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MARTENS' DIMENSION THEOREM FOR CURVES OF EVEN GONALITY

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Dedicated to Professor Hiroki Sato on his sixtieth birthday

ABSTRACT. For a smooth projective irreducible algebraic curve C of odd gonality, the maximal possible dimension of the variety of special linear systems $W_d^r(C)$ is $d-3r$ by a result of M. Coppens et al. [4]. This bound also holds if C does not admit an involution. Furthermore it is known that if $\dim W_d^r(C) \geq d-3r-1$ for a curve C of odd gonality, then C is of very special type of curves by a recent progress made by G. Martens [11] and Kato-Keem [9]. The purpose of this paper is to pursue similar results for curves of even gonality which does not admit an involution.

1. Introduction

Let C be a smooth projective irreducible algebraic curve over the field of complex numbers $\mathbb{C}$ or a compact Riemann surface of genus g. The Jacobian variety $J(C)$ is a g-dimensional abelian variety which parameterizes all the line bundles of given degree d on C. We denote by $W_d^r(C)$ the locus in $J(C)$ corresponding to those line bundles of degree d with at least $r+1$ independent global sections. Then $W_d^r(C)$ is a subvariety of $J(C)$ and can be equivalently viewed as the subvariety consisting of all effective divisor classes of degree d which move in a linear system of projective dimension at least r.

If $d \leq g+r-2$, then H. Martens [12] showed that the maximal possible dimension of $W_d^r(C)$ is $d-2r$ and the maximum is attained if and only if C is hyperelliptic.

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If $\dim W_d^r(C)$ is near the maximum number, C is of low gonality – the gonality of C means the minimal sheet number of a covering over $\mathbb{P}^1$, denoted by $\text{gon}(C)$ – or a double covering of a curve (cf. Mumford [13], Keem [10], Coppens-Martens-Keem [4]).

In particular, by Coppens, Martens and Keem [4, Theorem 3.2.1], if $\dim W_d^r(C) > d-3r$ for $d \le g-2$, then C is a double covering of a curve. Thus, if C does not have an involution – an automorphism of order 2 – or is of odd gonality, their result induces a significant refinement of the theorem of H. Martens. The precise statement of their theorem is:

THEOREM A ([4, Theorem 3.2.1]). *Let C be a smooth curve of genus g. Let $d \le g-1$ and $r \ge 2$. If $\dim W_d^r(C) = d - 2r - j$ for some j $(0 \le j \le r-1)$, then C is either a double covering of a curve of genus j or an extremal curve of degree $3r-1$ in $\mathbb{P}^r$, in this case $j = r-1$.*

Concerning this bound, recently G. Martens [11] and Kato-Keem [9] gave characterizations of $W_d^r(C)$'s for curves C of odd gonality which attain the dimension $d-3r$ or $d-3r-1$:

THEOREM B (G. Martens [11]). *Let C be a smooth projective curve of genus g over the complex number field. Assume that the gonality of C is odd. Then $\dim W_d^r(C) \le d-3r$ for any $d < g$, and if equality occurs for some $d \le g-2$ and $r > 0$ then C is either trigonal, smooth plane sextic, birational to a plane curve of degree 7 (in this case only $g = 13$ and $g = 14$ occur) or an extremal space curve of degree 10.*

THEOREM C (Kato and Keem [9]). *Let C be a smooth irreducible projective curve of genus g over the complex number field. Assume the gonality of C is odd and $\dim W_d^r(C) = d-3r-1$ for some $d \le g-4$ and $r > 0$. Then C is 5-gonal with $10 \le g \le 18$, $g = 20$ or 7-gonal of genus 21; furthermore C is a smooth plane sextic (resp. octic) in case $\text{gon}(C) = 5,\ g = 10$ (resp. $\text{gon}(C) = 7,\ g = 21$).*

In the present paper, we shall pursue similar results for curves of even gonality. In case of even gonality, Theorem A suggests us it is natural to assume that the curve does not have an involution.

