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INEQUALITIES FOR THE HILBERT TRANSFORM OF FUNCTIONS WHOSE DERIVATIVES ARE CONVEX

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Abstract. Using the well known Hermite-Hadamard integral inequality for convex functions, some inequalities for the finite Hilbert transform of functions whose first derivatives are convex are established. Some numerical experiments are performed as well.

1. Introduction

Let $\Omega = (-1, 1)$ where $1 \leq p < \infty$, the usual $\mathfrak{L}^p$-space with respect to the Lebesgue measure λ restricted to the open interval Ω will be denoted by $\mathfrak{L}^p(\Omega)$.

We define a linear operator T (see [24]) from the vector space $\mathfrak{L}^1(\Omega)$ into the vector space of all λ-measurable functions on Ω as follows. Let $f \in \mathfrak{L}^1(\Omega)$. The Cauchy principle value

$$\frac{1}{\pi} PV \int_{-1}^{1} \frac{f(\tau)}{\tau - t} d\tau = \lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

exists for λ-almost every $t \in \Omega$.

We denote the left-hand side of (1.1) by $(Tf)(t)$ for each $t \in \Omega$ for which $(Tf)(t)$ exists. The so-defined function Tf, which we call the *finite Hilbert Transform* of f, is defined λ-almost everywhere on Ω and is λ-measurable; (see for example [1, Theorem 8.1.5]). The resulting linear operator T will be called the *finite Hilbert transform operator* or Cauchy kernel operator.

It is known that $\mathfrak{L}^1(\Omega)$ is not invariant under T, namely, $T(\mathfrak{L}^1(\Omega)) \not\subset \mathfrak{L}^1(\Omega)$ [17, Proof of Theorem 1 (b)].

The following basic results are well known and their proofs may be found in Propositions 8.1.9 and 8.2.1 of [1] respectively.

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THEOREM 1. (M. Riesz) *Let $1 < p < \infty$. Then $T(\mathfrak{L}^p(\Omega)) \subset \mathfrak{L}^p(\Omega)$ and the linear operator*

$$T_p : f \mapsto Tf,\ f \in \mathfrak{L}^p(\Omega)$$

on $\mathfrak{L}^p(\Omega)$ is continuous.

THEOREM 2. (Parseval) *Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Then*

$$\int_{-1}^{1} (fTg + gTf)\, d\lambda = 0 \tag{1.2}$$

for every $f \in \mathfrak{L}^p(\Omega)$ and $g \in \mathfrak{L}^q(\Omega)$.

We introduce the following definition.

DEFINITION 1. A function $f : \Omega \to \mathbb{C}$ is said to be *α-Hölder continuous* $(0 < \alpha \leq 1)$ in a subinterval Ω_0 of Ω if there exists a constant $c > 0$, dependent upon Ω_0, such that

$$|f(s) - f(t)| \leq c\,|s - t|^{\alpha}, \quad s, t \in \Omega_0. \tag{1.3}$$

A function on Ω is said to be *locally α-Hölder continuous* if it is α-Hölder continuous in every compact subinterval of Ω. We denote by $H^{\alpha}_{loc}(\Omega)$ the space of all locally α-Hölder continuous functions on Ω.

The class of Hölder continuous functions on Ω is independent because the finite Hilbert transform of such a function exists everywhere on Ω (see [15, Section 3.2] or [21, Lemma II.1.1]).

This is in contrast to the λ-almost everywhere existence of the finite Hilbert transform of functions in $\mathfrak{L}^1(\Omega)$.

There are continuous functions $f \in \mathfrak{L}^1(\Omega)$ such that $(Tf)(t)$ does not exist at some point $t \in \Omega$. An example is given by the function f defined by (see [24])

$$f(t) = \begin{cases} 0 & \text{if } \ -1 < t \leq 0, \\ \frac{1}{\ln t - \ln 2} & \text{if } \ 0 < t < 1. \end{cases}$$

It readily follows that $(Tf)(0)$ does not exist.

In paper [24] it is proved amongst others the following result.

THEOREM 3. (Okada-Elliot) *The space $\mathfrak{L}^p(\Omega) \cap H^{\alpha}_{loc}(\Omega)$ is invariant under the finite Hilbert transform operator T and the restriction of T to that space is continuous whenever $1 < p < \infty$. However, this is not true when $p = 1$.*

We consider the finite Hilbert transform on the open interval (a,b)

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau, \quad t \in (a,b).$$

The following theorem holds (see [11]).

