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A CHARACTERIZATION OF WEIGHTED BERGMAN-PRIVALOV SPACES ON THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. Let B denote the unit ball in $\mathbb{C}^n$, and ν the normalized Lebesgue measure on B. For $\alpha > -1$, define $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, $z \in B$. Here c_α is a positive constant such that $\nu_\alpha(B) = 1$. Let $H(B)$ denote the space of all holomorphic functions in B. For $p \geq 1$, define the Bergman-Privalov space $(AN)^p(\nu_\alpha)$ by

$$(AN)^p(\nu_\alpha) = \{f \in H(B) : \int_B \{\log(1 + |f|)\}^p d\nu_\alpha < \infty\}.$$

In this paper we prove that a function $f \in H(B)$ is in $(AN)^p(\nu_\alpha)$ if and only if $(1 + |f|)^{-2}\{\log(1 + |f|)\}^{p-2}|\tilde{\nabla} f|^2 \in L^1(\nu_\alpha)$ in the case $1 < p < \infty$, or $(1 + |f|)^{-2}|f|^{-1}|\tilde{\nabla} f|^2 \in L^1(\nu_\alpha)$ in the case $p = 1$, where $\tilde{\nabla} f$ is the gradient of f with respect to the Bergman metric on B. This is an analogous result to the characterization of the Hardy spaces by M. Stoll [18] and that of the Bergman spaces by C. Ouyang-W. Yang-R. Zhao [13].

1. Introduction

Let $n \geq 1$ be a fixed integer. Let $H(B)$ denote the space of all holomorphic functions in the unit ball $B \equiv B_n$ of the complex n-dimensional Euclidean space $\mathbb{C}^n$. Let ν denote the normalized Lebesgue measure on B. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n + \alpha + 1)/\Gamma(n + 1)\Gamma(\alpha + 1)$ and $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, $z \in B$. Note that $\nu_\alpha(B) = 1$. For each $\alpha \in (-1, \infty)$ and $p \in [1, \infty)$, we define the *weighted Bergman-Privalov space* $(AN)^p(\nu_\alpha)$ by

$$(AN)^p(\nu_\alpha)=\left\{f \in H(B) : \|f\|_{(AN)^p(\nu_\alpha)} \equiv \left[\int_B \{\log(1+|f|)\}^p d\nu_\alpha\right]^{\frac{1}{p}}< \infty\right\}.$$

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In [20], the *Privalov space* $N^p(B)$ $(1 < p < \infty)$ is defined by

$$N^p(B) = \left\{ f \in H(B) : \|f\|_{N^p(B)} \equiv \sup_{0 \leq r < 1} \left[\int_S \{\log(1 + |f_r|)\}^p d\sigma \right]^{\frac{1}{p}} < \infty \right\},$$

where σ is the normalized Lebesgue measure on the unit sphere $S \equiv \partial B$ and $f_r(z) = f(rz)$ for $0 \leq r < 1$, $z \in \mathbb{C}^n$ with $rz \in B$. In the case $n = 1$, the space $N^p(B_1)$ were firstly considered by I. I. Privalov in [14]. The properties of the spaces $N^p(B_n)$ $(n \geq 1)$ were studied in [3, 4, 5, 7, 8, 11, 17, 20, 21]. As regards the Bergman-Privalov spaces $(AN)^p(\nu)$, M. Stoll [17, p.157] gave the definition of them in the case $n = 1$. The studies on the spaces $(AN)^p(\nu)$ $(n \geq 1)$ were in [9, 10, 16, 17].

In 1995, C. Ouyang, W. Yang and R. Zhao [13] gave the following characterization of the Bergman spaces $A^p(B) \equiv H(B) \cap L^p(\nu)$, $0 < p < \infty$:

THEOREM ([13]). *Let $f \in H(B)$ and $0 < p < \infty$. Then $f \in A^p(B)$ if and only if $|f|^{p-2}|\tilde{\nabla} f|^2 \in L^1(\nu)$. Here $\tilde{\nabla} f$ is the gradient of f with respect to the Bergman metric on B. Moreover, if $f \in A^p(B)$, then*

$$\lim_{r \uparrow 1} (1 - r^2)^{n+1} \int_{rB} |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 (1 - |z|^2)^{-n-1} \, d\nu(z) = 0,$$

where $rB = \{z \in B : |z| < r\}$, $0 < r < 1$.

This characterization of the Bergman spaces is the same type as that of the Hardy spaces by M. Stoll [18]. The purpose of the present paper is to give a characterization of the Bergman-Privalov spaces $(AN)^p(\nu_\alpha)$, $1 \leq p < \infty$, which is of the M. Stoll-C. Ouyang-W. Yang-R. Zhao's type. By taking the limit as $\alpha \downarrow -1$, we also have a characterization of the Privalov spaces $N^p(B)$, $1 < p < \infty$.

2. Notations

Let $\mathcal{M}$ denote the group of biholomorphic maps of B onto itself. For each $a \in B$, let $\varphi_a \in \mathcal{M}$ be the involution described in [15, p.25]. Let λ be the measure on B defined by

$$d\lambda(z) = \frac{1}{(1 - |z|^2)^{n+1}} d\nu(z), \; z \in B.$$

Then λ is the invariant volume measure induced by the Bergman metric on B. Thus

$$\int_B f\,d\lambda = \int_B (f \circ \psi)\,d\lambda$$

for each $f \in L^1(\lambda)$ and all $\psi \in \mathcal{M}$ ([15, Theorem 2.2.6]). For $f \in C^2(B)$ and $a \in B$, define

$$\tilde{\Delta} f(a) = \frac{1}{n+1}\Delta(f \circ \varphi_a)(0).$$

Then as in [15, Theorem 4.1.3],

$$\tilde{\Delta} f(a) = \frac{4}{n+1}(1-|a|^2)\sum_{i,j=1}^{n}(\delta_{ij} - a_i\bar{a}_j)\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(a).$$