In this paper, we use standard notation for divisors, linear series, invertible sheaves and line bundles on C as follows: As usual, g_d^r is a linear series of dimension r and degree d on C, which may be possibly incomplete. If D is a divisor on C, we write $|D|$ for the associated complete linear series on C. By K_C or K we denote a canonical divisor on C, and $|K_C|$ is the canonical linear series on C. The series $|K - g_d^r|$ is called the dual series of g_d^r. If L is a line bundle (or an invertible sheaf) we abbreviate the notation $H^i(C, L)$ (resp. $\dim H^i(C, L)$) by $H^i(L)$ (resp. $h^i(L)$) for simplicity. Also, for a divisor D on C we write $H^i(D), h^i(D)$ instead of $H^i(C, \mathcal{O}_C(D))$, $\dim H^i(C, \mathcal{O}_C(D))$. Since the Jacobian variety $J(C) \cong \mathrm{Pic}^0(C)$ is an abelian variety, it will cause no confusion to denote the addition on $J(C)$ by $+$. In particular for two non-empty subsets A and B of $J(C)$ we set $A \pm B := \{a \pm b | a \in A, b \in B\}$. A base-point-free g^r_d on C defines a morphism $f : C \to \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$. If f is birational onto its image $f(C)$ the given g^r_d is called simple. In case the given g^r_d is not simple, let C' be the normalization of $f(C)$. Then there is a morphism (a non-trivial covering map) $C \to C'$ and we use the same notation f for this covering map of some degree k induced by the original morphism $f : C \to \mathbb{P}^r$.

We recall the notions of the Clifford index and the Clifford dimension. For a line bundle L on C, the Clifford index of L, denoted by Cliff(L), is defined by $\mathrm{Cliff}(L) = \deg L - 2h^0(L) + 2$. The Clifford index of C is defined as

$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(L) | h^0(L) \geq 2, h^1(L) \geq 2\}.$$

We say that a line bundle L computes the Clifford index of C if $h^0(L) \geq 2, h^1(L) \geq 2$ and $\mathrm{Cliff}(L) = \mathrm{Cliff}(C)$. Then the Clifford dimension of C is defined as follows:

$$\mathrm{Cliff}\dim(C) = \min\{h^0(L) - 1 | L \text{ computes the Clifford index of } C\}.$$

Two quantities, the gonality and the Clifford index are related closely. We shall mention it in Lemmas 7 and 8.

2. Lemmas

In this section, we prepare several lemmas some of which are well known. Throughout this section, except for Lemma 9, let C be a smooth curve of genus g.

First, we give lemmas which admit genus bounds. For positive integers d, r, let $m = \left[\frac{d-1}{r-1}\right]$, $m_1 = \left[\frac{d-1}{r}\right]$, $\varepsilon = d - m(r-1) - 1$, $\varepsilon_1 = d - m_1 r - 1$ and $\mu_1 = \left[\frac{\varepsilon_1}{r-1}\right]$. Set

$$\begin{aligned}
\pi(d, r) &= \frac{m(m-1)}{2}(r-1) + m\varepsilon \\
\pi_1(d, r) &= \frac{m_1(m_1-1)}{2} r + m_1(\varepsilon_1 + 1) + \mu_1.
\end{aligned}$$

LEMMA 1 (Castelnuovo's bound). *Assume C admits a base-point-free and simple linear series g^r_d. Then $g \leq \pi(d, r)$.*

LEMMA 2 ([1, §7]). *If C admits infinite number of base-point-free simple linear series g^r_d's, then $g \leq \pi(d+1, r+1)$.*

The following is a special case of the so-called Castelnuovo-Severi inequality.

LEMMA 3 (Castelnuovo-Severi bound [2, Theorem 3.5]). *Assume there exist two curves C_1 and C_2 of genus g_1 and g_2, respectively, so that C is a k_i-sheeted covering of C_i $(i = 1, 2)$. If k_1 and k_2 are coprime, then $g \leq (k_1 - 1)(k_2 - 1) + k_1 g_1 + k_2 g_2$.*

The following lemma is a description of nearly extremal curves in projective space.

LEMMA 4 ([7, Theorem 3.15]). *Assume C admits a base-point-free simple linear series g^r_d. If $g > \pi_1(d, r)$ and $d \geq 2r + 1$, then C lies on a surface of degree $r - 1$ in $\mathbb{P}^r$.*

We also need the following lemmas which are easy consequences of [3, III, Exercise B-6]; note that there is a misprint in the exercise. The correct formula should be $r(\mathcal{D} + \mathcal{E}) \geq r(\mathcal{D}) + 2r(\mathcal{E}) - r(\mathcal{E} - \mathcal{D}) - 1$.