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be α-H-Hölder continuous on (a,b), i.e.,*

(1.4)
$$|f(t) - f(s)| \le H|t-s|^{\alpha} \quad \text{for all } t,s \in (a,b), \ \alpha \in (0,1], \ H > 0.$$

Then we have the estimate

$$(1.5) \qquad \left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) \right| \le \frac{H}{\alpha\pi}\left[(t-a)^{\alpha} + (b-t)^{\alpha}\right] \le \frac{H2^{1-\alpha}}{\alpha\pi}(b-a)^{\alpha}$$

for all $t \in (a,b)$.

The following result holds for monotonic functions (see [11]).

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t \in (a,b)$, then*

$$(1.6) \qquad (Tf)(a,b;t) \ge (\le) \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right)$$

for all $t \in (a,b)$.

Now, if we assume that the mapping $f : (a,b) \to \mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t \in (a,b)$.

The following result holds (see [11]).

THEOREM 6. *Let $f : (a,b) \to \mathbb{R}$ be a convex mapping on (a,b). Then we have*

$$(1.7) \qquad \frac{1}{\pi}\left[l(t)(b-t) + \int_a^t l(s)\,ds + f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \le (Tf)(a,b;t) \le \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + l(t)(t-a) + \int_t^b l(s)\,ds\right],$$

where $l(s) \in \left[f'_-(s), f'_+(s)\right]$, $s \in (a,b)$.

The following more practical result also holds [11]:

COROLLARY 1. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable convex function on (a,b). Then we have the inequality*

$$\begin{aligned} (1.8) \qquad & \frac{1}{\pi}\left[f(t) - f(a) + f'(t)(b-t) + f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \\ \le\ & (Tf)(a,b;t) \\ \le\ & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + f(b) - f(t) + f'(t)(t-a)\right] \end{aligned}$$

for all $t \in (a,b)$.

In this paper, by the use of the well known Hermite-Hadamard integral inequality for convex functions, we point out some inequalities for the finite Hilbert transform of functions whose first derivatives are convex. Some numerical experiments for particular examples of such functions are performed as well.

For a comprehensive number of results on the numerical approximation of the Cauchy principal value integrals, see [2]–[10], [13]–[14], [16], [18]–[20], [22]–[23], [25]–[27].

2. An inequality on the interval (a,b)

The following result holds.

THEOREM 7. *Assume that the differentiable function $f : (a,b) \to \mathbb{R}$ is such that f' is convex on (a,b). Then the Hilbert transform $(Tf)(a,b;\cdot)$ exists in every point $t \in (a,b)$ and:*

$$\begin{aligned} (2.1) \qquad & \frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right) - f\left(\frac{t+a}{2}\right)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\ \le\ & (Tf)(a,b;t) \\ \le\ & \frac{1}{2\pi}\left[f(b) - f(a) + (b-a)f'(t)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \end{aligned}$$

for any $t \in (a,b)$.

Proof. The existence of the Hilbert transform in each point $t \in (a,b)$ follows by the fact that f is locally Lipschitzian on (a,b).

Since f' is convex, we have, by the Hermite-Hadamard inequality, that

$$(2.2) \qquad f'\left(\frac{t+\tau}{2}\right) \le \frac{1}{\tau - t}\int_t^\tau f'(u)\,du \le \frac{f'(t) + f'(\tau)}{2}$$

for all $t, \tau \in (a,b)$, $t \neq \tau$, giving

$$f'\left(\frac{t+\tau}{2}\right) \leq \frac{f(\tau)-f(t)}{\tau-t} \leq \frac{f'(t)+f'(\tau)}{2} \tag{2.3}$$

for all $t, \tau \in (a,b)$, $t \neq \tau$.

Applying the PV in t, i.e., $\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)(\cdot)$, we get

(2.4)

$$PV\int_a^b f'\left(\frac{t+\tau}{2}\right)d\tau \leq PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau \leq PV\int_a^b \frac{f'(t)+f'(\tau)}{2}d\tau.$$

Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left(\frac{t+\tau}{2}\right)d\tau\right)\\
=&\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon} f'\left(\frac{t+\tau}{2}\right)d\tau+\int_{t+\varepsilon}^b f'\left(\frac{t+\tau}{2}\right)d\tau\right]\\
=&\lim_{\varepsilon\to 0+} 2\left[\left(f\left(\frac{2t-\varepsilon}{2}\right)-f\left(\frac{t+a}{2}\right)\right)+\left(f\left(\frac{t+b}{2}\right)-f\left(\frac{2t+\varepsilon}{2}\right)\right)\right]\\
=&\ 2\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left[\frac{f'(t)+f'(\tau)}{2}d\tau\right]\\
=&\ \frac{1}{2}\lim_{\varepsilon\to 0+}\left[f'(t)(t-\varepsilon-a)+f'(t)(b-t-\varepsilon)\right.\\
&\qquad\left.+f(t-\varepsilon)-f(a)+f(b)-f(t+\varepsilon)\right]\\
=&\ \frac{1}{2}\left[f(b)-f(a)+(b-a)f'(t)\right],
\end{aligned}$$

then by (2.4), we may state that:

$$\begin{aligned}
\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right] &\leq \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau\\
&\leq \frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'(t)\right]
\end{aligned} \tag{2.5}$$

for all $t \in (a,b)$.