The operator $\tilde{\Delta}$ is invariant under $\mathcal{M}$, that is, $\tilde{\Delta}(f\circ\psi) = (\tilde{\Delta} f)\circ\psi$ for all $\psi \in \mathcal{M}$ ([15, Theorem 4.1.2]). Let $\tilde{\nabla}$ denote the gradient with respect to the Bergman metric on B ([19, p.27]). Then as in [19, p.30],

$$|\tilde{\nabla} f(a)|^2 = \frac{2}{n+1}(1-|a|^2)\left[\sum_{j=1}^{n}\left|\frac{\partial f}{\partial z_j}(a)\right|^2 - \left|\sum_{j=1}^{n} a_j \frac{\partial f}{\partial z_j}(a)\right|^2\right], \ a \in B.$$

An upper semicontinuous function $u : B \longrightarrow [-\infty, \infty)$, $u \not\equiv -\infty$, is said to be $\mathcal{M}$-*subharmonic* if for each $a \in B$

$$u(a) \le \int_B u(\varphi_a(r\zeta))\,d\sigma(\zeta),\ 0 < r < 1.$$

A continuous function u defined in B is said to be $\mathcal{M}$-*harmonic* if equality holds in the above inequality. A function u in B is said to be $\mathcal{M}$-*superharmonic* if $-u$ is $\mathcal{M}$-subharmonic.

By [19, §6.2], the invariant Green's function on B is given by $G(z,a) = g(\varphi_a(z))$ for $(z,a) \in B \times B$, where

$$g(z) = \frac{n+1}{2n}\int_{|z|}^{1}(1-t^2)^{n-1}t^{-2n+1}\,dt.$$

Note that g is $\mathcal{M}$-harmonic in $B \setminus \{0\}$, and $\mathcal{M}$-superharmonic in B. Let f be an $\mathcal{M}$-subharmonic function in B. The *Riesz measure* of f is the non-negative regular Borel measure μ_f in B which satisfies

$$\int_B \psi\,d\mu_f = \int_B f\tilde{\Delta}\psi\,d\lambda$$

for all $\psi \in C_c^2(B)$. Here $C_c^2(B)$ is the class of twice continuously differentiable functions in B with compact support. If f is in $C^2(B)$, then by Green's identity [19, Proposition 3.1] $d\mu_f = \tilde{\Delta} f d\lambda$.

3. Preliminaries

LEMMA 1. *Suppose $\alpha \in (-1,\infty)$, $p \in [1,\infty)$ and $f \in (AN)^p(\nu_\alpha)$. Then*

$$\lim_{r\uparrow 1} \|f_r - f\|_{(AN)^p(\nu_\alpha)} = 0.$$

Proof. (cf. [6, Lemma 1.1]) Pick $\varepsilon > 0$. Then there exists an $r_0 \in (0,1)$ such that

$$\int_{B\setminus r_0B} \{\log(1+|f|)\}^p \, d\nu_\alpha < \varepsilon.$$

Since $\{\log(1+|f|)\}^p$ is subharmonic in B, for any $r \in (0,1)$

$$(1) \qquad \int_{B\setminus r_0B} \{\log(1+|f_r|)\}^p \, d\nu_\alpha \le \int_{B\setminus r_0B} \{\log(1+|f|)\}^p \, d\nu_\alpha < \varepsilon.$$

The uniform continuity of $\{\log(1+|f|)\}^p$ on the compact set $r_0\bar{B}$ implies that

$$(2) \qquad \lim_{r\uparrow 1} \int_{r_0B} \{\log(1+|f_r - f|)\}^p \, d\nu_\alpha = 0.$$

(1) and (2) prove the lemma. □

LEMMA 2. *Suppose $f \in H(B)$ and $p \in (1,\infty)$. then*

$$\lim_{\alpha\downarrow -1} \|f\|_{(AN)^p(\nu_\alpha)} = \|f\|_{N^p(B)}.$$

Proof. (cf. [1, p.25]) First we consider the case $\|f\|_{N^p(B)} < \infty$. It follows from the subharmonicity of $\{\log(1+|f|)\}^p$ that $\|f\|_{(AN)^p(\nu_\alpha)} \le \|f\|_{N^p(B)} < \infty$ for any $\alpha \in (-1,\infty)$. Pick $\varepsilon > 0$. By [20, Theorem 4], it holds that

$$(3) \qquad \lim_{r\uparrow 1} \|f_r - f\|_{N^p(B)} = 0.$$

By (3), there exists an $r_0 \in (0,1)$ such that for $r \in [r_0, 1)$

$$(4) \qquad \|f_r - f\|_{N^p(B)} < \varepsilon.$$

Fix $r_1 \in (r_0, 1)$. Since $\{\log(1+|f_{r_1}|)\}^p \in C(\bar{B})$, it holds that

$$\lim_{\alpha \downarrow -1} \|f_{r_1}\|_{(AN)^p(\nu_\alpha)} = \|f_{r_1}\|_{N^p(B)}. \tag{5}$$

(See [1, §0.3].) Using (4), we have for $\alpha \in (-1, \infty)$,

$$\begin{aligned}
&\|f_{r_1} - f\|^p_{(AN)^p(\nu_\alpha)} = \int_B \{\log(1+|f_{r_1} - f|)\}^p \, d\nu_\alpha \\
&= c_\alpha 2n \left(\int_0^{\frac{r_0}{r_1}} + \int_{\frac{r_0}{r_1}}^1 \right) t^{2n-1}(1-t^2)^\alpha \, dt \int_S \{\log(1+|f_{r_1 t} - f_t|)\}^p \, d\sigma \\
&\le c_\alpha 2n \int_0^{\frac{r_0}{r_1}} t^{2n-1}(1-t^2)^\alpha \, dt \cdot (2\|f\|_{N^p(B)})^p \\
&\quad + c_\alpha 2n \int_{\frac{r_0}{r_1}}^1 t^{2n-1}(1-t^2)^\alpha (\|f_{r_1 t} - f\|_{N^p(B)} + \|f - f_t\|_{N^p(B)})^p \, dt \\
&\le (2\|f\|_{N^p(B)})^p c_\alpha 2n \int_0^{\frac{r_0}{r_1}} (1-t^2)^\alpha \, dt + (2\varepsilon)^p c_\alpha 2n \int_{\frac{r_0}{r_1}}^1 t^{2n-1}(1-t^2)^\alpha \, dt \\
&< (2\|f\|_{N^p(B)})^p \frac{2n\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+2)} \{1 - (1 - \frac{r_0}{r_1})^{\alpha+1}\} + (2\varepsilon)^p.
\end{aligned}$$