LEMMA 5. *Assume C admits a base-point-free simple linear series g^r_d. Let $\rho = \dim |2g^r_d|$. If $d \geq 2r - 1$, then $\rho \geq 3r - 1$ and if $d \geq \rho - 1$, then $\dim |3g^r_d| \geq 2\rho - 1$.*

LEMMA 6. *Assume C admits a base-point-free simple linear series g^r_d and a pencil g^1_n. If $\dim |g^r_d - g^1_n| = r - \rho$, then $\dim |g^r_d + g^1_n| \geq r + \rho$. Equivalently, if $\dim |g^r_d + g^1_n| = r + \rho$, then $\dim |g^r_d - g^1_n| \leq r - \rho$.*

The following lemmas are concerned with the gonality and the Clifford index.

LEMMA 7 ([5, Theorem 3]). *Let $c = \mathrm{Cliff}(C)$. Then any linear series g^r_d $(d \leq g - 1)$ computing c is of degree $d \leq 2(c + 2)$ unless C is hyperelliptic or bi-elliptic.*

LEMMA 8 ([6]). *Let $c = \mathrm{Cliff}(C)$ and $r = \mathrm{Cliff}\dim(C)$. Then,*
1) $\mathrm{gon}(C) = c + 2$ *if and only if* $r = 1$.
2) $\mathrm{gon}(C) = c + 3$ *if and only if* $r \geq 2$.
3) *If* $r = 2$, *then C is a smooth plane curve of degree* $c + 4$.
4) *If* $3 \leq r \leq 9$, *then* $g = 4r - 2$ *and* $c = 2r - 3$.

LEMMA 9. *Let C be a smooth plane curve of degree $n \geq 7$. Let g^r_d ($d < n(n-4)$) be a special linear series on C. If* $\mathrm{Cliff}(g^r_d) = \mathrm{Cliff}(C) = n-4$, *then* $(d,r) = (n,2)$. *If* $\mathrm{Cliff}(g^r_d) = n-3$, *then* $(d,r) = (n-1,1), (n+1,2)$ *or* $((n-4)n-1, (n-4)(n-1)/2-1)$ *if* $n \geq 8$. *In case $n = 7$, in addition to the above possibilities, there exists another possibility that* $(d,r) = (14,5)$.

Proof. This lemma is a straightforward consequence of the following Noether bound (cf. [8]): Write $d = kn - e$ with $1 \leq k \leq n-4$, $0 \leq e \leq n-1$. Then

$$r \leq \begin{cases} \dfrac{(k-1)(k+2)}{2} & \text{if} \quad e > k+1 \\ \dfrac{k(k+3)}{2} - e & \text{if} \quad e \leq k+1. \end{cases}$$

Assume $e > k+1$. If $k = 1$, then $r = 0$, whence we may assume $k \geq 2$. Then,

$$\begin{aligned} d - 2r - (n-3) &\geq kn - e - (k-1)(k+2) - n + 3 \\ &\geq kn - (n-1) - (k-1)(k+2) - n + 3 \\ &= (k-2)(n-k-3) \geq 0. \end{aligned}$$

Equality occurs if $k = 2$, $e = n-1$ or $k = n-3$, $e = n-1$, i.e. if $d = n+1$ or $d = (n-4)n+1$ and r attains the upper bound.

Assume $e \leq k+1$. Then,

$$\begin{aligned} d - 2r - (n-4) &\geq kn - e - k(k+3) + 2e - n + 4 \\ &\geq kn - k(k+3) - n + 4 \\ &= (k-1)(n-k-4) \geq 0. \end{aligned}$$

Thus, $d - 2r - (n-4) = 0$ holds only if $k = 1$, $e = 0$ or $k = n-4$, $e = 0$, i.e. if $d = n$ or $d = (n-4)n$. The latter case is excluded by the assumption. $d - 2r - (n-4) = 1$ holds if $k = 1$, $e = 1$ or $k = n-4$, $e = 1$, i.e. if $d = n+1$ or $d = (n-4)n - 1$ and r attains the upper bounds. In case $n = 7$, in addition to the above cases, $d - 2r - (n-4) = 1$ holds if $k = 2$, $e = 0$, i.e. $d = 14$ and $r = 5$. This proves the lemma. □

3. Case $d - 3r$

In this section, we shall treat the case $\dim W^r_d(C) = d - 3r$.