As for the function $f_0(t)=1$, $t \in (a,b)$, we have

$$(Tf)(a,b;t)=\frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\quad t\in(a,b),$$

then obviously

$$\begin{aligned}(2.6)\ (Tf_0)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau \\ &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

for any $t \in (a,b)$.

Finally, by (2.5) and (2.6), we may obtain (3.1). □

The inequality (2.1) in Theorem 7 may be used to obtain different analytic inequalities for functions $f : [a,b] \to \mathbb{R}$ whose derivatives are convex on (a,b) and the Hilbert Transform $(Tf)(a,b;\cdot)$ is known.

For example, the following proposition holds.

Proposition 1. *For any $a,b \in \mathbb{R}$, $a < b$ and $t \in (a,b)$, we have the inequality:*

$$\begin{aligned}(2.7)\qquad & \ln\left(\frac{b-t}{t-a}\right) + 2\left(e^{\frac{b-t}{2}} - e^{\frac{a-t}{2}}\right) \\ \le\ & E_i(b-t) - E_i(a-t) \\ \le\ & \ln\left(\frac{b-t}{t-a}\right) + \frac{1}{2}\left[e^{b-t} - e^{a-t} + (b-a)\right],\end{aligned}$$

where E_i is defined in (2.8).

Proof. If we consider the function $f(t) = e^t$, $t \in (a,b)$, then f' is convex on (a,b),

$$(Tf)(a,b;t) = \frac{e^t}{\pi}\left[E_i(b-t) - E_i(a-t)\right],$$

where E_i is defined by

$$(2.8)\qquad E_i(z) := PV\int_{-\infty}^{z} \frac{e^t}{t}dt, \quad z \in \mathbb{R},$$

$$\begin{aligned}&\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right) - f\left(\frac{t+a}{2}\right)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\ &\qquad = \frac{2}{\pi}\left[e^{\frac{t+b}{2}} - e^{\frac{t+a}{2}}\right] + \frac{e^t}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

and

$$\frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'(t)\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$
$$=\frac{1}{2\pi}\left[e^{b}-e^{a}+(b-a)e^{t}\right]+\frac{e^{t}}{\pi}\ln\left(\frac{b-t}{t-a}\right).$$

Using (2.1) and dividing by e^t, we deduce (2.7). □

The following inequality also holds.

PROPOSITION 2. *For any $x>0$, we have the inequality*

$$2\sinh\left(\frac{1}{2}x\right)\le E_i(x)\le\frac{1}{2}\sinh(x)+\frac{1}{2}x. \tag{2.9}$$

Proof. If in (2.7) we put $t=\frac{a+b}{2}$, then we deduce:

$$\begin{aligned}2\left(e^{\frac{b-a}{4}}-e^{-\frac{b-a}{4}}\right)&\le E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)\\&\le\frac{1}{2}\left[e^{\frac{b-a}{2}}-e^{-\frac{b-a}{2}}+b-a\right].\end{aligned}$$

If we denote $x:=\frac{b-a}{2}$, then we get

$$2\left(e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}\right)\le E_i(x)-E_i(-x)\le\frac{1}{2}\left[e^{x}-e^{-x}+2x\right]. \tag{2.10}$$

However,

$$\begin{aligned}-E_i(-x)&=E_i(x),\\2\left(e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}\right)&=4\sinh\left(\frac{1}{2}x\right)\end{aligned}$$

and

$$\frac{1}{2}\left(e^{x}-e^{-x}+2x\right)=\sinh(x)+x$$

and then, by (2.10), we deduce (2.9). □

If we choose another function, for instance, $f:(a,b)\subset(0,\infty)\to\mathbb{R}$, $f(t)=-\frac{1}{t}$, then obviously f' is convex on (a,b), and we may state the following result as well.

PROPOSITION 3. *For any $0<a<b$ and $t\in(a,b)$, we have the inequality:*

$$\frac{2tG^2}{G^2+t^2}\le L\le\frac{t^2+2At+G^2}{4t}\quad\textit{for any}\quad t\in(a,b), \tag{2.11}$$

where $A=\frac{a+b}{2}$, $G=\sqrt{ab}$ and $L=\frac{b-a}{\ln b-\ln a}$ (the logarithmic mean).