Hence

$$\limsup_{\alpha \downarrow -1} \|f_{r_1} - f\|_{(AN)^p(\nu_\alpha)} \le 2\varepsilon. \tag{6}$$

(4), (5) and (6) prove the lemma in the case $\|f\|_{N^p(B)} < \infty$.

Suppose now that $\|f\|_{N^p(B)} = \infty$. Pick an arbitrary number $M \in (0, \infty)$. Then there exists an $\rho_0 \in (0,1)$ such that for $r \in [\rho_0, 1)$

$$\int_S \{\log(1+|f_r|)\}^p \, d\sigma > 2M.$$

Using this, we have for $\alpha \in (-1, \infty)$

$$\begin{aligned}
\|f\|^p_{(AN)^p(\nu_\alpha)} &\geq c_\alpha 2n \int_{\rho_0}^1 r^{2n-1}(1-r^2)^\alpha \, dr \int_S \{\log(1+|f_r|)\}^p \, d\sigma \\
&\geq c_\alpha 2n \int_{\rho_0}^1 r^{2n-1}(1-r^2)^\alpha \, dr \cdot 2M \\
&= 2M \left\{ 1 - c_\alpha 2n \int_0^{\rho_0} r^{2n-1}(1-r^2)^\alpha \, dr \right\} \\
&\geq 2M \left\{ 1 - \frac{2n\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^{\rho_0} (1-r)^\alpha \, dr \right\} \\
&\geq 2M \left(1 - \frac{2n\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+2)} \{1-(1-\rho_0)^{\alpha+1}\} \right).
\end{aligned}$$

This implies that $\lim_{\alpha\downarrow -1} \|f\|_{(AN)^p(\nu_\alpha)} = +\infty = \|f\|_{N^p(B)}$. □

By a simple computation we can prove the following lemma and its corollary.

LEMMA 3. *Suppose $f \in H(B)$, $1 \leq p < \infty$ and $\varepsilon > 0$. Then*

$$\tilde{\Delta}(\{\log(1+|f|)\}^p) = \frac{p}{2}\frac{1}{(1+|f|)^2}\left[(p-1) + \frac{\log(1+|f|)}{|f|}\right] \times \{\log(1+|f|)\}^{p-2}|\tilde{\nabla} f|^2$$

in $B \setminus Z(f)$, where $Z(f) = \{z \in B : f(z) = 0\}$, and

$$\begin{aligned}
&\tilde{\Delta}\left(\left[\log(1+(|f|^2+\varepsilon)^{1/2})\right]^p\right) \\
&= \frac{p}{2}(|f|^2+\varepsilon)^{-3/2}\{1+(|f|^2+\varepsilon)^{1/2}\}^{-2}\left[\log(1+(|f|^2+\varepsilon)^{1/2})\right]^{p-2} \\
&\quad \times \Big[(p-1)|f|^2(|f|^2+\varepsilon)^{1/2} + \{|f|^2 + 2\varepsilon(1+(|f|^2+\varepsilon)^{1/2})\} \\
&\quad \times \{\log(1+(|f|^2+\varepsilon)^{1/2})\}\Big]|\tilde{\nabla} f|^2
\end{aligned}$$

in B.

COROLLARY 1. *Suppose $f \in H(B)$ and $1 \leq p < \infty$. Then*

$$\lim_{\varepsilon\downarrow 0} \tilde{\Delta}\left(\left[\log(1+(|f|^2+\varepsilon)^{1/2})\right]^p\right) = \tilde{\Delta}(\{\log(1+|f|)\}^p)$$

in $B \setminus Z(f)$.

LEMMA 4. *Let* $f \in H(B) \setminus \{0\}$ *and* $1 \le p < \infty$. *Then*

(a) $\tilde{\Delta}(\{\log(1+|f|)\}^p) \in L^1_{loc}(\lambda)$.

(b) $d\mu_{\{\log(1+|f|)\}^p} = \tilde{\Delta}(\{\log(1+|f|)\}^p) d\lambda$.

Proof. Put $v = \{\log(1+|f|)\}^p$ in B. For $\varepsilon \in (0,1)$, let v_ε be defined by $v_\varepsilon = \left[\left(\log(1+(|f|^2+\varepsilon)^{1/2})\right)\right]^p$ in B. And define

$$\varphi_\varepsilon(t) = (t+\varepsilon)^{-3/2}\{1+(t+\varepsilon)^{1/2}\}^{-2}\Big[(p-1)t(t+\varepsilon)^{1/2} + \{t+2\varepsilon(1+(t+\varepsilon)^{1/2})\}\{\log(1+(t+\varepsilon)^{1/2})\}\Big]$$

for $t > -\varepsilon$. Then $v_\varepsilon \in C^\infty(B)$ and

$$(7) \qquad v_\varepsilon \to v \text{ uniformly on compact subsets of } B \text{ as } \varepsilon \downarrow 0.$$

