the above equation. For the finite element discretization, the domain, $0 < t \leq 30$, is divided into three and six hundred elements of equal time steps of $\Delta t = 0.1$ and 0.05, respectively. For all the cases studied, third-degree Legendre polynomials were used as basis functions in the calculation for the hierarchic approximations. The adoption of hierarchical basis functions (Park and Kapania 1998) will enable the system matrices to be nearly diagonal and hence imply better conditioning than those with standard ones.

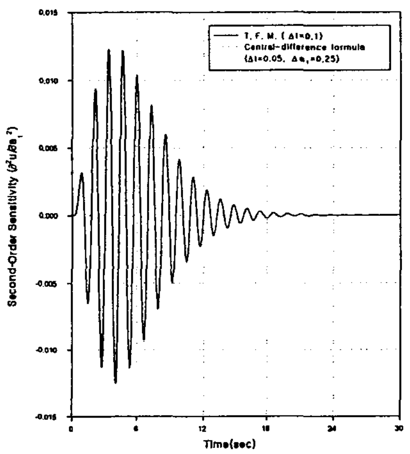

Fig. 8 Second-order response sensitivity with respect to a_1

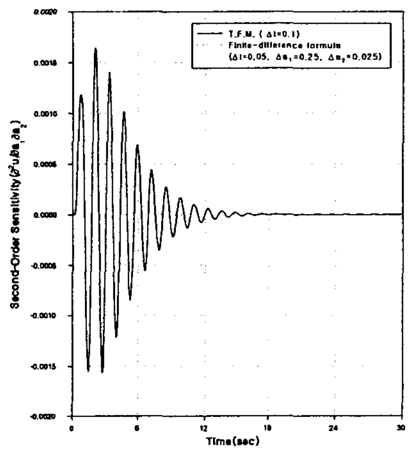

Fig. 9 Second-order response sensitivity with respect to a_1 and a_2

Fig. 8 shows the second-order sensitivity with respect to a_1. For comparison, the finite-difference methods used to verify the results for the response sensitivities. The sensitivities were obtained in two steps: At first, the responses were calculated using Newmark $\beta = 1/4$ method with a parameter, a_1, a 1% positively and negatively perturbed values from its nominal one while holding the other parameters constant. Then the sensitivity analysis was carried out using a finite-difference scheme. Fig. 9 shows the second-order sensitivity of transient response to system parameters a_1 and a_2. Also, the results show the interactions between parameters. In the same way, we may easily obtain the sensitivity results for the remaining system parameters.

The results obtained from the proposed method are not dependent upon the parameter step size while the results from the finite-difference approximation are very sensitive to the values of the parameter step size used. This is caused by the two sources of error: truncation and condition errors.

Two-degree-of-freedom system having cubic nonlinearities

As a second example, a two-degree-of-freedom system having cubic nonlinearities, associated with many physical systems such as the vibration of strings, beams and plates, and presented by Nayfeh and Mook (1979), is considered.

$$\ddot{u}_1 = -\omega_1^2 u_1 - 2\mu_1 \dot{u}_1 - \alpha_1 u_1^3 - \alpha_2 u_1^2 u_2 - \alpha_3 u_1 u_2^2 - \alpha_4 u_2^3$$
$$\ddot{u}_2 = -\omega_2^2 u_2 - 2\mu_2 \dot{u}_2 - \alpha_5 u_1^3 - \alpha_6 u_1^2 u_2 - \alpha_7 u_1 u_2^2 - \alpha_8 u_2^3 \quad 0 < t \leq 30 \tag{31}$$

with initial conditions

$$u_1(0) = 1.5 \qquad \dot{u}_1(0) = 0.0$$
$$u_2(0) = -1.0 \qquad \dot{u}_2(0) = 0.0$$

and the following values of system parameters

$$\omega_1^2 = 25.0, \quad \mu_1 = 0.35, \quad \alpha_1 = 5.0, \quad \alpha_2 = 0.5 \quad \alpha_3 = 0.25, \quad \alpha_4 = 3.0$$
$$\omega_2^2 = 17.0, \quad \mu_2 = 0.25, \quad \alpha_5 = 2.5, \quad \alpha_6 = 0.75 \quad \alpha_7 = 0.2, \quad \alpha_8 = 5.0$$

Note the nonzero initial values in displacements for the problem. For the time finite element formulation the domain with a range of time $0 < t \leq 30$ is divided into six hundred elements of equal time steps of $\Delta t = 0.05$. Once again, third-degree Legendre polynomials were used as basis functions in the response and sensitivity calculations.