Proof. For the function $f:(a,b)\to\mathbb{R}$, $f(t)=-\frac{1}{t}$, we have

$$(Tf)(a,b;t)=\frac{1}{\pi t}\left[\ln\left(\frac{b}{a}\right)-\ln\left(\frac{b-t}{t-a}\right)\right],$$

$$\begin{aligned}&\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ =\;&\frac{4}{\pi}\cdot\frac{b-a}{(t+a)(t+b)}-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'(t)\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ =\;&\frac{b-a}{2\pi}\left[\frac{1}{ab}+\frac{1}{t^2}\right]-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right).\end{aligned}$$

Now, if we use (2.1), we may write:

$$\begin{aligned}\frac{4}{\pi}\cdot\frac{b-a}{(t+a)(t+b)}-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)&\le\frac{1}{\pi t}\ln\left(\frac{b}{a}\right)-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)\\ &\le\frac{b-a}{2\pi}\left(\frac{t^2+ab}{abt^2}\right)-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

which is equivalent to:

$$\frac{4t}{(t+a)(t+b)}\le\frac{\ln b-\ln a}{b-a}\le\frac{t^2+ab}{2tab}.$$

Using the fact that $L:=\frac{b-a}{\ln b-\ln a}$, we deduce (2.11). □

COROLLARY 2. *We have the inequality*

$$G\le L\le\frac{G+A}{2}. \tag{2.12}$$

REMARK 1. The first inequality is a well known result as the following sequence of inequalities hold

$$G\le L\le I\le A.$$

The second inequality is equivalent with:

$$L(a,b)\le\left[A\left(\sqrt{a},\sqrt{b}\right)\right]^2, \tag{2.13}$$

which is interesting in itself.

3. An inequality on an equidistant division of (a,b)

The following lemma is interesting in itself.

LEMMA 1. *Let $g : [a,b] \to \mathbb{R}$ be a convex function. Then for $n \geq 1$ and $t, \tau \in [a,b]$, $t \neq \tau$, we have the inequality:*

$$\begin{aligned} (3.1) \qquad & \frac{1}{n}\sum_{i=0}^{n-1} g\left[t + \left(i+\frac{1}{2}\right)\cdot\frac{t-\tau}{n}\right] \\ \leq\ & \frac{1}{\tau - t}\int_t^\tau g(u)\,du \\ \leq\ & \frac{1}{2n}\sum_{i=0}^{n-1}\left[g\left(t + i\cdot\frac{\tau-t}{n}\right) + g\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]. \end{aligned}$$

Proof. Consider the equidistant partitioning of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau < t$) given by

$$(3.2) \qquad E_n : x_i = t + i\cdot\frac{\tau - t}{n}, \quad i = \overline{0,n}.$$

Then, applying the Hermite-Hadamard inequality, we may write that:

$$g\left(\frac{x_i + x_{i+1}}{2}\right) \leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} g(u)\,du \leq \frac{g(x_i)+g(x_{i+1})}{2}$$

i.e.,

$$\begin{aligned} g\left(t + \left(i+\frac{1}{2}\right)\cdot\frac{t-\tau}{n}\right) \leq\ & \frac{n}{\tau-t}\int_{x_i}^{x_{i+1}} g(u)\,du \\ \leq\ & \frac{1}{2}\left[g\left(t+i\cdot\frac{\tau-t}{n}\right) + g\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]. \end{aligned}$$

Dividing by n and summing over i from 0 to $n-1$, we deduce the desired inequality (3.1). □

The following generalization of Theorem 7 holds.

THEOREM 8. *Assume that $f:(a,b)\to\mathbb{R}$ fulfills the hypothesis of Theorem 7. Then for all $n\geq 1$, we have the double inequality:*

$$
\begin{aligned}
(3.3)\qquad & \frac{b-a}{n\pi}\sum_{i=0}^{n-1}\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right]\\
& +\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\
\leq\ & (Tf)(a,b;t)\\
\leq\ & \frac{f(b)-f(a)+f'(t)(b-a)}{2n\pi}\\
& +\frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-i\cdot\frac{t-a}{n},t+i\cdot\frac{b-t}{n}\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)
\end{aligned}
$$

for any $t\in(a,b)$, where $[f;c,d]$ denotes the divided difference $\frac{f(c)-f(d)}{c-d}$.