THEOREM 1. *Let C be an even gonal curve of genus g which does not have an involution. Assume there exist positive integers r and $d \leq g-2$, so that* $\dim W^r_d(C) = d - 3r \geq 0$. *Then, C is 4-gonal and one of the*

following holds:
i) C *is a smooth plane quintic* $(g = 6)$ *and* $\dim W_4^1(C) = 1$.
ii) C *is a plane curve of degree* 6 *and of genus* 8 *or* 9 *and* $\dim W_6^2(C) = 0$.
iii) C *is an extremal curve of degree* $3r$ *in* $\mathbb{P}^r$ $(g = 3r+3)$ *and* $\dim W_{3r}^r(C) = 0$.
iv) C *lies on a smooth normal scroll in* $\mathbb{P}^r$, $p_a(C) = 3r + 3$, $g = 3r + 2$ *and* $\dim W_{3r}^r(C) = 0$.
v) C *lies on a cone over a rational normal curve in* $\mathbb{P}^{r-1}$ *for* $r = 3$ *or* 4, $g = 3r + 2$ *and* $\dim W_{3r}^r(C) = 0$.
vi) C *is an extremal curve of degree* $3r + 2$ *in* $\mathbb{P}^{r+1}$ $(g = 3r + 3)$ *and* $\dim W_{3r+1}^r(C) = W_{3r+2}^{r+1}(C) - W(C)$ *is of dimension* 1.

REMARK. In the statements i) – vi) of the theorem, it is obvious that $W_d^r(C) + W_n(C)$ also satisfies the dimension hypothesis as long as $d + n \leq g - 2$. The same remark will be valid for the statement of Theorem 2, too.

Proof. Let Z be an irreducible component of $W_d^r(C)$ of dimension $d - 3r$ and let $g_d^r(z)$ be the linear series associated to an element $z \in Z$. Using the same procedure as in the proof of [11, 9], we may assume $g_d^r(z)$ is complete and base point free for general $z \in Z$.

First, assume $g_d^r(z)$ is compounded for a general $z \in Z$. Then $g_d^r(z)$ induces an n-sheeted covering map π of C onto a smooth curve C' of genus g' with $n|d$ and $n \geq 3$. Then $g_d^r(z)$ is the pull back of a base point free complete linear series $g_{d/n}^r$ on C' with respect to π.

Assume $g_{d/n}^r$ is non-special. Then $g' = \frac{d}{n} - r$. Let $g' = 0$. Then $\frac{d}{n} - r = g' = 0$ and $Z \subset r \cdot W_n^1(C)$. Hence, by H. Martens' theorem [12] and the hypothesis, we have

$$n - 3 \geq \dim W_n^1(C) \geq d - 3r = (n - 3)r.$$

Since $g' = 0$, we have $n \geq 4$, whence $r = 1$. Thus, $\dim W_d^1(C) = d - 3$. Then, by Mumford's theorem [13], C is a smooth plane quintic.

Let $g' > 0$. By de Franchis' theorem, we may assume that the map $W_{d/n}^r(C') \xrightarrow{\pi^*} Z$ is dominant. Hence, $d - 3r = \dim Z \leq \dim W_{d/n}^r(C') = g' = \frac{d}{n} - r$. It follows that $2nr \geq (n-1)d \geq 3(n-1)r$. Thus $n \leq 3$, namely, $n = 3$ and $d = 3r$. This implies $g' = \frac{d}{3} - r = 0$ which is a contradiction.

Assume $g^r_{d/n}$ is special. By H. Martens' theorem [12], we have

$$d-3r=\dim Z\leq \dim W^r_{d/n}(C')\leq \frac{d}{n}-2r.$$

Hence, we have $3(n-1)r\leq (n-1)d\leq nr$. Thus $n<3$. This is a contradiction.

Next, we consider the case that $g^r_d(z)$ is simple for a general $z\in Z$, whence $r\geq 2$.

By the Accola-Griffiths-Harris theorem [7, p.73], one has the following inequality;

$$d-3r\leq \dim W^r_d(C)\leq \dim T_{|D|}W^r_d(C)\leq h^0(2D)-3r \quad \text{for} \quad D\in g^r_d(z).$$

Hence, we have $d\leq h^0(2D)=2d+1-g+h^1(2D)$ and $h^1(2D)\geq 1$.

First, we assume $h^1(2D)\geq 2$. Then

$$\mathrm{Cliff}(C)\leq \mathrm{Cliff}(2D)=2d-2(h^0(2D)-1)\leq 2.$$

If $\mathrm{Cliff}(C)\leq 1$, then C is either hyperelliptic, trigonal or a smooth plane quintic. The former two cases conflict with our assumption. Since $r\geq 2$, a smooth plane quintic does not occur. Hence, $\mathrm{Cliff}(C)=2$ and $|2D|$ computes the Clifford index. Moreover, we have $h^0(2D)=d$ and $\mathrm{gon}(C)=4$.

