J. Korean Math. Soc. **39** (2002), No. 5, pp. 693–708

# ON THE RICCI CURVATURE OF SUBMANIFOLDS IN THE WARPED PRODUCT $L \times_f F$

YOUNG-MI KIM AND JIN SUK PAK

ABSTRACT. The warped product $L \times_f F$ of a line $L$ and a Kaehler manifold $F$ is a typical example of Kenmotsu manifold. In this paper we determine submanifolds of $L \times_f F$ which are tangent to the structure vector field and satisfy certain conditions concerning with Ricci curvature and mean curvature.

## 1. Fundamental equations on Kenmotsu manifold

A *Kenmotsu manifold* ([7]) is a $(2m+1)$-dimensional Riemannian manifold which has an almost contact metric structure $(\phi, \xi, \eta, g)$ satisfying

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \tag{1.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \qquad g(\xi, X) = \eta(X), \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$(\widetilde{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{1.4}$$

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi \tag{1.5}$$

for any vector fields $X$ and $Y$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to $g$. A typical example of Kenmotsu manifold is the warped product $L \times_f F$, where $F$ is a Kaehler manifold and $f(t) = ce^t$ ($c$ is a nonzero constant) a function on a line $L$. In fact a Kenmotsu structure $(\phi, \xi, \eta, g)$ on $L \times_f F$ is given as follows. Denote by $(J, G)$ the Kaehler structure of $F$ and let $(t, x_1, \cdots, x_{2m})$ be a local coordinate of

Received September 4, 2001. Revised January 28, 2002.
2000 Mathematics Subject Classification: 53B40.
Key words and phrases: Kenmotsu manifold, totally real submanifold, Ricci curvature.

$L \times_f F$ where $t$ and $(x_1, \cdots, x_{2m})$ are the local coordinates of $L$ and $F$, respectively. We define a Riemannian metric tensor $g$, a vector field $\xi$ and a 1-form $\eta$ as follows.

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)G_{(x)} \end{pmatrix},$$

$$\xi = d/dt, \quad \eta(X) = g(X, \xi).$$

We also define a $(1,1)$-tensor field $\phi$ by

$$\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\phi}_{(t,x)} \end{pmatrix},$$

where

$$\tilde{\phi}_{(t,x)} = (exp(t\xi))_* J_x (exp(-t\xi))_*.$$

Then we can easily verify that the aggregate $(\phi, \xi, \eta, g)$ satisfies (1.1)-(1.5) (for more details, see [7]).

We notice that Kenmotsu structure is normal but not Sasakian in the sense of [1, 9, 11] and especially is not compact because of (1.5). Moreover, in order that a Kenmotsu manifold has (point wise) constant $\phi$-holomorphic sectional curvature $c$, it is necessary and sufficient that its curvature tensor $\widetilde{R}$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\end{aligned}$$

for any vector fields $X, Y, Z$ ([7]). In the sequel we will denote such a manifold by $\widetilde{M}^{2m+1}(c)$.

REMARK. An example of Kenmotsu manifold with constant $\phi$-holom orphic sectional curvature is the warped product $L \times_f F(k)$, where $F(k)$ denotes a Kaehler manifold with constant holomorphic sectional curvature $k$. Moreover, if a Kenmotsu manifold is a space of constant $\phi$-holomorphic sectional curvature $c$, then it is a space of constant curvature $c = -1$ (for details, see [7]). As already shown in [7, 10] the warped product $L \times_f CE^m$ is a Kenmotsu manifold of constant curvature $c = -1$ whose automorphism group has the maximum dimension, where $CE^m$ denotes the complex Euclidean space with $\dim_C = m$.

## 2. Fundamental properties on submanifolds of $L \times_f F$

Let $M$ be an $n$-dimensional submanifold of a Kenmotsu manifold $\widetilde{M}$ in which the structure vector field $\xi$ is tangent to $M$. Denoting by $\nabla$ and $\nabla^{\perp}$ the induced connections on $M$ and the normal bundle $T^{\perp}M$ of $M$ respectively, we have the equations of Gauss and Weingarten

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.2}$$

for tangent vector fields $X, Y$ and normal vector field $N$ to $M$, where $h$ and $A_N$ denote the second fundamental form and the shape operator in the direction of $N$ which are related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.3}$$

We first notice that (1.5) and (2.3) yield

$$A_N \xi = 0 \tag{2.4}$$

for any normal vector field $N$ to $M$ since the structure vector field $\xi$ is tangent to $M$.

