

ASYMPTOTIC BEHAVIOR OF HARMONIC MAPS AND EXPONENTIALLY HARMONIC FUNCTIONS

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ABSTRACT. Let M be a Riemannian manifold with asymptotically non-negative curvature. We study the asymptotic behavior of the energy densities of a harmonic map and an exponentially harmonic function on M . We prove that the energy density of a bounded harmonic map vanishes at infinity when the target is a Cartan-Hadamard manifold. Also we prove that the energy density of a bounded exponentially harmonic function vanishes at infinity.

1. Introduction

Liouville type theorems for Riemannian manifolds have been studied for a long time. It turned out that they work well especially in the case of non-negative Ricci curvature of the domain manifold ([3], [4], [6]). On the other hand, if the domain manifold has negative curvature, they does not hold in general any more ([1], [2]). The condition of the domain manifold in this article is in between.

We deal with the manifolds having asymptotically non-negative Ricci curvature as the domain manifolds. Precisely, let M be a complete non-compact Riemannian manifold and x_0 be a point in M . Denote r by the distance function of M from x_0 . Then M has *asymptotically* non-negative Ricci curvature is defined that $Ric_M(x)$, the Ricci curvature of M , satisfies that $Ric_M(x) \geq -k(r(x))$, where $k : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+$ is a non-increasing function with $\lim_{r \rightarrow \infty} k(r) = 0$. Note that we impose no restriction, even for the decaying rate of the Ricci curvature, in the

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interior of the domain manifolds. The key results of this paper are the followings.

THEOREM 1.1. *Let M be a complete non-compact Riemannian manifold with asymptotically non-negative Ricci curvature, and N be a complete simply connected Riemannian manifold of non-positive sectional curvature. Then for every harmonic map $u : M \rightarrow N$ such that the image of u is contained in a compact subset of N , $|\nabla u|(x) \rightarrow 0$ as $r(x) \rightarrow \infty$.*

Next, we prove the similar theorem for exponentially harmonic functions. The notion of exponentially harmonic maps was first posed by J. Eells. The *exponential energy* of a map $\phi : (M, g) \rightarrow (N, h)$ is defined by

$$\mathcal{E}(\phi) := \int_M e^{|\mathrm{d}\phi|^2/2} \mathrm{dVol},$$

and ϕ is *exponentially harmonic* if it is a smooth extremal of the exponential energy functional \mathcal{E} . But in the case of exponentially harmonic functions, as we can see in [6], the sectional curvature of the domain manifold should be controlled to get a proper result. Asymptotically non-negative sectional curvature is similarly defined as the following: M is said to have *asymptotically non-negative sectional curvature* if there is a non-increasing function $k : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow \infty} k(r) = 0$ such that the sectional curvature of M , $\mathrm{Sec}_M(x)$ satisfies $\mathrm{Sec}_M(x) \geq -k(r(x))$.

THEOREM 1.2. *Let M be a complete non-compact Riemannian manifold such that M has asymptotically non-negative sectional curvature. If $\phi : M \rightarrow \mathbb{R}$ is a bounded exponentially harmonic function, then $|\nabla \phi|$ vanishes at infinity.*

Both of Theorem 1.1 and Theorem 1.2 were proved in the case that M has non-negative Ricci curvature and non-negative sectional curvature in [3] and [6] respectively. In fact, they proved in those papers that the energy densities are constantly 0, hence the maps are constant. But in our case, the classical Liouville theorem cannot be expected. The bounded harmonic functions on a connected sum of $\mathbb{S}^{n-1} \times [0, \infty)$'s (smoothing at 0 in any way) would be the counterexamples.

But in Theorem 3.2, we prove that asymptotically constant bounded harmonic maps should be constant on M .

2. Preliminaries

To make this article self-contained, we recall the basic tensor formulas which are used in this article.