We can also easily see that

$$(8) \qquad 0 < \varphi_\varepsilon(t) < p+2 \text{ if } t \ge 0.$$

By Lemma 3, (8) and Corollary 1, it holds that for $\varepsilon \in (0,1)$, in B

$$\begin{aligned} 0 \le \tilde{\Delta} v_\varepsilon &= \frac{p}{2}\varphi_\varepsilon(|f|^2)\left[\log(1+(|f|^2+\varepsilon)^{1/2})\right]^{p-2}|\tilde{\nabla} f|^2 \\ &\le \frac{p}{2}(p+2)\left[\log(1+(|f|^2+\varepsilon)^{1/2})\right]^{p-2}|\tilde{\nabla} f|^2 \qquad (9) \end{aligned}$$

and, in $B \setminus Z(f)$

$$(10) \qquad \lim_{\varepsilon \downarrow 0} \tilde{\Delta} v_\varepsilon = \tilde{\Delta} v.$$

Since f is holomorphic and $f \not\equiv 0$ in B,

$$(11) \qquad \lambda(Z(f)) = 0.$$

By Lemma 3, in $B \setminus Z(f)$

$$(12) \qquad \{\log(1+|f|)\}^{p-2}|\tilde{\nabla} f|^2 = \frac{2}{p}\frac{(1+|f|)^2}{(p-1)+\frac{\log(1+|f|)}{|f|}}\tilde{\Delta} v.$$

When $1 \le p \le 2$, it follows from (12) that in $B \setminus Z(f)$

$$(13) \quad \{\log(1+|f|)\}^{p-2}|\tilde{\nabla} f|^2 \le (\log 2)^{p-2}|\tilde{\nabla} f|^2 + \frac{2}{p}\frac{4}{(p-1)+\log 2}\tilde{\Delta} v.$$

By (9) and (13), for $\varepsilon \in (0,1)$, in B

$$(14) \quad 0 \le \tilde{\Delta} v_\varepsilon \le \frac{p}{2}(p+2)\left[(\log 2)^{p-2}|\tilde{\nabla} f|^2 + \frac{2}{p}\frac{4}{(p-1)+\log 2}\tilde{\Delta} v\right]$$

if $1 \le p \le 2$. In the case $2 < p < \infty$, by (9), for $\varepsilon \in (0,1)$, in B

$$0 \le \tilde{\Delta} v_\varepsilon \le \frac{p}{2}(p+2)\left[\log(1+(|f|^2+1)^{1/2})\right]^{p-2} |\tilde{\nabla} f|^2. \tag{15}$$

Let K be any compact subset of B, and let $\psi \in C_c^2(B)$ with $\psi \ge 0$ in B such that $\psi \equiv 1$ on K. Using (10), (11), Fatou's lemma and Green's identity, we have

$$\begin{aligned} 0 \le \int_K \tilde{\Delta} v \, d\lambda &\le \liminf_{\varepsilon \downarrow 0} \int_K \tilde{\Delta} v_\varepsilon \, d\lambda = \liminf_{\varepsilon \downarrow 0} \int_K \psi \tilde{\Delta} v_\varepsilon \, d\lambda \\ &\le \liminf_{\varepsilon \downarrow 0} \int_B \psi \tilde{\Delta} v_\varepsilon \, d\lambda = \liminf_{\varepsilon \downarrow 0} \int_B v_\varepsilon \tilde{\Delta} \psi \, d\lambda. \end{aligned} \tag{16}$$

Since $\tilde{\Delta}\psi$ is a continuous function with compact support in B, by (7)

$$\liminf_{\varepsilon \downarrow 0} \int_B v_\varepsilon \tilde{\Delta} \psi \, d\lambda = \int_B v \tilde{\Delta} \psi \, d\lambda < \infty. \tag{17}$$

(16) and (17) show the assertion (a), that is,

$$\tilde{\Delta} v \in L^1_{loc}(\lambda). \tag{18}$$

In order to prove (b), pick $\phi \in C_c^2(B)$. By (14), (15) and (18), the functions $\{\tilde{\Delta} v_\varepsilon : 0 < \varepsilon < 1\}$ are dominated by a function in $L^1_{loc}(\lambda)$. By Lebesgue's dominated convergence theorem, Green's identity and (7), we have

$$\int_B \phi \tilde{\Delta} v \, d\lambda = \lim_{\varepsilon \downarrow 0} \int_B \phi \tilde{\Delta} v_\varepsilon \, d\lambda = \lim_{\varepsilon \downarrow 0} \int_B v_\varepsilon \tilde{\Delta} \phi \, d\lambda = \int_B v \tilde{\Delta} \phi \, d\lambda.$$

This proves (b). □

LEMMA 5. *Let f be a non-negative measurable function in B. Then the following inequalities hold:*

$$\begin{aligned} \frac{1}{2} \int_B f(z)(1-|z|^2)^{\beta+1} \, d\lambda(z) &\le \int_0^1 dt \int_{tB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \\ &\le \int_B f(z)(1-|z|^2)^{\beta+1} \, d\lambda(z) \end{aligned} \tag{19}$$

for all $\beta \in \mathbb{R}$. And

$$(1-r^2)^{\beta+1} \int_{rB} f \, d\lambda \le 2 \int_r^1 dt \int_{tB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \tag{20}$$

for all $r \in (0,1)$ and $\beta \in \mathbb{R}$ with $\beta \ge 0$.

Proof. For $\beta \in \mathbb{R}$,

$$\int_0^1 dt \int_{tB} f(z)(1-|z|^2)^\beta \, d\lambda(z) = \int_B f(z)(1-|z|^2)^{\beta+1} \frac{1}{1+|z|} \, d\lambda(z).$$

Since $\frac{1}{2} < \frac{1}{1+|z|} \le 1$ for all $z \in B$, we get (19).