For a tangent vector field $X$ and normal vector field $N$ to $M$, we put

$$\phi X = PX + FX \quad \text{and} \quad \phi N = tN + t^{\perp}N, \tag{2.5}$$

where $PX$ and $tN$ denote the tangential component of $\phi X$ and $\phi N$, respectively. Then we can easily see that $P$ and $t^{\perp}$ are skew-symmetric endomorphisms acting on $T_pM$ and $T_p^{\perp}M$, respectively. If $\phi$ maps $T_pM$ into $T_pM$ for each $p \in M$ and the structure vector field $\xi$ is tangent to $M$, then $M$ is said to be *invariant* in $\widetilde{M}$. On the other side, if $\phi$ maps $T_pM$ into $T_p^{\perp}M$ for each point $p \in M$ and $\xi$ is tangent to $M$, then $M$ is said to be *totally real* (or *anti-invariant*) in $\widetilde{M}$ (cf. [11]).

If the Kenmotsu manifold $\widetilde{M}$ has (point wise) constant $\phi$-holomorphic sectional curvature c, then the equation of Gauss for $M$ is given by

(2.6)

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{c-3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \frac{c+1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) \\ &- g(Y,Z)\eta(X)\eta(W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\ &+ 2g(X,\phi Y)g(\phi Z,W)\} + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \end{aligned}$$

for tangent vector fields $X, Y, Z, W$ to $M$.

The mean curvature vector field $H$ of $M$ in $\widetilde{M}$ is defined by $H = \frac{1}{n}\text{trace } h$. The Ricci tensor $S$ and the scalar curvature $\rho$ at a point $p \in M$ are given respectively by $S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$ and $\rho = \sum_{i=1}^{n} S(e_i, e_i)$, where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of the tangent space $T_pM$. For a submanifold $M$ of $\widetilde{M}(c)$, by taking contracting on (2.6) we have the following basic formula:

$$(2.7)\ \ \rho = \frac{(n-1)}{4}\{c(n-2) - 3n - 2\} + \frac{3(c+1)}{4}\|P\|^2 + n^2\|H\|^2 - |h|^2,$$

where $|h|^2$ denotes the squared norm of the second fundamental form.

## 3. Ricci tensor of submanifolds in Kenmotsu manifold

In his paper [5], Chen proved that there exists a basic inequality on Ricci tensor $S$ for an $n$-dimensional submanifold $M$ in a real space form $R^m(c)$; namely,

$$S \leq ((n-1)c + \frac{n^2}{4}\|H\|^2)g$$

with the equality holding if and only if either $M$ is a totally geodesic submanifold or $n = 2$ and $M$ is a totally umbilical submanifold.

In this section we will investigate the inequality for an $n$-dimensional submanifold $M$ of $\widetilde{M}^{2m+1}(c)$ whose structure vector field $\xi$ is tangent to $M$. In order to do that we need a lemma due to Chen ([2, 3, 4]).

LEMMA C. ([2, 3, 4]) *Let $a_1, \cdots, a_n, d$ be $n+1$ ($n \geq 2$) real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i{}^2 + d)$$

*then $2a_1a_2 \geq d$ with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.*

For a submanifolds $M$ in $\widetilde{M}^{2m+1}(c)$, we have the following.

THEOREM 3.1. *Let $M$ be a submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field $\xi$ is tangent to $M$. Then the Ricci tensor $S$ of $M$*

*satisfies*

(3.1)

$$S(X,X) \leq \frac{n^2\|H\|^2}{4} + \frac{-3n+2}{4} + \frac{3(c+1)}{2}\|PX\|^2 + \frac{(n-2)c}{4} - \frac{(n-2)(c+1)}{4}\eta^2(X)$$

*for any unit vector* $X \in T_pM$. *The equality holds identically if and only if* $M$ *is totally geodesic in* $\widetilde{M}^{2m+1}(c)$.