Choose local orthonormal frames $\{e_\alpha\}$ in a neighborhood of $x \in M$ and $\{f_i\}$ in a neighborhood of $u(x) \in N$. Let $\{\theta_\alpha\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha\}$ and $\{f_i\}$ respectively. The connection forms $\{\theta_{\alpha\beta}\}$ and $\{\omega_{ij}\}$ are defined by

$$\begin{aligned} d\theta_\alpha &= \sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta, & \theta_{\alpha\beta} + \theta_{\beta\alpha} &= 0, \\ d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} &= 0. \end{aligned}$$

Define $u_{i\alpha}$ by the equation

$$u^* \omega_i = \sum_\alpha u_{i\alpha} \theta_\alpha,$$

and

$$e(u) = \sum_{i,\alpha} u_{i\alpha}^2$$

is the energy density of u . The covariant derivatives $u_{i\alpha\beta}$ is defined by the equation

$$\sum_\beta u_{i\alpha\beta} \theta_\beta = du_{i\alpha} + \sum_j u_{j\alpha} u^* \omega_{ji} + \sum_\beta u_{i\beta} \theta_{\beta\alpha}.$$

Then u is harmonic if and only if $\sum u_{i\alpha\alpha} = 0$ for all i .

The Bochner type formula for a harmonic map u is given by

$$(2.1) \quad \frac{1}{2} \Delta e(u) = \sum_{i,\alpha,\beta} u_{i\alpha\beta}^2 - \sum_{i,j,k,l,\alpha,\beta} R_{ijkl}^N u_{i\alpha} u_{j\beta} u_{k\alpha} u_{l\beta} + \sum_{\alpha,\beta,i} Ric_{\alpha\beta}^M u_{i\alpha} u_{i\beta},$$

where R_{ijkl}^N is the curvature tensor of N and $Ric_{\alpha\beta}^M$ is the Ricci tensor of M .

Let f be any smooth function defined on N . Then for a harmonic map u ,

$$(2.2) \quad \Delta(f \circ u) = f_{ij} u_{i\alpha} u_{j\alpha}.$$

For the exponentially harmonic functions, let $Q^{ij}(\phi) = \delta^{ij} + \phi_i \phi_j$. Due to Duc and Eells ([5]), ϕ is an *exponentially harmonic function* if

$$\sum_{i,j} Q^{ij}(\phi) \phi_{ij} = 0.$$

Applying ∇_k to the above equation, we have

$$\sum_{i,j} (Q^{ij}(\phi) \phi_{ijk} + 2\phi_i \phi_{jk} \phi_{ij}) = 0.$$

Since

$$\phi_{ij} = \phi_{ji} \quad \text{and} \quad \phi_{ijk} - \phi_{ikj} = -\phi_l R_{iljk},$$

we have the Bochner type formula for an exponentially harmonic function,

$$\begin{aligned} \sum_{i,j} Q^{ij} e_{ij} &= 2 \sum_{i,j,k} Q^{ij} (\phi_{kij} \phi_k + \phi_{ki} \phi_{kj}) \\ &= 2 \sum_{i,j,k} Q^{ij} (\sum_l \phi_k \phi_l R_{iljk} + \phi_{ijk} \phi_k) \\ &\quad + 2 \sum_{i,k} \phi_{ki}^2 + 2 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &= 2 \sum_{i,j,k,l} Q^{ij} \phi_k \phi_l R_{iljk} - 4 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &\quad + 2 \sum_{i,k} \phi_{ki}^2 + 2 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &= 2 \sum_{i,j,k,l} Q^{ij} \phi_k \phi_l R_{iljk} + 2|\nabla d\phi|^2 - \frac{1}{2}|de|^2. \end{aligned}$$

For a function f on \mathbb{R} , we have

$$(2.3) \quad Q^{ij}(f \circ \phi)_{ij} = Q^{ij}(f'' \phi_i \phi_j + f' \phi_{ij}) \geq f''(e + e^2).$$