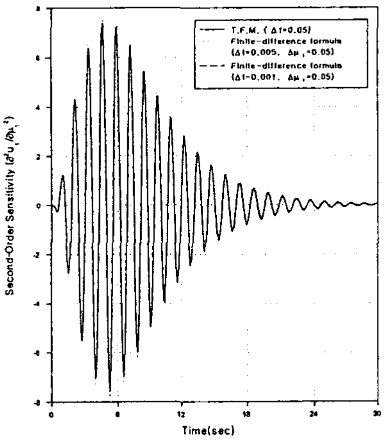

Fig. 10 Second-order response sensitivity with respect to μ_1 in the case of u_1

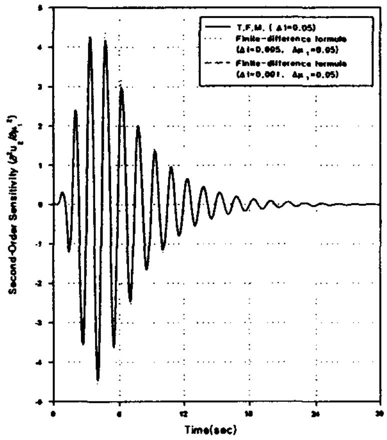

Fig. 11 Second-order response sensitivity with respect to μ_1 in the case of u_2

Plots of the second-order sensitivities with respect to μ_1 of the two responses, u_1 and u_2, are presented in Figs. 10 and 11, respectively. Note that the proposed method used ten to twenty times larger time step than the one used in finite-difference approximations. So it is easily seen that the proposed method works very effectively for the example. Note further that the comparison of Figs. 10 and 11 shows the parameter μ_1 impacts the response u_1 more than it does u_2.

Summary and conclusion

The bilinear formulation is extended to solve nonlinear transient problems. This method is easily extended to the time-finite-element formulation for the initial value problems by adopting a constraint on the test functions, $v(T_0) \neq 0$ and $v(T_f) = 0$. The bilinear formulation can be proved to be a convergent method, provided that the formulation has appropriate constraint on the test functions. The use of a Lagrange multiplier method for applying the constraint $v(T_f) = 0$ eliminates various numerical difficulties faced in earlier implementation of the method. Throughout this paper, Legendre polynomials are used as basis functions. Although not presented here, results, in some cases, were also obtained using other polynomials such as Chebyshev, Hermite, and Integral form of Legendre polynomials and were found to be almost identical to those obtained using Legendre polynomials. An advantage of the use of the orthogonal functions as basis functions is that the resulting "stiffness matrix" is numerically well behaved. Also, it is convenient to use hierarchical form (Zienkiewicz and Morgan 1983) of basis functions since it allows additional higher order basis functions within elements without changing the mesh and without removing basis functions that are already in use. As a result, one need not calculate the entire matrix anew when higher-order basis functions are added to improve the accuracy of the approximation. The present approach is thus ideal for adaptive schemes. By using a time finite element formulation, not only the transient responses but also first- and second-order sensitivities of the transient response are calculated easily (by performing a direct differentiation of the resulting algebraic equations). An advantage of the present approach over the central difference approach is that one does not need to perform a convergence study to select an appropriate step size for obtaining the sensitivities. The numerical results for the presented examples show very good agreement between the results obtained using the present approach and those available either exactly or obtained from central difference approximations. Based on the results presented here, the proposed method appears to be a good choice for calculating both the transient response and its sensitivity with respect to various system parameters for linear and nonlinear, and damped and undamped systems.