*Proof.* Let $M$ be a submanifold of $\widetilde{M}^{2m+1}(c)$ . Then it follows from (2.7) that

(3.2)

$$\rho = \frac{(n-1)(n-2)c}{4} - \frac{(3n+2)(n-1)}{4} + \frac{3(c+1)}{4}\|P\|^2 + n^2\|H\|^2 - |h|^2.$$

We put

(3.3)

$$\delta = \rho - \frac{(n-1)(n-2)c}{4} + \frac{(3n+2)(n-1)}{4} - \frac{3(c+1)}{4}\|P\|^2 - \frac{n^2\|H\|^2}{2}.$$

Then from (3.2) and (3.3) we find

$$n^2\|H\|^2 = 2(\delta + |h|^2). \tag{3.4}$$

Assume that $H \neq 0$. Let $\{e_1, e_2, \cdots, e_{2m+1}\}$ be an orthonormal basis of $T_p\widetilde{M}$ such that

(1) $e_1, \cdots, e_n$ are tangent to M,
(2) $e_{n+1} = \frac{H}{\|H\|}$.

Putting $a_i = h_{ii}^{n+1}, i = 1, \cdots, n$ and using (3.4), we get

(3.5)

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = 2\left\{\delta + \sum_{i=1}^{n}(a_i)^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i,j\leq n}(h_{ij}^{r})^2\right\}.$$

Equation (3.5) is equivalent to

(3.6)

$$\Big(\sum_{i=1}^{3}\bar{a}_i\Big)^2 = 2\left\{\delta + \sum_{i=1}^{3}(\bar{a}_i)^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i,j\leq n}(h_{ij}^{r})^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta\right\},$$

where $\bar{a}_1 = a_1, \bar{a}_2 = a_2 + a_3 + \cdots + a_{n-1}, \bar{a}_3 = a_n$. Applying Lemma C to (3.6) (for $n = 3$), we have $2\bar{a}_1\bar{a}_2 \geq d$ with equality holding if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$, where we put

$$d = \delta + \sum_{1\leq i\neq j\leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i,j\leq n} (h_{ij}^r)^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta.$$

This inequality is equivalent to

$$\sum_{1\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta \geq \delta + 2 \sum_{1\leq i<j\leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i,j\leq n} (h_{ij}^r)^2,$$

which yields, by (3.3)

(3.7)

$$\frac{(n-1)(n-2)c}{4} - \frac{(3n+2)(n-1)}{4} + \frac{3(c+1)}{4}\|P\|^2 + \frac{n^2\|H\|^2}{2}$$
$$\geq \rho - \sum_{1\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta + 2 \sum_{1\leq i<j\leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i,j\leq n} (h_{ij}^r)^2.$$

Using (2.6), we have

(3.8)

$$\rho - \sum_{1\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta + 2 \sum_{1\leq i<j\leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i,j\leq n} (h_{ij}^r)^2$$
$$= \sum_{1\leq i\neq j\leq n-1} R(e_i, e_j, e_j, e_i) + 2S(e_n, e_n) - \sum_{1\leq\alpha\neq\beta\leq n-1} a_\alpha a_\beta$$
$$+ 2 \sum_{1\leq i<j\leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i,j\leq n} (h_{ij}^r)^2$$
$$= \frac{(n-1)(n-2)c}{4} - \frac{3(n-1)(n-2)}{4} - \frac{(c+1)(2n-4)}{4} + 2S(e_n, e_n)$$
$$+ \frac{(c+1)(2n-4)}{4}\eta^2(e_n) + \frac{3(c+1)}{4}\|P\|^2 - \frac{3(c+1)}{2}\|Pe_n\|^2$$
$$+ 2\sum_{i=1}^{n-1}(h_{in}^{n+1})^2 + \sum_{r=n+2} \left\{(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + (\sum_{\alpha=1}^{n-1} h_{\alpha\alpha}^r)^2 \right\}.$$