3. Asymptotic behavior of harmonic maps

For an arbitrary point $z \in M$, let $2a$ be the distance from x_0 to z and $\gamma : [0, 2a] \rightarrow M$ be a geodesic with $\gamma(0) = x_0$ and $\gamma(2a) = z$. Now

consider the ball $B_z(a)$ around z with the radius a . By the triangle inequality, the distance between x_0 and each point in the ball is greater than a , which means that the Ricci curvature has a lower bound $-k(a)$ in the ball. Denote s by the distance function of M from z . For the target manifold N , let $y_0 \in N$ lie outside $u(M)$ and ρ denote the distance function of N from y_0 . Since u has a bounded image in N , we can take $b > 0$ such that $b > \sup\{\rho(u(x)) | x \in B_{\gamma(2a)}(a)\}$, for all $z \in M$. Also we have a lower bound $\beta > 0$ such that $\beta < \inf\{b^2 - \rho^2 \circ u(x) | x \in B_{\gamma(2a)}(a)\}$ for all $z \in M$. Note that b and β are independent of $z \in M$. Now take a function Φ in $B_z(a)$ such that

$$\Phi = \frac{(a^2 - s^2)^2 |\nabla u|^2}{(b^2 - \rho^2 \circ u)^2}.$$

Φ is actually defined in the moving balls. But it is essentially the same function as appeared in [3] if we regard $z \in M$ above as a fixed point, so we can use similar calculation as in [3] by considering that $-k(a)$ is the lower bound of the Ricci curvature in $B_z(a)$, which is shown in Proposition 3.1. But in order to prove the theorem we should analyze the result in other way. From now on, we will use $B_{\gamma(2a)}(a)$ instead of $B_z(a)$.

PROPOSITION 3.1. *Let M and N be the same as in Theorem 1.1. And let $u : M \rightarrow N$ be a harmonic map whose image is contained in a compact subset of N . Then in $B_{\gamma(2a)}(a)$,*

$$\Phi \leq C_m \max \left\{ \frac{k(a)(a^2 - s^2)^2}{b^2 - \rho^2 \circ u}, \frac{a^2}{b^2 - \rho^2 \circ u}, \frac{(a^2 - s^2)(1 + k(a)a)}{b^2 - \rho^2 \circ u}, \frac{a^2 b^2}{(b^2 - \rho^2 \circ u)^2} \right\} \Big|_{\text{at } \bar{x}},$$

where \bar{x} is a maximum point of Φ in $B_{\gamma(2a)}(a)$ and C_m is a constant depending only on the dimension of M .

Proof. Since Φ vanishes on the boundary of $B_{\gamma(2a)}(a)$, Φ assumes its maximum at a point \bar{x} in the interior of the ball. By the maximum principle, at the point \bar{x} we have

$$\begin{aligned} \nabla \Phi &= 0, \\ \Delta \Phi &< 0. \end{aligned}$$

They are actually calculated as

$$\begin{aligned}
 (3.1) \quad 0 &= \frac{-2ds^2}{a^2 - s^2} + \frac{d|\nabla u|^2}{|\nabla u|^2} + \frac{2d(\rho^2 \circ u)}{b^2 - \rho^2 \circ u}, \\
 0 &\geq \frac{-2\Delta s^2}{a^2 - s^2} - \frac{2|ds^2|^2}{(a^2 - s^2)^2} + \frac{\Delta|\nabla u|^2}{|\nabla u|^2} \\
 (3.2) \quad &- \frac{|d|\nabla u|^2|^2}{|\nabla u|^4} + \frac{2\Delta(\rho^2 \circ u)}{b^2 - \rho^2 \circ u} + \frac{2|d(\rho^2 \circ u)|^2}{(b^2 - \rho^2 \circ u)^2}.
 \end{aligned}$$

The following formulas enable us to convert the above into an inequality not containing the second derivative terms of u .