Fix $r \in (0,1)$ and $\beta \ge 0$. Then

$$\begin{aligned} (1-r^2)^{\beta+1} \int_{rB} f \, d\lambda &\le (1-r^2) \int_{rB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \\ &\le 2 \int_{rB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \int_r^1 dt. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} &\int_r^1 dt \int_{tB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \\ &= \int_{rB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \int_r^1 dt + \int_{B \setminus rB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \int_{|z|}^1 dt \\ &\ge \int_{rB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \int_r^1 dt. \end{aligned} \tag{22}$$

(20) follows from (21) and (22). □

COROLLARY 2. *Suppose that f is a non-negative measurable function in B, $\beta \in [0,\infty)$ and*

$$\int_B f(z)(1-|z|^2)^{\beta+1} \, d\lambda(z) < \infty. \tag{23}$$

Then

$$\lim_{r \uparrow 1} \left[(1-r^2)^{\beta+1} \int_{rB} f \, d\lambda \right] = 0. \tag{24}$$

Proof. By (23) and (19), we have

$$\lim_{r \uparrow 1} \left[\int_r^1 dt \int_{tB} f(z)(1-|z|^2)^\beta \, d\lambda(z) \right] = 0. \tag{25}$$

(24) follows from (20) and (25). □

4. Main results

The proof of the following theorem goes along the same line as that of C. Ouyang-W. Yang-R. Zhao's theorem ([13, Theorem 1]).

THEOREM 1. *Let* $-1 < \alpha < \infty$ *and* $1 \le p < \infty$.

(a) *Every* $f \in H(B) \setminus \{0\}$ *satisfies the following inequalities*:

$$\begin{aligned} &\frac{a_n \Gamma(n+\alpha+1)}{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)} \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^\alpha \, d\nu(z) \\ &\quad + \{\log(1+|f(0)|)\}^p \\ &\le \|f\|^p_{(AN)^p(\nu_\alpha)} \\ &\le \frac{b_n 2^{\alpha+\alpha^+}\Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+2)} \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^\alpha \, d\nu(z) \\ &\quad + \int_S \{\log(1+|f_{\frac{1}{2}}|)\}^p \, d\sigma, \end{aligned}$$

where

$$a_n = \frac{n+1}{2^{n+2}\Gamma(n+1)}, \qquad b_n = \frac{2^{3n-1}(n+1)}{\Gamma(n+1)}.$$

(b) *A function* $f \in H(B) \setminus \{0\}$ *is in* $(AN)^p(\nu_\alpha)$ *if and only if*

$$\int_B \tilde{\Delta}(\{\log(1+|f|)\}^p) \, d\nu_\alpha < \infty.$$

(c) *Suppose* $1 < p < \infty$ *and* $f \in H(B)$. *Then* $f \in (AN)^p(\nu_\alpha)$ *if and only if*

$$\int_B \frac{\{\log(1+|f|)\}^{p-2}}{(1+|f|)^2} |\tilde{\nabla} f|^2 \, d\nu_\alpha < \infty.$$

(d) *A function* $f \in H(B) \setminus \{0\}$ *is in* $(AN)^1(\nu_\alpha)$ *if and only if*

$$\int_B \frac{|\tilde{\nabla} f|^2}{|f|(1+|f|)^2} \, d\nu_\alpha < \infty.$$

(e) *If* $f \in (AN)^p(\nu_\alpha) \setminus \{0\}$, *then*

$$\lim_{r \uparrow 1} \left[(1-r^2)^{n+1+\alpha} \int_{rB} \tilde{\Delta}(\{\log(1+|f|)\}^p) \, d\lambda \right] = 0.$$

(f) *Suppose* $1 < p < \infty$ *and* $f \in (AN)^p(\nu_\alpha)$. *Then*

$$\lim_{r \uparrow 1} \left[(1-r^2)^{n+1+\alpha} \int_{rB} \frac{\{\log(1+|f|)\}^{p-2}}{(1+|f|)^2} |\tilde{\nabla} f|^2 \, d\lambda \right] = 0.$$

(g) *If* $f \in (AN)^1(\nu_\alpha) \setminus \{0\}$, *then*

$$\lim_{r \uparrow 1} \left[(1-r^2)^{n+1+\alpha} \int_{rB} \frac{|\tilde{\nabla} f|^2}{|f|(1+|f|)^2} \, d\lambda \right] = 0.$$

Proof. For each $\varepsilon > 0$, let

$$v_\varepsilon = \left[\log(1 + (|f|^2 + \varepsilon)^{1/2}) \right]^p.$$

Then $v_\varepsilon \in C^\infty(B)$. For $0 < \delta < r < 1$, let $\Omega_{\delta r} = \{z \in B : \delta < |z| < r\}$. By Green's formula as in [2, §3.3]

$$\begin{aligned} &\int_{\Omega_{\delta r}} \left[\{g - g(re_1)\} \tilde{\Delta} v_\varepsilon - v_\varepsilon \tilde{\Delta} \{g - g(re_1)\} \right] d\tilde{\tau} \\ &= \int_{\partial \Omega_{\delta r}} \left[\{g - g(re_1)\} \frac{\partial v_\varepsilon}{\partial \tilde{n}} - v_\varepsilon \frac{\partial}{\partial \tilde{n}} \{g - g(re_1)\} \right] d\tilde{\sigma}, \end{aligned}$$

where $d\tilde{\tau}$ is the volume element on B determined by the Bergman metric, $d\tilde{\sigma}$ is the surface area element on $\partial \Omega_{\delta r}$ determined by the Bergman metric, and $\frac{\partial}{\partial \tilde{n}}$ denotes the outward normal differentiation along $\partial \Omega_{\delta r}$ with respect to the Bergman metric. Note that $\tilde{\Delta}(\{g - g(re_1)\}) = 0$ in $\Omega_{\delta r}$, $g - g(re_1) = 0$ on $rS \equiv \{z \in \mathbb{C}^n : |z| = r\}$, and $g - g(re_1) = g(\delta e_1) - g(re_1)$ on δS. Thus