Combining (3.7) and (3.8) yields

(3.9)

$$S(e_n,e_n)+\sum_{1\le i<n}(h_{in}^{n+1})^2+\sum_{r=n+2}\Big[\sum_{i=1}^{n-1}(h_{in}^r)^2+\frac{1}{2}\{(h_{nn}^r)^2+(\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r)^2\}\Big]$$
$$\le\frac{n^2\|H\|^2}{4}+\frac{-6n+4}{8}+\frac{3(c+1)}{4}\|Pe_n\|^2+\frac{(2n-4)}{8}\{c-(c+1)\eta^2(e_n)\}$$

and consequently

(3.10)

$$\begin{aligned}S(e_n,e_n)\ &\le\frac{n^2\|H\|^2}{4}+\frac{-3n+2}{4}+\frac{3(c+1)}{4}\|Pe_n\|^2\\&+\frac{(n-2)c}{4}-\frac{(n-2)(c+1)}{4}\eta^2(e_n).\end{aligned}$$

Moreover, it is clear from (3.9) that the equality holds if and only if

(3.11)

$$h_{\alpha n}^{n+1}=0,\quad h_{in}^r=0,\quad \sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r=0$$
$$\text{for } 1\le\alpha\le n-1,\ 1\le i\le n,\ n+2\le r\le 2m+1.$$

Since Lemma C yields that $2\bar{a}_1\bar{a}_2=d$ if and only if $\bar{a}_1+\bar{a}_2=\bar{a}_3$, (3.6) also implies that the equality holds if and only if $\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^{n+1}=h_{nn}^{n+1}$. Since $e_n$ can be any unit tangent vector of $M^n$, (3.10) implies the inequality (3.1). Now, assume that for all unit tangent vector $e_i$ the equality sign of (3.1) holds identically. Then we have

$$\begin{aligned}&h_{ij}^{n+1}=0\quad(1\le i\ne j\le n),\\&h_{ij}^r=0\quad(1\le i,j\le n,n+2\le r\le 2m+1),\\&\sum_{k\ne i}h_{kk}^{n+1}=h_{ii}^{n+1},\end{aligned}$$

from which together with (2.4), we conclude that $M$ is totally geodesic.□

COROLLARY 3.2. *Let $M$ be a totally real submanifold of $\widetilde{M}^{2m+1}(c)$. Then the Ricci tensor $S$ of $M$ satisfies*

$$S(X,X) \leq \frac{n^2\|H\|^2}{4} + \frac{-3n+2}{4} + \frac{(n-2)c}{4} - \frac{(n-2)(c+1)}{4}\eta^2(X)$$

*for any unit vector $X \in T_pM$. The equality holds identically if and only if $M$ is totally geodesic in $\widetilde{M}^{2m+1}(c)$.*

COROLLARY 3.3. *Let $M$ be a submanifold of the warped product $L \times_f CE^m$ whose structure vector field $\xi$ is tangent to $M$. Then the Ricci tensor $S$ of $M$ satisfies*

$$S \leq (-n+1+\frac{n^2\|H\|^2}{4})g. \tag{3.12}$$

*The equality holds identically if and only if $M$ is totally geodesic in $L \times_f CE^m$.*

## 4. Ricci curvature and squared mean curvature

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_pM$. Suppose $L$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L$. We choose an orthonormal basis $\{e_1, \cdots, e_k\}$ of $L$ such that $e_1 = X$. Define the Ricci curvature $Ric_L$ of $L$ at $X$ by

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. Such a curvature is simply called a *$k$-Ricci curvature* ([5]). The scalar curvature $\tau$ of the $k$-plane section $L$ is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}.$$

For each integer $k$, $2 \leq k \leq n$, two Riemannian invariants $\theta_k$, $\bar{\theta}_k$ on the $n$-dimensional Riemannian manifold $M$ is defined by

$$\theta_k(p) = \frac{1}{k-1}\inf_{L,X} Ric_L(X), \; p \in M, \tag{4.1}$$

where $L$ runs over all $k$-plane sections in $T_pM$ and $X$ runs over all unit vectors in $L$.

$$(4.2) \qquad \bar{\theta}_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \ p \in M,$$

where $L$ runs over all $k$-plane sections in $T_pM$ which is orthogonal to $\xi$ and $X$ runs over all unit vectors in $L$.