First, the equation (3.1) gives

$$\left| \frac{d|\nabla u|^2}{|\nabla u|^2} \right|^2 \leq \frac{4|ds^2|^2}{(a^2 - s^2)^2} + \frac{8|ds^2| \times |d\rho^2| \times |\nabla u|}{(a^2 - s^2)(b^2 - \rho^2 \circ u)} + \frac{4|d(\rho^2 \circ u)|^2}{(b^2 - \rho^2 \circ u)^2}.$$

The Bochner formula and Schwartz inequality give

$$\frac{\Delta|\nabla u|^2}{|\nabla u|^2} \geq \frac{1}{2} \frac{|d|\nabla u|^2|^2}{|\nabla u|^4} - 2k(a) \quad \text{in } B_{\gamma(2a)}(a).$$

And the Hessian comparison theorem with the curvature assumption implies

$$\begin{aligned}
 \Delta s^2 &\leq C_m(1 + k(a)s) \quad \text{in } B_{\gamma(2a)}(a), \\
 \Delta(\rho^2 \circ u) &\geq |\nabla u|^2.
 \end{aligned}$$

Finally applying these estimates to the above inequality (3.2), we can get the following quadratic inequality with respect to the energy density:

$$\begin{aligned}
 0 &\geq \frac{-C_m(1 + k(a)a)}{(a^2 - s^2)} - \frac{12a^2}{(a^2 - s^2)^2} \\
 &\quad - \frac{16ab|\nabla u|}{(a^2 - s^2)(b^2 - \rho^2 \circ u)} + \frac{|\nabla u|^2}{(b^2 - \rho^2 \circ u)} - k(a).
 \end{aligned}$$

And this is followed by the upper bound of Φ in terms of s and $k(a)$ up to multiplying by constant which depends only on the dimension of M . Precisely,

$$\begin{aligned}
 \Phi(\bar{x}) = \frac{(a^2 - s^2)^2|\nabla u|^2}{(b^2 - \rho^2 \circ u)^2}(\bar{x}) &\leq C_m \max \left\{ \frac{k(a)(a^2 - s^2)^2}{b^2 - \rho^2 \circ u}, \frac{a^2}{b^2 - \rho^2 \circ u}, \right. \\
 &\quad \left. \frac{(a^2 - s^2)(1 + k(a)a)}{b^2 - \rho^2 \circ u}, \frac{a^2b^2}{(b^2 - \rho^2 \circ u)^2} \right\} \Big|_{\text{at } \bar{x}}.
 \end{aligned}$$

Since $\Phi(x) \leq \Phi(\bar{x})$ for all $x \in B_{\gamma(2a)}(a)$, it completes the proof. □

Proof of Theorem 1.1. To conclude the result in our case, we should confine the domain of Φ to $B_{\gamma(2a)}(\frac{a}{2})$. In $B_{\gamma(2a)}(\frac{a}{2})$ the estimate in Proposition 3.1 still holds and we have a bound $a^2 - s^2 > \frac{a^2}{2}$. With this bound we have

$$\Phi = \frac{(a^2 - s^2)^2 |\nabla u|^2}{(b^2 - \rho^2 \circ u)^2} \geq \frac{a^4 |\nabla u|^2}{4b^4}.$$

Combining this inequality and Proposition 3.1, we have

$$|\nabla u|^2 \leq C \max \left\{ k(a), \frac{1}{a^2}, \frac{1 + k(a)a}{a^2}, \frac{b^2}{\beta a^2} \right\} \text{ in } B_{\gamma(2a)}\left(\frac{a}{2}\right),$$

where $C = C_m 4 \frac{b^4}{\beta}$. Now at every z with $r(z) = 2a \geq 2$,

$$|\nabla u(z)|^2 \leq C \max \left\{ k\left(\frac{r(z)}{2}\right), \frac{1}{r(z)^2}, \frac{b^2}{\beta r(z)^2} \right\}.$$

Note that the constant C here does not depend on $r(z)$. So we have that $|\nabla u(z)| \rightarrow 0$ as $r(z) \rightarrow \infty$. \square

It is known that there can be infinitely many bounded harmonic maps from M to N in general. Now we are interested in finding out how the asymptotic behavior of u determines properties of u on the whole domain. The following theorem tells that if $u(x)$ goes to a constant point $y_0 \in N$ as $r(x) \rightarrow \infty$, then u should be constant. For the sake of completeness, let us define a parabolic end by an end of M which does not admit any positive Green's function satisfying the Neumann boundary condition on the boundary of the end, and define a nonparabolic end otherwise.