References

Achar, N. S., Gaonkar, G. H., 1993, "Helicopter trim analysis by shooting and finite element methods with optimally damped Newton iterations," *AIAA J.*, Vol. 31, pp. 225-234

Adelman, H. M., Haftka, R. T., 1986, "Sensitivity analysis of discrete structural systems," *AIAA J.*, Vol. 24, pp. 823-832

Argyris, J. H., Scharpf, D. W., 1969, "Finite element in time and space," *J. of the Royal Soc.*, Vol. 73, pp. 1041-1044

Atluri, S. N., 1973, "An assumed stress hybrid finite element model for linear elastodynamic analysis," *AIAA J.*, Vol. 11, pp. 1028-1031

Atluri, S. N., Cazzani, A., 1995, "Rotations in computational solid mechanics," *Archive of Computational Methods in Engineering*, Vol. 2, pp. 49-138

Bailey, C. D., 1975a, "A new look at Hamilton's principle," *Found. of Physics.*, Vol. 5, pp. 433-451

Bailey, C. D., 1975b, "Application of Hamilton's law of varying action," *AIAA J.*, Vol. 13, pp. 1154-1157

Bailey, C. D., 1976a, "The method of Ritz applied to the equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, pp. 235-247

Bailey, C. D., 1976b, "Hamilton, Ritz, and elastodynamics," *J. of Applied Mechanics*, Vol. 43, pp. 684-688

Baruch, M., Riff, R., 1982, "Hamilton principle, Hamilton's law 6^n correct formulations," *AIAA J*, Vol. 20, pp. 687-692

Bindolino, G. and Mantegazza, P., 1987, "Aeroelastic Derivatives as a Sensitivity Analysis of Nonlinear Equations," *AIAA J.*, Vol. 25, pp. 1145-1146

Borri, M., Ghiringhelli, G. L., Lanz, M., Mantegazza, P. and Merlini, T., 1985, "Dynamic response of mechanical systems by a weak Hamiltonian formulation," *Comput. & Struct.*, Vol. 20, pp. 495-508

Borri, M., Mello, F. and Atluri, S. N., 1990, "Variational approaches for dynamics and time-finite-elements: numerical studies," *Comp. Mech.*, Vol. 7, pp. 49-76

Borri, M., Mello, F. and Atluri, S. N., 1991, "Primal and mixed forms of Hamilton's principle for constrained rigid body systems: numerical studies," *Comp. Mech.*, Vol. 7, pp. 205-220

Burden, R. L. and Faires, J. D., 1993, *Numerical analysis*, PWS-KENT Publishing, Boston, pp.

544-551

Chen. X., Tamma, K. K. and Sha, D., 1993, "Virtual-pulse time integral methodology: A new explicit approach for computational dynamics - Theoretical developments for general non-linear structural dynamics," 34th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference. La Jolla, California, April 19-22, 1993.

Fried, I., 1969, "Finite element analysis of time dependent pheonomena," *AIAA J.*, Vol. 7, pp. 1170-1173

Gurtin, M. E., 1964a, "Variational principles for linear elastodynamics," *Arch. Rational Mec. Anal.*, Vol. 16, pp. 34-50

Gurtin, M. E., 1964b, "Variational principles for linear initial value problem," *Quarter. of Applied Math.*, Vol. 22, pp. 252-256

Haftka, R. T., 1982, "Second-Order Sensitivity Derivatives in Structural Systems," *AIAA J.*, Vol. 20, pp. 1765-1766.

Haftka, R. T. and Adelman, H. M., 1989, "Recent developments in structural sensitivity analysis," *Struct. Optim.*, Vol. 1, pp. 137-151

Haftka, R. T. and Gurdal, Z., 1992, *Elements of Structural Optimization*, Kluwer Academic Publishers, Dordrecht, pp. 398-403

Haug, E. J., 1981, "Second-Order Design Sensitivity Analysis of Structural Systems," *AIAA J.*, Vol. 19, pp. 1087-1088.