Proof. If we write the inequality (3.1) for f', then we have

$$
\begin{aligned}
(3.4)\qquad & \frac{1}{n}\sum_{i=0}^{n-1}f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]\\
\leq\ & \frac{f(\tau)-f(t)}{\tau-t}\\
\leq\ & \frac{1}{2n}\sum_{i=0}^{n-1}\left[f'\left(t+i\cdot\frac{\tau-t}{n}\right)+f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\\
=\ & \frac{1}{2n}\left[f'(t)+\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right.\\
& \left.+\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)+f'(\tau)\right]\\
=\ & \frac{1}{2n}\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right],
\end{aligned}
$$

since it is obvious that

$$\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right).$$

Applying the PV over t, i.e., $\lim_{\varepsilon\to0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)$ to the inequality (3.4), we deduce

$$
\begin{aligned}
(3.5)\qquad & \frac{1}{n}\sum_{i=1}^{n-1} PV\int_a^b f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau \\
\le\ & PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau \\
\le\ & \frac{1}{2n}PV\int_a^b\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau.
\end{aligned}
$$

Now, as

$$
\begin{aligned}
& PV\int_a^b f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau \\
=\ & \lim_{\varepsilon\to0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau\right) \\
=\ & \lim_{\varepsilon\to0+}\frac{n}{i+\frac{1}{2}}\left[f\left(t-\left(i+\frac{1}{2}\right)\cdot\frac{\varepsilon}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right)\right. \\
& \left.+f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\varepsilon}{n}\right)\right] \\
=\ & \frac{n}{i+\frac{1}{2}}\left[f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right)\right] \\
=\ & (b-a)\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& PV\int_a^b\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau \\
=\ & \lim_{\varepsilon\to0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau \\
=\ & \lim_{\varepsilon\to0+}\left[f'(t)(t-\varepsilon-a)+f'(t)(b-t-\varepsilon)+f(t-\varepsilon)-f(a)\right. \\
& +f(b)-f(t+\varepsilon)+2\sum_{i=1}^{n-1}\frac{n}{i}\left[f\left(t-\frac{i\varepsilon}{n}\right)-f\left(t+i\cdot\frac{a-t}{n}\right)\right.
\end{aligned}
$$

$$+ f\left(t+i\cdot\frac{b-t}{n}\right) - f\left(t+\frac{i\varepsilon}{n}\right)\Bigg]\Bigg]$$
$$= f(b) - f(a) + f'(t)(b-a)$$
$$+ 2(b-a)\sum_{i=1}^{n-1}\left[f; t - i\cdot\frac{t-a}{n}, t+i\cdot\frac{b-t}{n}\right],$$

then by (3.5) we deduce

$$\begin{aligned}
&\frac{b-a}{n}\sum_{i=0}^{n-1}\left[f; t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n}, t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right] \\
&\leq PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau \\
&\leq \frac{f(b)-f(a)+f'(t)(b-a)}{2n} \\
&\quad + \frac{b-a}{n}\sum_{i=1}^{n-1}\left[f; t-i\cdot\frac{t-a}{n}, t+i\cdot\frac{b-t}{n}\right].
\end{aligned} \tag{3.6}$$

Using the identity (2.6) and the inequality (3.6), we obtain the desired result (3.3). □

4. The case of nonequidistant partitioning

The following lemma holds.

LEMMA 2. *Let $g : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and $t,\tau \in [a,b]$ with $t \neq \tau$. If $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$, then we have the inequality:*

$$\begin{aligned}
&\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, g\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t + \frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right] \\
&\leq \frac{1}{\tau-t}\int_t^\tau g(u)\,du \\
&\leq \frac{1}{2}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\left\{g\left[(1-\lambda_i)t+\lambda_i\tau\right] + g\left[(1-\lambda_{i+1})t+\lambda_{i+1}\tau\right]\right\}.
\end{aligned} \tag{4.1}$$

Proof. Consider the partitioning of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$I_n : x_i = (1-\lambda_i)t + \lambda_i\tau, \quad \left(i = \overline{0,n}\right).$$

Then, obviously,

$$\frac{x_i + x_{i+1}}{2} = \left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right) t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot \tau, \quad \left(i = \overline{0, n-1}\right)$$

and

$$x_{i+1} - x_i = (\tau - t)(\lambda_{i+1} - \lambda_i), \quad \left(i = \overline{0, n-1}\right).$$

Applying the Hermite-Hadamard inequality on $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$, we may write that

$$\begin{aligned} & g\left[\left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right) t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot \tau\right] \\ \leq\ & \frac{1}{(\tau - t)(\lambda_{i+1} - \lambda_i)} \int_{x_i}^{x_{i+1}} g(u)\, du \\ \leq\ & \frac{1}{2} \left\{ g\left[(1 - \lambda_i) t + \lambda_i \tau\right] + g\left[(1 - \lambda_{i+1}) t + \lambda_{i+1} \tau\right] \right\} \end{aligned}$$

for any $i = \overline{0, n-1}$.

If we multiply with $\lambda_{i+1} - \lambda_i > 0$ and sum over i from 0 to $n-1$, we deduce the desired inequality (4.1). □

The following theorem holds.