THEOREM 3.2. *Let M , N , and u be the same as in Theorem 1.1. If*

$$\lim_{r(x) \rightarrow \infty} u(x) = y_0 \quad \text{for some } y_0 \in N,$$

then $u \equiv y_0$ on M . Furthermore, suppose that M has at least one nonparabolic end. If on each nonparabolic end E of M ,

$$\lim_{r(x) \rightarrow \infty, x \in E} u(x) = y_0 \quad \text{for some } y_0 \in N,$$

then $u \equiv y_0$ on M .

Proof. Let $f(x) = k(0)\rho_{y_0}^2 \circ u(x)$ for $x \in M$, where $\rho_{y_0}(y) = \text{dist}_N(y, y_0)$. Note that f is bounded since u has a bounded image in N . By (2.2) and Hessian comparison for Cartan-Hadamard manifold N , we have $\Delta f \geq 2k(0)|\nabla u|^2$. Now consider the function $|\nabla u|^2 + f$ on M . By (2.1),

$$\frac{1}{2}\Delta(|\nabla u|^2 + f) \geq -k|\nabla u|^2 + k(0)|\nabla u|^2 \geq 0.$$

That is, $|\nabla u|^2 + f$ is a bounded non-negative subharmonic function on M . Note that by Theorem 1.1 $|\nabla u|^2$ vanishes at the infinity, so it is bounded. Hence by the maximum principle, $|\nabla u|^2 + f$ attains its maximum at the infinity of some ends of M . Since $|\nabla u|^2 + f = 0$ at the infinity by Theorem 1.1 and our assumption, $\sup_M[|\nabla u|^2 + f] = 0$. Hence $|\nabla u|^2 \equiv 0$ and $f \equiv 0$ on M as both are non-negative. Therefore $u \equiv \text{const}$.

Assume that M has at least one nonparabolic end, and $\lim_{r(x) \rightarrow \infty} u(x) = y_0$ only on the nonparabolic ends. We claim that $|\nabla u|^2 + f$ has its maximum at the infinity of a nonparabolic end unless it is constant. In fact, by the same argument in [8], the maximum cannot be attained at the infinity of a parabolic end unless it is constant. Namely otherwise, $\max_M[|\nabla u|^2 + f] - [|\nabla u|^2 + f]$ will be a positive superharmonic function on the parabolic end attaining its minimum at the infinity. But a parabolic end does not admit such a function ([7]). It is a contradiction. So $|\nabla u|^2 + f$ attains its maximum at the infinity of some nonparabolic ends. But $f = 0$ at the infinity of every nonparabolic end, so $\sup_M[|\nabla u|^2 + f] = 0$, i.e., $|\nabla u|^2 + f \equiv 0$. Hence $u \equiv \text{const}$. \square

Now if we impose that M has finite volume, then we can recover a classical Liouville type theorem. Before we state the theorem, recall that the following special case of theorems in [9].

PROPOSITION 3.3. *Let u be a non-negative smooth subharmonic function on M . Then $\int_M u^2 = \infty$, unless u is a constant function.*

Proof. See [9]. \square

THEOREM 3.4. *Let M , N , and u be the same as in Theorem 1.1. Suppose that M has finite volume. Then $u \equiv \text{const}$ on M .*

Proof. Let κ_1, κ_2 be two positive constants satisfying $k(0) \leq \kappa_1 < \kappa_2$. For $x_0 \in M$, define $f_i = \kappa_i \rho_{u(x_0)}^2 \circ u$. Then we have

$$\frac{1}{2} \Delta(|\nabla u|^2 + f_i) \geq -k|\nabla u|^2 + \kappa_i |\nabla u|^2 \geq 0, \text{ for } i = 1, 2.$$