Herrera, I. and Bielak, J., 1974, "A simplified version of Gurtin's variational principles," *Arch. Rational Mec. Anal.*, Vol. 53, pp. 131-149

Hitzl, D. L. and Levinston, D. A., 1980, "Application of Hamilton's law of varying action to the restricted three-body problem," *Celestial Mech.*, Vol. 22, pp. 255-266

Kapania, R. K. and Park, S., 1997, "Parametric Identification of Nonlinear Structural Dynamic Systems using Time Finite Element Method," *AIAA J.*, Vol. 35, pp. 719-726

Kim, S. J. and Cho, J. Y., "Peneralized Weighted Residual Method for the Initial Value Problems," *AIAA J.*, Vol. 35, pp. 172-177

Mello, F., Borri, M. and Atluri, S. N., 1990, "Time finite element methods for large rotational dynamics of multibody systems," *Comput. & Struct.*, Vol. 37, pp. 231-240

Park, S., and Kapania, R. K., 1998, "Comparison of various Orthogonal Polynomials in the hp-Version Time Finite Element Method," *AIAA J.*, Vol. 36, pp. 651-655

Peters, D. A. and Izadpanah, A. P., 1988, "hp-version finite elements for the space-time domain," *Comp. Mech.*, Vol. 3, pp. 73-88

Pian, T. H. H. and O'Brien, T. F., 1957, "Transient response of continuous structures using assumed time functions," 9th Congress International Mechnique Applied, University Bruxelles. 7, pp. 350-359

Reddy, J. N., 1975, "A note on mixed variational principles for initial value problems," *Quarter. of Applied Math.*, Vol. 27, pp. 123-132

Reddy, J. N., 1976, "Modified Gurtin's variational principles in the linear dynamic theory of viscoelasticity," *Inter. J. of Solids Strut.*, Vol. 12, pp. 227-235

Riff, R. and Baruch, M., 1984a, "Stability of time finite elements," *AIAA J.*, Vol. 22, pp. 1171-1173

Riff, R. and Baruch, M., 1984b, "Time finite element discretization of Hamilton's law of varying action," *AIAA J.*, Vol. 22, pp. 1310-1318

Sandhu, R. S. and Pister, K. S., 1972, "Variational methods in continuum mechanics," Proceedings of International Conference on Variational Methods in Engineering, Southampton University, England. pp. 1.13-1.25

Shampine, L. F., 1994, *Numerical solution of ordinary differential equations*, Chapman & Hall, New York, pp. 398-403

Simkins, T. E., 1978, "Unconstrained variational statements for initial and boundary-value problems," *AIAA J.*, Vol. 16, pp. 559-563

Simkins, T. E., 1981, "Finite elements for initial value problems in dynamics," *AIAA J.*, Vol. 19, pp. 1357-1362

Smith, C. V. Jr., 1979, Comment on "Unconstrained variational statements for initial and boundary-value problems," *AIAA J.*, Vol. 17, pp. 126-127

Tiersten, H. F., 1968, "Natural boundary and initial conditions from a modification of Hamilton's principle," *J. of Math. Physics.,* Vol. 9, pp. 1445-1450

Tomovic, R., 1963, *Sensitivity analysis of dynamics systems,* McGraw-Hill Book, New York, pp. 82-89

Tonti, E., 1972, "A systematic approach to the search for variational principles," Proceedings of International Conference on Variational Methods in Engineering, Southampton University, England, pp. 1.1-1.12

Van Belle, H., 1982, "Higher Order Sensitivities in Structural Systems," *AIAA J.,* Vol. 20, pp. 286-288.

Wang, B. P. and Lu, C. M., 1993, "A new method for transient response sensitivity analysis in structural dynamics," 34th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, La Jolla, California, April 19-22

Wu, J. J., 1977, "Solution to initial value problems by use of finite elements-unconstrained variational formulations," *J. of Sound and Vibration,* Vol. 53, pp. 341-356

Zienkiewicz, O. C. and Parekh, C. J., 1970, "Transient field problems-two and three dimensional analysis by isoparametric finite elements," *Int. J. of Num. Meth. Eng.,* Vol. 2, pp. 61-71

Zienkiewicz, O. C. and Morgan, K., 1983, *Finite elements and approximation,* John Wiley & Sons, New York, pp. 231-265

Zienkiewicz, O. C. and Wood, W. L., 1987, *Transient response analysis in Finite element handbook (H. Kardestuncer and D.H. Norrie, eds.),* McGraw-Hill Book, New York, pp. 2.275-2.314