For a submanifold $M$ in a Riemannian manifold the relative null space of $M$ at $p$ is defined by

$$(4.3) \qquad N_p = \{X \in T_pM | h(X,Y) = 0 \text{ for all } Y \in T_pM\}.$$

Recently Chen ([5]) established a relationship between *k-Ricci curvature* and the squared mean curvature for submanifold in a real space form. In this section we investigate *k-Ricci curvature* for submanifold of Kenmotsu manifold with constant $\phi$-holomorphic sectional curvature whose structure vector field $\xi$ is tangent to the submanifold.

THEOREM 4.1. *Let $M$ be an $n$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ $(c \le -1)$ whose structure vector field $\xi$ is tangent to $M$. Then*

(1) *For each unit vector $X \in T_pM$, we have*

$$(4.4) \qquad \begin{aligned} \|H\|^2 \ge & \frac{4}{n^2}\{Ric(X) + (n-1) - \frac{3(c+1)}{4}\|PX\|^2 \\ & - \frac{(n-2)(c+1)}{4} + \frac{(n-2)(c+1)}{4}\eta^2(X)\}. \end{aligned}$$

(2) *If $H(p) = 0$, then a unit tangent vector $X$ at $p$ satisfies the equality case of (4.4) if and only if $X \in N_p$.*

(3) *The equality case of (4.4) holds identically for all unit tangent vector at $p$ if and only if $p$ is a totally geodesic point.*

*Proof.* (1) Let $X \in T_pM$ be a unit tangent vector $X$ at $p$. We choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for $T_pM$ and $\{e_{n+1}, \cdots, e_{2m+1}\}$ for $T_p{}^{\perp}M$ with $e_1 = X$. Then, from (2.7), we have

(4.5)

$$n^2\|H\|^2 = \rho + |h|^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

It follows from (4.5) that

(4.6)

$$\begin{aligned}
n^2\|H\|^2 &= \rho + \sum_{r=n+1}^{2m+1}\{(h_{11}^r)^2 + ({h_{22}}^r + \cdots + h_{nn}^r)^2 + 2\sum_{1\le i<j\le n}(h_{ij}^r)^2\} \\
&\quad - \frac{3(c+1)}{4}\|P\|^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r \\
&\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4} \\
&= \rho + \frac{1}{2}\sum_{r=n+1}^{2m+1}\{(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\} \\
&\quad + 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{3(c+1)}{4}\|P\|^2 \\
&\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.
\end{aligned}$$

On the other hand, (2.6) implies

(4.7)

$$\begin{aligned}
K_{ij} &= \sum_{r=n+1}^{2m+1}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\} + \frac{c-3}{4} \\
&\quad - \frac{c+1}{4}\{\eta^2(e_i) + \eta^2(e_j)\} + \frac{3(c+1)}{4} g^2(e_i, \phi e_j)
\end{aligned}$$

and consequently

(4.8)

$$\begin{aligned}
\sum_{2\le i<j\le n} K_{ij} &= \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\} + \frac{(n-1)(n-2)(c-3)}{8} \\
&\quad + \frac{3(c+1)}{8}\|P\|^2 - \frac{3(c+1)}{4}\|Pe_1\|^2 \\
&\quad - \frac{(c+1)(n-2)}{4}\{1 - \eta^2(e_1)\}.
\end{aligned}$$

Taking account of (4.8) into (4.6), we get

(4.9)