Hence $|\nabla u|^2 + f_i$ is non-negative subharmonic functions on M . On the other hand, since $|\nabla u|^2 + f_i$ is bounded,

$$\begin{aligned} & \int_M [|\nabla u|^2 + f_i]^2 d\text{Vol} \\ & \leq \max_M [|\nabla u|^2 + f_i]^2 \int_M d\text{Vol} \\ & < \infty. \end{aligned}$$

So by Proposition 3.3, we have

$$|\nabla u|^2 + f_i \equiv C_i, \text{ for some non-negative constants } C_i.$$

By subtracting, we have

$$(\kappa_2 - \kappa_1) \rho_{u(x_0)}^2 \circ u \equiv C_2 - C_1,$$

i.e.,

$$\rho_{u(x_0)}^2 \circ u \equiv \frac{C_2 - C_1}{\kappa_2 - \kappa_1}.$$

Since $\rho_{u(x_0)}^2 \circ u(x_0) = 0$, we have $\rho_{u(x_0)}^2 \circ u \equiv 0$. Hence $u(x) = u(x_0)$ for all $x \in M$. □

4. Asymptotic behavior of exponentially harmonic functions

Here the situation is similar to the case of Theorem 1.1. Again, for an arbitrary point $z \in M$, let $2a$ be the distance from x_0 to z and $\gamma : [0, 2a] \rightarrow M$ be a geodesic with $\gamma(0) = x_0$ and $\gamma(2a) = z$. Take the ball $B_{\gamma(2a)}(a)$ around $\gamma(2a)$ with the radius a . Then the sectional curvature has a lower bound $-k(a)$ in the ball. Denote s by the distance function of M from $\gamma(2a)$. Consider a function F in $B_{\gamma(2a)}(a)$ such that

$$F = \frac{(a^2 - s^2)^2 |\nabla \phi|^2}{b^2 - \phi^2},$$

where b is a constant with $b > 3 \max_M |\phi|$. Note that we can take b such that $b^2 > \phi^2 + \beta$ for some constant $\beta > 0$ also.

Proof of Theorem 1.2. Let \bar{x} be a maximum point of F . Then by the maximum principle, we have

$$\begin{aligned}\nabla F(\bar{x}) &= 0, \\ Q^{ij} F_{ij}(\bar{x}) &\leq 0.\end{aligned}$$

And they are

$$\begin{aligned}(4.1) \quad 0 &= \frac{-2ds^2}{a^2 - s^2} + \frac{de(\phi)}{e(\phi)} + \frac{d(\phi^2)}{b^2 - \phi^2}, \\ &\geq \frac{-2Q^{ij}(s^2)_{ij}}{a^2 - s^2} - \frac{2Q^{ij}(s^2)_i(s^2)_j}{(a^2 - s^2)^2} + \frac{Q^{ij}e(\phi)_{ij}}{e(\phi)} \\ (4.2) \quad &- \frac{Q^{ij}e(\phi)_ie(\phi)_j}{e(\phi)^2} + \frac{Q^{ij}(\phi^2)_{ij}}{b^2 - \phi^2} + \frac{Q^{ij}(\phi^2)_i(\phi^2)_j}{(b^2 - \phi^2)^2}.\end{aligned}$$

Now we will replace all the terms including Q^{ij} in the above inequality (4.2) by the next four formulas. First we have

$$Q^{ij}(\phi^2)_{ij} = Q^{ij}(2\phi_i\phi_j + 2\phi\phi_{ij}) = 2Q^{ij}\phi_i\phi_j \geq 2(e(\phi) + e(\phi)^2),$$

and

$$Q^{ij}\eta_i\eta_j = |d\eta|^2 + (\phi_i\eta_i)^2 \text{ for a function } \eta.$$

The Hessian comparison under the sectional curvature condition gives

$$Q^{ij}D_{ij}s^2 \leq C_m(1 + k(a)s)(1 + e(\phi)) \text{ in } B_{\gamma(2a)}(a).$$