THEOREM 9. *Assume that $f : (a, b) \to \mathbb{R}$ fulfills the hypothesis of Theorem 7. Then for all $n \geq 1$, and $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$, we have the inequality*

$$\begin{aligned} & \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) + \frac{b-a}{\pi} \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \\ & \times \left[f; \left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right) t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot b, \right. \\ & \qquad \left. \left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right) t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot a \right] \\ \leq\ & (Tf)(a, b; t) \\ \leq\ & \frac{1}{2\pi} \left\{ \lambda_1 (b-a) f'(t) + (1 - \lambda_{n-1}) \left[f(b) - f(a)\right] \right\} \\ & + \frac{b-a}{2\pi} \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \left[f; (1 - \lambda_i) t + \lambda_i b, (1 - \lambda_i) t + \lambda_i a\right] \\ & + \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) \end{aligned} \tag{4.2}$$

for any $t \in (a,b)$.

Proof. If we write the inequality (4.1) for f', then we have

$$
\begin{aligned}
(4.3)\quad & \sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right] \\
\leq\ & \frac{f(\tau)-f(t)}{\tau-t} \\
\leq\ & \frac{1}{2}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\left\{f'[(1-\lambda_i)t+\lambda_i\tau]+f'[(1-\lambda_{i+1})t+\lambda_{i+1}\tau]\right\} \\
=\ & \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'[(1-\lambda_i)t+\lambda_i\tau]\right. \\
& \left.+\sum_{i=0}^{n-2}(\lambda_{i+1}-\lambda_i)\, f'[(1-\lambda_{i+1})t+\lambda_{i+1}\tau]+(1-\lambda_{n-1})\, f'(\tau)\right] \\
=\ & \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'[(1-\lambda_i)t+\lambda_i\tau]\right. \\
& \left.+\sum_{i=1}^{n-1}(\lambda_i-\lambda_{i-1})\, f'[(1-\lambda_i)t+\lambda_i\tau]+(1-\lambda_{n-1})\, f'(\tau)\right] \\
=\ & \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'[(1-\lambda_i)t+\lambda_i\tau]\right. \\
& \left.+(1-\lambda_{n-1})\, f'(\tau)\right].
\end{aligned}
$$

Applying the PV over t, i.e., $\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^{b}\right)$ to the inequality (4.3) we deduce

$$
\begin{aligned}
(4.4)\quad & \sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, PV\int_a^b f'\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right]d\tau \\
\leq\ & PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau
\end{aligned}
$$

$$\leq \frac{1}{2}\left[\lambda_1 (b-a) f'(t) + \sum_{i=1}^{n-1} (\lambda_{i+1} - \lambda_i) PV \int_a^b f'[(1-\lambda_i)t + \lambda_i \tau] d\tau \right.$$
$$\left. + (1-\lambda_{n-1})(f(b) - f(a))\right].$$

Since

$$\begin{aligned} & PV \int_a^b f'\left[\left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right)t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot \tau\right] d\tau \\ = & \frac{2}{\lambda_i + \lambda_{i+1}} \left(f\left[\left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right)t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot b\right]\right. \\ & \left. - f\left[\left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right)t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot a\right]\right) \\ = & (b-a)\left[f; \left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right)t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot b,\right. \\ & \left. \left(1 - \frac{\lambda_i + \lambda_{i+1}}{2}\right)t + \frac{\lambda_i + \lambda_{i+1}}{2} \cdot a\right] \end{aligned}$$

and

$$\begin{aligned} & PV \int_a^b f'[(1-\lambda_i)t + \lambda_i \tau] d\tau \\ & \qquad = (b-a)[f; (1-\lambda_i)t + \lambda_i b, (1-\lambda_i)t + \lambda_i a], \end{aligned}$$

then by (4.4) we deduce the desired inequality (4.2). □

REMARK 2. It is obvious that for $\lambda_i = \frac{i}{n}$ $\left(i = \overline{0,n}\right)$, we recapture the inequality (3.3).

The following corollary also holds.