$$\begin{aligned}
n^2\|H\|^2 \geq& \rho + \frac{1}{2}\sum_{r=n+1}^{2m+1}(h_{11}{}^r + h_{22}{}^r + \ldots + h^r_{nn})^2 - 2\sum_{2\leq i<j\leq n} K_{ij} \\
&+ \frac{(n-1)(n-2)(c-3)}{4} + \frac{3(c+1)}{4}\|P\|^2 - \frac{3(c+1)}{2}\|Pe_1\|^2 \\
&- \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-1)(n-2)c}{4} + \frac{(3n+2)(n-1)}{4} \\
&+ 2\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(h^r_{1j})^2 - \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\} \\
\geq& \rho + \frac{1}{2}n^2\|H\|^2 - 2\sum_{2\leq i<j\leq n} K_{ij} + 2(n-1) - \frac{3(c+1)}{2}\|Pe_1\|^2 \\
&- \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\},
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{1}{2}n^2\|H\|^2 \geq& \rho - 2\sum_{2\leq i<j\leq n} K_{ij} + 2(n-1) - \frac{3(c+1)}{2}\|Pe_1\|^2 \\
&- \frac{(c+1)(n-2)}{2}\{1-\eta^2(e_1)\},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\frac{1}{2}n^2\|H\|^2 \geq& 2Ric(X) - \frac{3(c+1)}{2}\|PX\|^2 + 2(n-1) \\
&- \frac{(c+1)(n-2)}{2}\{1-\eta^2(X)\}.
\end{aligned}$$

(2) Assume that $H(p) = 0$. The equality holds in (4.4) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, \quad r = n+1, \cdots, 2m+1. \end{cases}$$

Then $h^r_{1j} = 0$, for all $j = 1, 2, \cdots, n$ and $r = n+1, \cdots, 2m+1$, which means that $X \in N_p$.

(3) The equality case of (4.4) holds for all unit tangent vector at $p$ if and only if

$$\begin{cases} h^r_{ij} = 0, \ i \neq j \quad \text{and} \quad r = n+1, \cdots, 2m+1, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \ i = 1, \cdots, n \text{ and } r = n+1, \cdots, 2m+1, \end{cases}$$

which implies by (2.4) that $p$ is a totally geodesic point. □

THEOREM 4.2. *Let $M$ be an $n$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field $\xi$ is tangent to $M$. Then*

$$\|H\|^2 \geq \frac{\rho}{n(n-1)} - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n+2}{4n}. \tag{4.10}$$

*Proof.* Let $p \in M$ and let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for $T_pM$. From (2.6) we have

(4.11)

$$n^2\|H\|^2 = \rho + |h|^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

We choose an orthonormal basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots e_{2m+1}\}$ at $p$ such that $e_{n+1}$ is parallel to the mean curvature vector $H(p)$ and $e_1, \cdots, e_n$ diagonalize the shape operator $A_{n+1}$, then

$$A_{n+1} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

$$A_r = (h^r_{ij}) \quad \text{with} \quad \text{trace} A_r = 0, \quad r = n+2, \cdots, 2m+1,$$

which and (4.11) imply

(4.12)

$$n^2\|H\|^2 = \rho + \sum_{i=1}^{n} {a_i}^2 + \sum_{r=n+1}^{2m+1} \sum_{1\leq i \neq j \leq n} (h^r_{ij})^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}.$$

On the other hand

$$0 \leq \sum_{1 \leq i < j \leq n} (a_i - a_j)^2 = (n-1) \sum_{1 \leq i \leq n} {a_i}^2 - 2 \sum_{1 \leq i < j \leq n} a_i a_j,$$

which yields

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} {a_i}^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j \leq n \sum_{i=1}^{n} {a_i}^2,$$

which implies $\sum_{i=1}^{n} {a_i}^2 \geq n\|H\|^2$. Thus we have from(4.12)

$$\begin{aligned} n^2\|H\|^2 &\geq \rho + \sum_{i=1}^{n} {a_i}^2 - \frac{3(c+1)}{4}\|P\|^2 \\ &\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4} \\ &\geq \rho + n\|H\|^2 - \frac{3(c+1)}{4}\|P\|^2 \\ &\quad - \frac{(n-2)(n-1)c}{4} + \frac{(3n+2)(n-1)}{4}, \end{aligned}$$

or equivalently

$$\|H\|^2 \geq \frac{\rho}{n(n-1)} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 - \frac{(n-2)c}{4n} + \frac{(3n+2)}{4n}.$$

□

Corollary 4.3. *Let $M$ be an $n(\geq 2)$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)(c \leq -1)$ whose structure vector field $\xi$ is tangent to $M$. Then*

$$\|H\|^2 \geq \frac{1}{n^2}\{\rho + |h|^2 + n(n-1)\}.$$

*The equality holds identically if and only if either $c = -1$ or $n = 2$ and $M$ is totally real in the ambient manifold.*