If $2d\leq g-1$, then by Lemma 7, we have $d\leq 4$ which contradicts $r\geq 2$. Hence, we have $2d\geq g$. Consider the dual of $2g^r_d=g^{d-2}_{2d}$; $|K_C-2g^r_d|=g^{g-d-2}_{2g-2-2d}$. Let $\rho=g-d-2$ and $\delta=2g-2-2d$. Again by Lemma 7, we have $\delta\leq 8$. Since $\dim W^\rho_\delta(C)\leq \delta-3\rho$, we have $\rho\leq 2$.

Case $\rho=2$. We have $d=g-4$, whence $\delta=6$. Since $\mathrm{Cliff}(C)=2$, $g^2_6=|K_C-2g^r_d|$ is simple. This linear series induces a plane curve of degree 6 with singular points because of $\mathrm{gon}(C)=4$. Thus, $g\leq 9$ and $d=g-4\leq 5$. On the other hand, since $\dim W^r_d(C)\leq \dim W^2_6(C)=0$, we have $d=3r$ which is impossible.

Case $\rho=1$. We have $d=g-3$, whence $\delta=4$. Since $\dim W^1_4(C)\leq 1$, we have $d=3r$ or $3r+1$. In case $d=3r+1$, we have $\dim W^1_4(C)=1$, whence C is a smooth plane quintic. Then $g=6$ and $d=3$. This is absurd. In case $d=3r$, if $r=2$, a simple g^2_6 induces a plane curve of degree 6. Since C is 4-gonal and $d=g-3$, the plane curve has exactly one ordinary node or cusp, i.e., $g=9$. If $r\geq 3$, a simple g^r_{3r} induces an extremal curve of degree $3r$ in $\mathbb{P}^r$ which is 4-gonal.

Next, we assume $h^1(2D)\leq 1$. In this case $d=g-2$. Since $2D=K_C-P-Q$ for some $P,Q\in C$, we have $0\leq \dim Z\leq 2$. Hence, $d=3r,3r+1$ or $3r+2$. Let $d=3r$. If $r=2$, then C is a plane curve of degree 6 with at most 2 singularities of multiplicity 2. Let $r\geq 3$

and C' be the model of C of degree $3r$ in $\mathbb{P}^r$ induced by g^r_{3r}. Since $g > \pi_1(3r, r) = 3r + 1$, by Lemma 4, C' lies on a surface S of degree $r - 1$ in $\mathbb{P}^r$. Since the Veronese surface in $\mathbb{P}^5$ cannot contain curves of odd degree, S is either a cone over a rational normal curve in $\mathbb{P}^{r-1}$ or a rational normal scroll in $\mathbb{P}^r$.

In case S is a rational normal cone with vertex v over a rational normal curve in $\mathbb{P}^{r-1}$. Let $m \geq 0$ be the multiplicity of C' at v. Let n be the degree of the (base-point-free) pencil cut out on C' by the ruling of the cone S. Considering a sufficiently general hyperplane $H \subset \mathbb{P}^r$ passing through the vertex v, one sees that $H \cap S$ is a union of $r - 1$ lines through v and hence

$$d - m = 3r - m = n(r - 1).$$

Since $n \geq 4$ and $r \geq 3$, the possibility of n, m, r are $(n, m, r) = (4, 0, 4)$ or $(4, 1, 3)$.

Assume that S is a smooth rational normal scroll. Recall that $\mathrm{Pic}(S)$ is freely generated by the classes H of a hyperplane section of S, and L of a line of the ruling. Put $C' \sim \alpha H + \beta L$. Since $g = 3r+2$, $p_a(C') = 3r+2$ or $3r + 3$. By the adjunction formula we have a system of equations

$$\begin{aligned} p_a(C') &= \frac{(\alpha - 1)(\alpha - 2)}{2}(r - 1) + (r - 2 + \beta)(\alpha - 1), \\ d &= (r - 1)\alpha + \beta. \end{aligned}$$

If $p_a(C') = 3r+2$, there is no integral solution of α, β. If $p_a(C') = 3r+3$, then we have $((r-1)\alpha - (3r+1))(\alpha - 4) = 0$. Thus, C' is always 4-gonal.