$$\begin{aligned} &\int_{\Omega_{\delta r}} \{g - g(re_1)\} \tilde{\Delta} v_\varepsilon \, d\tilde{\tau} \\ &= -\{g(\delta e_1) - g(re_1)\} \int_{\delta S} \frac{\partial v_\varepsilon}{\partial \tilde{n}} \, d\tilde{\sigma}_\delta - \int_{rS} v_\varepsilon \frac{\partial g}{\partial \tilde{n}} \, d\tilde{\sigma}_r + \int_{\delta S} v_\varepsilon \frac{\partial g}{\partial \tilde{n}} \, d\tilde{\sigma}_\delta, \end{aligned} \tag{26}$$

where $d\tilde{\sigma}_t$ is the surface area element on tS. By [2, p.18, (6)],

$$d\tilde{\tau}(z) = \frac{\omega_n (n+1)^n}{2n(1-|z|^2)^{n+1}} d\nu(z) = \frac{\omega_n (n+1)^n}{2n} d\lambda(z), \quad z \in B, \tag{27}$$

where ω_n denotes the Euclidean surface area of S. Since $\{g - g(re_1)\}$ $(\tilde{\Delta} v_\varepsilon) \in L^1(\tilde{\tau})$ on rB, (27) gives

$$\begin{aligned} \lim_{\delta \downarrow 0} \int_{\Omega_{\delta r}} \{g - g(re_1)\} \tilde{\Delta} v_\varepsilon \, d\tilde{\tau} &= \int_{rB} \{g - g(re_1)\} \tilde{\Delta} v_\varepsilon \, d\tilde{\tau} \\ &= \frac{\omega_n (n+1)^n}{2n} \int_{rB} \{g - g(re_1)\} \tilde{\Delta} v_\varepsilon \, d\lambda. \end{aligned} \tag{28}$$

By [2, p.20, (10)],

$$\frac{\partial g}{\partial \tilde{n}} = -\frac{(n+1)^{1/2}}{2n}\frac{(1-t^2)^n}{t^{2n-1}} \quad \text{on } tS \quad (0 < t < 1). \tag{29}$$

By [2, the bottom line of p.19], for $t \in (0,1)$ and $\zeta \in S$,

$$\begin{aligned} \left|\frac{\partial v_\varepsilon}{\partial \tilde{n}}(t\zeta)\right| &= \frac{2}{\sqrt{n+1}}(1-t^2)\left|\Re\left[\sum_{j=1}^{n} \zeta_j \frac{\partial v_\varepsilon}{\partial z_j}(t\zeta)\right]\right| \\ &\leq \frac{2}{\sqrt{n+1}}(1-t^2)\sum_{j=1}^{n}\left|\frac{\partial v_\varepsilon}{\partial z_j}(t\zeta)\right|. \end{aligned} \tag{30}$$

By [2, p.19, (7)], for $t \in (0,1)$ and $\zeta \in S$,

$$d\tilde{\sigma}_t(t\zeta) = \frac{\omega_n (n+1)^{n-1/2} t^{2n-1}}{(1-t^2)^n} d\sigma(\zeta). \tag{31}$$

By (30) and (31),

$$\begin{aligned} &\left|\{g(\delta e_1) - g(re_1)\}\int_{\delta S} \frac{\partial v_\varepsilon}{\partial \tilde{n}}\, d\tilde{\sigma}_\delta\right| \\ &\leq \frac{2\omega_n (n+1)^{n-1}\delta^{2n-1}}{(1-\delta^2)^{n-1}} g(\delta e_1)\int_S \sum_{j=1}^{n}\left|\frac{\partial v_\varepsilon}{\partial z_j}(\delta\zeta)\right| d\sigma(\zeta). \end{aligned} \tag{32}$$

Since $\lim_{\delta\downarrow 0}\delta^{2n-1}g(\delta e_1) = 0$ [19, p.65, (6.6)], it follows from (32) that

$$\lim_{\delta\downarrow 0}\{g(\delta e_1) - g(re_1)\}\int_{\delta S}\frac{\partial v_\varepsilon}{\partial \tilde{n}}\, d\tilde{\sigma}_\delta = 0. \tag{33}$$

Moreover, (29) and (31) give

$$\lim_{\delta\downarrow 0}\int_{\delta S} v_\varepsilon \frac{\partial g}{\partial \tilde{n}}\, d\tilde{\sigma}_\delta = -\frac{\omega_n (n+1)^n}{2n} v_\varepsilon(0) \tag{34}$$

and

$$-\int_{rS} v_\varepsilon \frac{\partial g}{\partial \tilde{n}}\, d\tilde{\sigma}_r = \frac{\omega_n (n+1)^n}{2n}\int_S (v_\varepsilon)_r\, d\sigma. \tag{35}$$

By (26), (28), (33), (34) and (35), we have

$$\int_{rB}\{g - g(re_1)\}\tilde{\Delta} v_\varepsilon\, d\lambda = \int_S (v_\varepsilon)_r\, d\sigma - v_\varepsilon(0). \tag{36}$$

Since $(v_\varepsilon)_r \to \{\log(1+|f_r|)\}^p$ uniformly on S as $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon\downarrow 0}\int_S (v_\varepsilon)_r\, d\sigma = \int_S \{\log(1+|f_r|)\}^p\, d\sigma. \tag{37}$$

By (36) and (37),

(38)
$$\lim_{\varepsilon\downarrow 0}\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v_\varepsilon\,d\lambda=\int_S\{\log(1+|f_r|)\}^p\,d\sigma-\{\log(1+|f(0)|)\}^p.$$