*Proof.* (4.11) says

$$\|H\|^2 = \frac{1}{n^2}\Big\{\rho + |h|^2 - \frac{3(c+1)}{4}\|P\|^2 - \frac{(n-2)(n-1)(c+1)}{4} + n(n-1)\Big\}$$

and consequently

$$\|H\|^2 \geq \frac{1}{n^2}\{\rho + |h|^2 + n(n-1)\}$$

since $c \leq -1$. □

THEOREM 4.4. *Let $M$ be an $n$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field $\xi$ is tangent to $M$. Then for any integer $k$, $2 \leq k \leq n$ and any point $p \in M$ we have*

$$\|H\|^2(p) \geq \theta_k(p) - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n+2}{4n}. \tag{4.13}$$

*Proof.* Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for $T_pM$. Denoting by $L_{i_1,\cdots,i_k}$ the $k$-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$, we have

$$\tau(L_{i_1,\cdots,i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\cdots,i_k\}} Ric_{L_{i_1,\cdots,i_k}}(e_i), \tag{4.14}$$

$$\frac{1}{2}\rho(p) = \frac{1}{{}_{n-2}C_{k-2}}\sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1,\cdots,i_k}). \tag{4.15}$$

Combining (4.1), (4.14) and (4.15), we obtain

$$\frac{1}{2}\rho(p) \geq \frac{n(n-1)}{2}\theta_k(p),$$

which together with (4.10) yields (4.13). □

THEOREM 4.5. *Let $M$ be an $n$-dimensional submanifold of $\widetilde{M}^{2m+1}(c)$ whose structure vector field $\xi$ is tangent to $M$. Then for any integer $k$, $2 \leq k \leq n$ and any point $p \in M$ we have*

$$\|H\|^2(p) \geq \frac{n-1}{n}\bar{\theta}_k(p) - \frac{(n-2)c}{4n} - \frac{3(c+1)}{4n(n-1)}\|P\|^2 + \frac{3n-6}{4n}. \tag{4.16}$$

*Proof.* Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for $T_pM$. Denoting by $L_{i_1,\cdots,i_k}$ the $k$-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$, we have

$$\tau(L_{i_1,\cdots,i_k}) = \frac{1}{2} \sum_{i \in \{i_1,\cdots,i_k\}} Ric_{L_{i_1,\cdots,i_k}}(e_i), \tag{4.17}$$

$$\frac{1}{2}\rho(p) = \frac{1}{{}_{n-2}C_{k-2}} \sum_{1\le i_1<\cdots<i_k\le n} \tau(L_{i_1,\cdots,i_k}). \tag{4.18}$$

Combining (4.2), (4.17) and (4.18), we find

$$\frac{1}{2}\rho(p) \ge -(n-1) + \frac{(n-1)^2}{2}\bar{\theta}_k(p),$$

which together with (4.10) implies (4.16). □

## References


[1] D. E. Blair, *Contact manifolds in Riemannian geometry Matsumoto*, Lecture Note in Mathematics **509**, Springer-Verlag, Berlin, 1976.

[2] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch Math. (Basel) **60** (1993), 568–578.

[3] ———, *A Riemannian invariant for submanifold in space forms and its applications*, Geometry and topology of submanifolds **6** (1994), 58–81.

[4] ———, *A Riemannian invariant and its applications to submanifold theory*, Results Math. **27** (1995), 17–26.

[5] ———, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension*, Glasg. Math. J. **41** (1999), 33–41.

[6] S. Funabashi, Y.-M. Kim and J. S. Pak, *On submanifolds of $L\times_f F$ satisfying Chen's basic equality* (preprint).

[7] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93–103.

[8] Y. H. Kim and D.-S. Kim, *A basic inequality for submanifolds in Sasakian space forms*, Houston J. Math. **25** (1999), 247–257.

[9] S. Sasaki, *Almost contact manifolds*, Lecture Note, Tohoku Univ., 1965.

[10] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J. **21** (1969), 21–38.

[11] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhauser, Boston, 1983.

Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: ymkim91@hanmail.net
jspak@bh.kyungpook.ac.kr