COROLLARY 3. *Assume that $f:(a,b)\to\mathbb{R}$ fulfills the hypothesis of Theorem 7. Then for $n\geq 1$ we have:*

$$\begin{aligned}(4.5)\quad &\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ &+\frac{b-a}{\pi}\left[\frac{1}{2^{n-1}}\left[f;\left(1-\frac{1}{2^n}\right)t+\frac{1}{2^n}b,\left(1-\frac{1}{2^n}\right)t+\frac{1}{2^n}a\right]\right]\\ &+\frac{b-a}{\pi}\sum_{i=1}^{n-1}\frac{1}{2^{n-1}}\left[f;\left(1-\frac{3}{2^{n-i}}\right)t+\frac{3}{2^{n-i}}b,\left(1-\frac{3}{2^{n-i}}\right)t+\frac{3}{2^{n-i}}a\right]\\ \leq\ &(Tf)(a,b;t)\\ \leq\ &\frac{1}{2\pi}\left\{\frac{(b-a)f'(t)}{2^{n-1}}+\frac{1}{2}[f(b)-f(a)]\right\}\\ &+\frac{b-a}{2\pi}\left(\frac{1}{2^{n-2}}-1\right)\\ &\quad\times\left[f;\left(1-\frac{1}{2^{n-1}}\right)t+\frac{1}{2^{n-1}}b,\left(1-\frac{1}{2^{n-1}}\right)t+\frac{1}{2^{n-1}}a\right]\\ &+3\cdot\frac{b-a}{2\pi}\sum_{i=2}^{n-1}\frac{1}{2^{n-i+1}}\\ &\quad\times\left[f;\left(1-\frac{1}{2^{n-i}}\right)t+\frac{1}{2^{n-i}}b,\left(1-\frac{1}{2^{n-i}}\right)t+\frac{1}{2^{n-i}}a\right]\\ &+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

for any $t\in(a,b)$.

The proof follows by Theorem 9 applied for $\lambda_0=0$, $\lambda_i=\frac{2^i}{2^n}$, $i=\overline{1,n}$. We omit the details.

5. Numerical experiments

Let us define the following bounds

$$\begin{aligned}L_n(f,a,b;t):=\ &\frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right]\\ &+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

called the *lower bound* and

$$\begin{aligned} U_n(f,a,b;t) \; : \; &= \frac{f(b)-f(a)+f'(t)(b-a)}{2n\pi} \\ &+ \frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-i\cdot\frac{t-a}{n},t+i\cdot\frac{b-t}{n}\right] \\ &+ \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \end{aligned}$$

called the *upper bound* for the Finite Hilbert Transform $(Tf)(a,b;t)$ (as shown by the inequality (3.3)). We also define the *left error* $LEr_n(f,a,b;t)$

$$LEr_n(f,a,b;t) := (Tf)(a,b;t) - L_n(f,a,b;t) \geq 0$$

and the *right error* $REr_n(f,a,b;t)$

$$REr_n(f,a,b;t) := U_n(f,a,b;t) - (Tf)(a,b;t) \geq 0$$

and will investigate them numerically for different functions f and natural numbers n.

If we consider the function $f:[-1,1]\to R, f(x):=\exp(x)$, then the exact finite Hilbert transform provided by Maple 6, is

$$(Tf)(-1,1;t) = (\exp(t)Ei(1-t) - \exp(t)Ei(-t-1))/\pi, t\in[-1,1].$$

The plot of the Hilbert transform is embodied in Figure 1.

If we plot in the same system of co-ordinates $LEr_n(f,a,b;t)$ and $REr_n(f,a,b;t)$, for $n=100$, then we observe that the distance between the exact Hilbert transform and its lower bound is smaller than the distance between the same Hilbert transform and its upper bound (see Figure 2). A theoretical investigation on this fact will be conducted in [12].

If we now consider the polynomial function $f:[-1,1]\to\mathbb{R},\ f(x)=x^5$, then its derivative is a convex function on $[-1,1]$. Its Hilbert transform is

$$(Tf)(-1,1;t) = (2t^4+2/5-t^5\ln(t+1)+t^5\ln(1-t)+2/3\cdot t^2)/\pi, t\in[-1,1]$$

and the plot is embodied in Figure 3.

If we plot in the same system of co-ordinates $LEr_n(f,a,b;t)$ and $REr_n(f,a,b;t)$, for $n=1000$, then we observe that the distance between the exact Hilbert transform and its lower bound is smaller than the distance between the same Hilbert transform and its upper bound (see Figure 4).