By Corollary 1, Fatou's lemma and (38),

$$\begin{aligned}\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v\,d\lambda&\le\liminf_{\varepsilon\downarrow 0}\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v_\varepsilon\,d\lambda\\&=\int_S\{\log(1+|f_r|)\}^p d\sigma-\{\log(1+|f(0)|)\}^p<\infty,\end{aligned}$$

where $v=\{\log(1+|f|)\}^p$. Thus

$$\{g-g(re_1)\}\tilde{\Delta}v\in L^1(\lambda,rB). \tag{39}$$

By (9) and (13) for any $\varepsilon\in(0,1]$,

$$\begin{aligned}&\frac{2}{p(p+2)}\{g-g(re_1)\}\tilde{\Delta}v_\varepsilon\\&\le\{g-g(re_1)\}\left[\log(1+|f|)\right]^{p-2}|\tilde{\nabla}f|^2\\&\le\{g-g(re_1)\}\left\{(\log 2)^{p-2}|\tilde{\nabla}f|^2+\frac{2}{p}\frac{4}{(p-1)+\log 2}\tilde{\Delta}v\right\}\\&=(\log 2)^{p-2}|\tilde{\nabla}f|^2\{g-g(re_1)\}+\frac{2}{p}\frac{4}{(p-1)+\log 2}\{g-g(re_1)\}\tilde{\Delta}v\end{aligned}\tag{40}$$

in $rB\setminus Z(f)$, if $1\le p\le 2$. Since $g-g(re_1)\in L^1_{loc}(\lambda)$, by (39),

$$\begin{aligned}&(\log 2)^{p-2}|\tilde{\nabla}f|^2\{g-g(re_1)\}+\frac{2}{p}\frac{4}{(p-1)+\log 2}\{g-g(re_1)\}(\tilde{\Delta}v)\\&\qquad\in L^1(\lambda,rB).\end{aligned}\tag{41}$$

If $2<p<\infty$, then for any $\varepsilon\in(0,1]$,

(42)
$$\frac{2}{p(p+2)}\{g-g(re_1)\}\tilde{\Delta}v_\varepsilon\le\{g-g(re_1)\}\left[\log(1+(|f|^2+1)^{1/2})\right]^{p-2}|\tilde{\nabla}f|^2$$

in $rB\setminus Z(f)$. Since $g-g(re_1)\in L^1_{loc}(\lambda)$,

$$\{g-g(re_1)\}\left[\log(1+(|f|^2+1)^{1/2})\right]^{p-2}|\tilde{\nabla}f|^2\in L^1(\lambda,rB). \tag{43}$$

By Corollary 1 and (40) $\sim$ (43), we can apply Lebesgue's dominated convergence theorem to (38). Hence we have

(44)
$$\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v\,d\lambda=\lim_{\varepsilon\downarrow 0}\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v_\varepsilon\,d\lambda=\int_S v_r\,d\sigma-v(0).$$

The left hand side of (44) is

$$\begin{aligned}\int_{rB}\{g-g(re_1)\}\tilde{\Delta}v\,d\lambda&=\frac{n+1}{2n}\int_{rB}\tilde{\Delta}v\,d\lambda\int_{|z|}^{r}(1-t^2)^{n-1}t^{-2n+1}\,dt\\&=\frac{n+1}{2n}\int_0^r(1-t^2)^{n-1}t^{-2n+1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda\\&\geq\frac{n+1}{2n}r^{-2n+1}\int_0^r(1-t^2)^{n-1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda.\end{aligned}\tag{45}$$

On the other hand, for $0<r<1$, there exists a positive integer k such that $1/2^k\leq r<1/2^{k-1}$. Thus

$$\begin{aligned}&\frac{n+1}{2n}\int_0^r(1-t^2)^{n-1}t^{-2n+1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda\\&=\frac{n+1}{2n}\left(\int_0^{\frac{1}{2^k}}+\int_{\frac{1}{2^k}}^{r}\right)(1-t^2)^{n-1}t^{-2n+1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda\\&\leq\frac{n+1}{2n}\int_0^{\frac{1}{2}}(1-t^2)^{n-1}t^{-2n+1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda\\&\quad+\frac{n+1}{2n}2^{k(2n-1)}\int_0^r(1-t^2)^{n-1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda\\&\equiv I_1+I_2.\end{aligned}\tag{46}$$

By (44) $\sim$ (46),

$$I_1=\int_{\frac{1}{2}B}\{g-g(2^{-1}e_1)\}\tilde{\Delta}v\,d\lambda=\int_S v_{\frac{1}{2}}\,d\sigma-v(0)<\infty.\tag{47}$$

Since $2^{k(2n-1)}<2^{2n-1}r^{-2n+1}$,

$$I_2\leq\frac{n+1}{2n}2^{2n-1}r^{-2n+1}\int_0^r(1-t^2)^{n-1}\,dt\int_{tB}\tilde{\Delta}v\,d\lambda.\tag{48}$$

By (44) $\sim$ (48) for any $r \in (0,1)$

$$
\begin{aligned}
&\beta_n r^{-2n+1} \int_0^r (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda + v(0) \\
(49) \qquad &\leq \int_S v_r\, d\sigma, \\
&\leq \gamma_n r^{-2n+1} \int_0^r (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda + \int_S v_{\frac{1}{2}}\, d\sigma,
\end{aligned}
$$

where $\beta_n = \frac{n+1}{2n}$ and $\gamma = \frac{n+1}{2n} 2^{2n-1}$. Note that

$$
\|f\|^p_{(AN)^p(\nu_\alpha)} = \int_B \{\log(1+|f|)\}^p d\nu_\alpha = c_\alpha 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha dr \int_S v_r d\sigma.
$$

It follows from this and (49) that

$$
\begin{aligned}
&\beta_n c_\alpha 2n \int_0^1 (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda \int_t^1 (1-r^2)^\alpha\, dr + v(0) \\
(50) \quad &\leq \|f\|^p_{(AN)^p(\nu_\alpha)} \\
&\leq \gamma_n c_\alpha 2n \int_0^1 (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda \int_t^1 (1-r^2)^\alpha\, dr + \int_S v_{\frac{1}{2}}\, d\sigma.
\end{aligned}
$$