With the curvature assumption again we have the following Bochner

type formula for exponentially harmonic functions in $B_{\gamma(2a)}(a)$:

$$\begin{aligned} \sum_{i,j} Q^{ij} e(\phi)_{ij} &= 2 \sum_{i,j,k} Q^{ij} (\phi_{kij} \phi_k + \phi_{ki} \phi_{kj}) \\ &= 2 \sum_{i,j,k} Q^{ij} (\sum_l \phi_k \phi_l R_{iljk} + \phi_{ij} \phi_k) \\ &\quad + 2 \sum_{i,k} \phi_{ki}^2 + 2 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &= 2 \sum_{i,j,k,l} Q^{ij} \phi_k \phi_l R_{iljk} - 4 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &\quad + 2 \sum_{i,k} \phi_{ki}^2 + 2 \sum_{k,i,j} \phi_i \phi_j \phi_{ki} \phi_{kj} \\ &= 2 \sum_{i,j,k,l} Q^{ij} \phi_k \phi_l R_{iljk} + 2|\nabla d\phi|^2 - \frac{1}{2}|de(\phi)|^2 \\ &\geq -2k(a)e(\phi) + \frac{|de(\phi)|^2}{2e(\phi)} - \frac{|de(\phi)|^2}{2}. \end{aligned}$$

In the above, $\frac{|de(\phi)|^2}{2}$ is a bad term. But from (4.1) we have

$$\frac{|de(\phi)|^2}{e(\phi)^2} \leq \frac{16s^2}{(a^2 - s^2)^2} + \frac{4\phi^2 e(\phi)}{(b^2 - \phi^2)^2},$$

and by the choice of b , we have

$$\frac{\phi^2}{b^2 - \phi^2} < \frac{1}{8}.$$

So it is dominated by

$$\frac{|de(\phi)|^2}{2} \leq \frac{8s^2 e(\phi)^2}{(a^2 - s^2)^2} + \frac{e(\phi)^3}{4(b^2 - \phi^2)}.$$

As combining all these estimates we can get

$$\begin{aligned} 0 \geq & \tilde{C}_m \frac{e(\phi)^2}{b^2 - \phi^2} - e(\phi) \left(\frac{1 + k(a)s}{a^2 - s^2} + \frac{s^2}{(a^2 - s^2)^2} + \frac{s^2}{(a^2 - s^2)^2(b^2 - \phi^2)} \right) \\ & - \left(\frac{1 + k(a)s}{a^2 - s^2} + \frac{s^2}{(a^2 - s^2)^2} + \frac{s^2}{(a^2 - s^2)^2(b^2 - \phi^2)} + k(a) \right). \end{aligned}$$

This also gives the following bound of F at \bar{x} and hence at every point in $B_{\gamma(2a)}(a)$;

$$F = \frac{(a^2 - s^2)^2 |\nabla\phi|^2}{(b^2 - \phi^2)} \leq C \max \left\{ a^{3+\frac{1}{2}}, \quad a^2, \quad 1, \quad a^4 \sqrt{k(a)} \right\}.$$

To confine the domain of F to $B_{\gamma(2a)}(\frac{a}{2})$ gives us a bound

$$\frac{(a^2 - s^2)^2 |\nabla\phi|^2}{(b^2 - \phi^2)} \geq \frac{a^4}{4b^2} |\nabla\phi|^2.$$

This implies

$$|\nabla\phi|^2 \leq C \max \left\{ a^{-\frac{1}{2}}, \quad a^{-2}, \quad \frac{1}{a^4}, \quad \sqrt{k(a)} \right\} \text{ in } B_{\gamma(2a)}\left(\frac{a}{2}\right),$$

where $C = C(m, b, \beta)$ is a constant. Since we are interested in the only asymptotic behavior, for simplicity, assume that $r(z) \geq 2$. Then we have

$$|\nabla\phi(z)|^2 \leq C \max \left\{ \sqrt{k\left(\frac{r(z)}{2}\right)}, \quad \sqrt{\frac{1}{r(z)}} \right\},$$

hence $|\nabla\phi(z)|^2 \rightarrow 0$ as $r(z)$ goes to ∞ . □

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