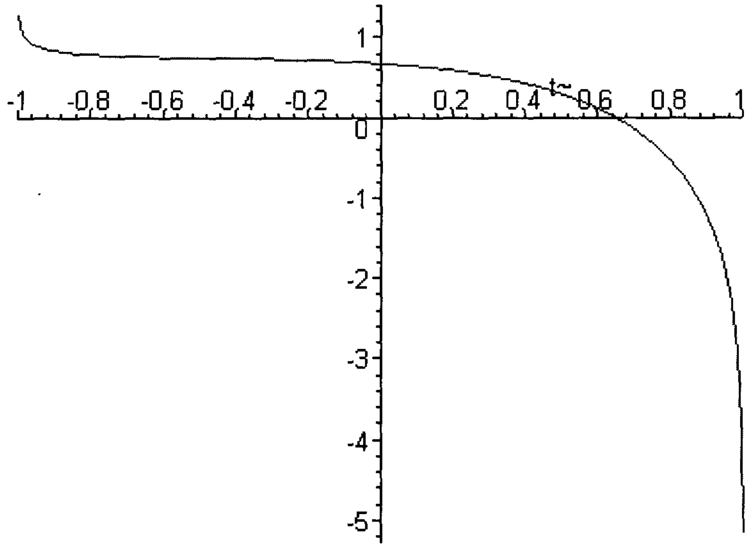

FIGURE 1

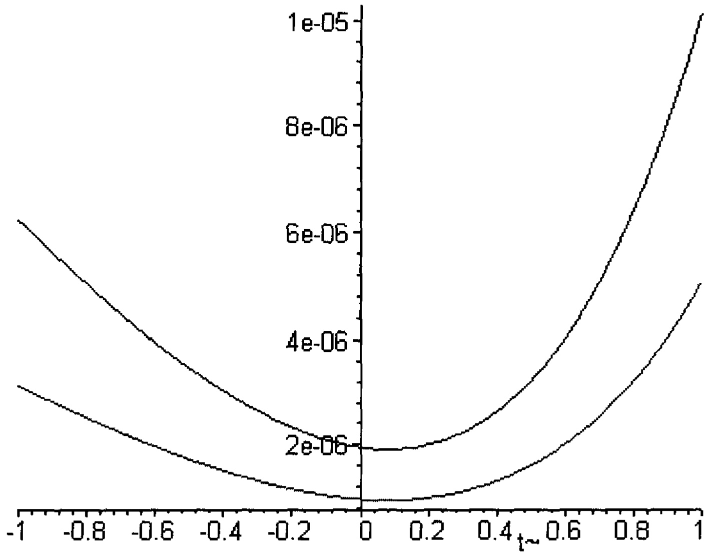

FIGURE 2

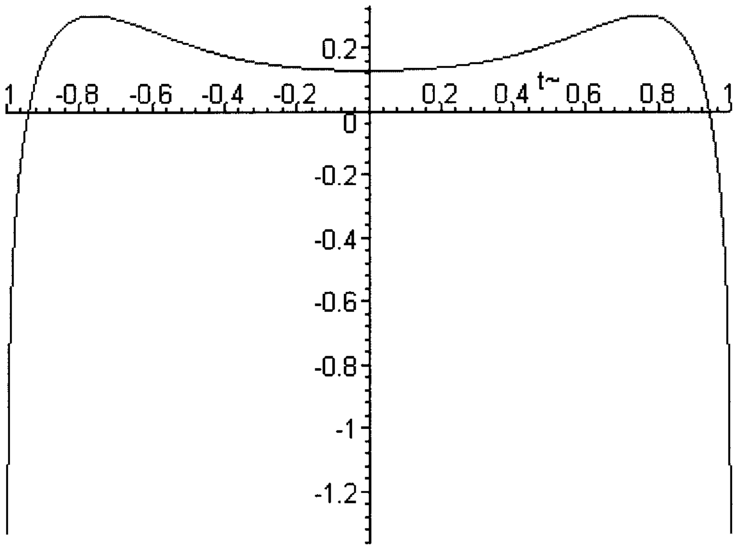

FIGURE 3

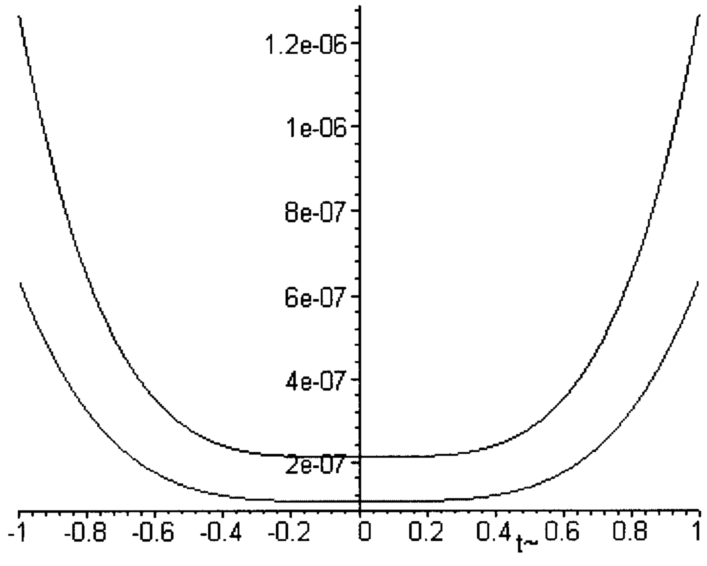

FIGURE 4

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