For $\alpha \in (-1,\infty)$ and $t \in (0,1)$, it holds that

$$
(51) \qquad \frac{(1-t^2)^{1+\alpha}}{2^{1+\alpha^+}(1+\alpha)} < \int_t^1 (1-r^2)^\alpha\, dr < \frac{2^{\alpha^+}(1-t^2)^{1+\alpha}}{1+\alpha},
$$

where $\alpha^+ = 0$ if $\alpha \leq 0$, $\alpha^+ = \alpha$ if $\alpha > 0$. By (51),

$$
\begin{aligned}
&\beta_n c_\alpha 2n \int_0^1 (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda \int_t^1 (1-r^2)^\alpha\, dr \\
(52) \qquad &\geq \beta_n c_\alpha 2n \frac{1}{2^{1+\alpha^+}(1+\alpha)} \int_0^1 (1-t^2)^{n+\alpha}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda \\
&= \frac{2n\beta_n c_\alpha}{2^{1+\alpha^+}(1+\alpha)} \int_B (\tilde{\Delta} v)(z)\, d\lambda(z) \int_{|z|}^1 (1-t^2)^{n+\alpha}\, dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\gamma_n c_\alpha 2n \int_0^1 (1-t^2)^{n-1}\, dt \int_{tB} \tilde{\Delta} v\, d\lambda \int_t^1 (1-r^2)^\alpha\, dr \\
(53) \qquad &\leq \frac{2^{\alpha^+} 2n\gamma_n c_\alpha}{1+\alpha} \int_B (\tilde{\Delta} v)(z)\, d\lambda(z) \int_{|z|}^1 (1-t^2)^{n+\alpha}\, dt.
\end{aligned}
$$

Moreover, we can easily show that

$$(54)\qquad \frac{(1-|z|^2)^{n+\alpha+1}}{2^{n+\alpha+1}(n+\alpha+1)} < \int_{|z|}^{1}(1-t^2)^{n+\alpha}\,dt \le \frac{2^{n+\alpha}(1-|z|^2)^{n+\alpha+1}}{n+\alpha+1}$$

for all $z \in B$. By (50), (52), (53) and (54), we obtain

$$\begin{aligned}
&\frac{a_n\Gamma(n+\alpha+1)}{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)}\int_B \tilde{\Delta}v(z)(1-|z|^2)^{\alpha}\,d\nu(z)+v(0)\\
&\le \|f\|^p_{(AN)^p(\nu_\alpha)}\\
&\le \frac{b_n 2^{\alpha+\alpha^+}\Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+2)}\int_B \tilde{\Delta}v(z)(1-|z|^2)^{\alpha}\,d\nu(z)+\int_S v_{\frac{1}{2}}\,d\sigma.
\end{aligned}$$

This proves (a). (b) follows from (a). (c) and (d) follow from Lemma 3 and (b). (e) follows from (b) and Corollary 2. (f) follows from (c) and Corollary 2. Furthermore, (g) follows from (d) and Corollary 2. □

THEOREM 2. *Suppose* $1 < p < \infty$.

(a) *A function* $f \in H(B)\setminus\{0\}$ *is in* $N^p(B)$ *if and only if*

$$\int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z)\frac{d\nu(z)}{1-|z|^2} < \infty.$$

(b) *For* $f \in H(B)$, $f \in N^p(B)$ *if and only if*

$$\int_B \frac{\{\log(1+|f(z)|)\}^{p-2}}{(1+|f(z)|)^2}|\tilde{\nabla}f(z)|^2\frac{d\nu(z)}{1-|z|^2} < \infty.$$

(c) *If* $f \in N^p(B)\setminus\{0\}$, *then*

$$\lim_{r\uparrow 1}\left[(1-r^2)^n\int_{rB}\tilde{\Delta}(\{\log(1+|f|)\}^p)\,d\lambda\right]=0.$$

(d) *If* $f \in N^p(B)$, *then*

$$\lim_{r\uparrow 1}\left[(1-r^2)^n\int_{rB}\frac{\{\log(1+|f|)\}^{p-2}}{(1+|f|)^2}|\tilde{\nabla}f|^2\,d\lambda\right]=0.$$

Proof. Let $f \in H(B)\setminus\{0\}$. The monotone convergence theorem gives

$$\begin{aligned}
&\lim_{\alpha\downarrow -1}\int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^{\alpha}\,d\nu(z)\\
(55)\qquad &= \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^{-1}\,d\nu(z).
\end{aligned}$$

And note that

$$\lim_{\alpha\downarrow -1} \frac{\Gamma(n+\alpha+1)}{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)} = \frac{2\Gamma(n)}{n}. \tag{56}$$

By Theorem 1(a), Lemma 2, (55) and (56), we have

$$\begin{aligned}
&\frac{a_n 2\Gamma(n)}{n} \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z) \frac{d\nu(z)}{1-|z|^2} + \{\log(1+|f(0)|)\}^p \\
&\leq \|f\|^p_{N^p(B)} \\
&\leq \frac{b_n\Gamma(n)}{2n} \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(z) \frac{d\nu(z)}{1-|z|^2} + \int_S \{\log(1+|f_{\frac{1}{2}}|)\}^p \, d\sigma.
\end{aligned}$$

This proves (a). (b) follows from Lemma 3 and (a). (c) follows from (a) and Corollary 2. Furthermore, (d) follows from (b) and Corollary 2. □

REMARK. C. Ouyang and J. Riihentaus proved the following theorem in [12, p.38, Corollary 2]:

THEOREM. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, convex function that is bounded from below and differentiable. Let E_∞ be the set of points t in $\mathbb{R}$ for which $\varphi''(t)$ exists and equal $+\infty$ (and which set is of Lebesgue measure zero). Suppose further that $|\varphi'(E_\infty)| = 0$. Then a holomorphic function f on B, $f \not\equiv 0$, belongs to the Hardy-Orlicz class $H_\varphi(B)$ if*

$$\int_B (1-|z|^2)^n \varphi''(\log|f(z)|) \frac{|\tilde{\nabla} f(z)|^2}{|f(z)|^2} \, d\lambda(z) < \infty. \tag{57}$$

The integrand in (57) is defined to be 0 in the case when its expression is not defined.

This result by C. Ouyang-J. Riihentaus is related to our Theorem 2 (b).

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