Let $d = 3r+1$. In this case, $\dim Z = 1$. For a general $z' \in Z$, we have $\dim |g^r_d(z) + g^r_d(z')| \geq 3r$ and $\dim |K_C - (g^r_d(z) + g^r_d(z'))| \geq 0$. Let φ, ψ be the map induced by $|K_C - g^r_d(z)|$ and $g^r_d(z')$ onto a curve $C' \subset \mathbb{P}^{r+1}$, $C_{z'} \subset \mathbb{P}^r$, respectively. Let π be the projection of C' to $C_{z'}$. Since there are infinitely many $z' \in Z$, we may assume that the center of π is not a singular point on C'. Then $\deg C' = \deg C_{z'} + 1 = d+1$, whence we have a linear series $g^{r+1}_{d+1} = g^{r+1}_{3r+2}$. Hence, by Theorem A, C is an extremal curve of degree $3r + 2$ in $\mathbb{P}^{r+1}$.

Let $d = 3r + 2$. In this case, $\dim Z = 2$. As in the preceding case, we have infinitely many linear series g^{r+1}_{3r+3}'s on C. Thus $\dim W^{r+1}_{3r+3}(C) \geq 1$ which is absurd because of $3r + 3 = g - 1$ and Theorem A.

This completes the proof. □

4. Case $d - 3r - 1$

In this section, we shall treat the case $\dim W^r_d(C) = d - 3r - 1$.

THEOREM 2. *Let C be an even gonal curve of genus g which does not have an involution. Assume there exist positive integers r and $d \leq g-4$, so that $\dim W^r_d(C) = d-3r-1 \geq 0$. Then, C is either 4-gonal or 6-gonal and one of the following holds:*
1) *C is 4-gonal.*
1-i) $\dim W^1_4(C) = 0$.
1-ii) *C is an extremal curve of degree $3r+5$ in $\mathbb{P}^{r+2}$ $(g = 3r+6)$ and $W^r_{3r+1}(C) \subset W^{r+2}_{3r+5}(C) - g^1_4$ is of dimension 0.*
1-iii) *C lies on a smooth normal scroll in $\mathbb{P}^{r+1}$, $p_a(C) = 3r+6$, $g = 3r+5$ and $W^r_{3r+1}(C) \subset K - W^{r+1}_{3r+3}(C) - g^1_4$ is of dimension 0.*
1-iv) *C lies on a quadric surface of degree 10 in $\mathbb{P}^3$, $g = 15$ and $\dim W^3_{10}$ $(C) = 0$.*
1-v) *C is an extremal curve of degree 13 in $\mathbb{P}^4$ $(g = 18)$ and $\dim W^4_{13}(C)$ $= 0$.*
2) *C is 6-gonal.*
2-i) *C is a plane curve of degree 8 with one ordinary node or cusp $(g = 20)$ and $\dim W^5_{16}(C) = 0$.*
2-ii) *C is a smooth plane septic $(g = 15)$ and $\dim W^2_7(C) = 0$.*

Proof. Let Z be an irreducible component of $W^r_d(C)$ of dimension $d - 3r - 1$ and let $g^r_d(z)$ be the linear series associated to an element $z \in Z$. Using the same procedure as in the proof of [11, 9], we may assume $g^r_d(z)$ is complete and base point free for general $z \in Z$.

First, assume $g^r_d(z)$ is compounded for a general $z \in Z$. Then $g^r_d(z)$ induces an n-sheeted covering map π of C onto a smooth curve C' of genus g' with $n|d$ and $n \geq 3$. Then $g^r_d(z)$ is the pull back of a base point free complete linear series $g^r_{d/n}$ on C' with respect to π.

Assume $g^r_{d/n}$ is non-special. Then $g' = \frac{d}{n} - r$. Let $g' = 0$. Then $\frac{d}{n} - r = g' = 0$ and $Z \subset r \cdot W^1_n(C)$. Hence, we have

$$n - 3 \geq \dim W^1_n(C) \geq d - 3r - 1 = (n-3)r - 1.$$

Thus, $(r, n) = (1, d)$ or $(2, 4)$. In case $(r, n) = (1, d)$, since $\dim W^1_d(C) = d - 4$, by Mumford's theorem [13], C is 4-gonal and $d = 4$. By the hypothesis, we have $g \geq d + 4 = 8$, whence C is not a smooth plane quintic. Since C does not have an involution, C is not bielliptic. Hence, for every 4-gonal C satisfying our assumption, we have $\dim W^1_4(C) =$

0. Let $(r,n) = (2,4)$. Then $d = 8$ and $1 = n - 3 \geq \dim W^r_d(C) \geq \dim W^1_4(C) \geq d - 3r - 1 = 1$. Hence, C is a smooth plane quintic which is impossible as like as the case $(r,n) = (1,d)$.

Let $g' > 0$. By de Franchis' theorem, we may assume that the map $W^r_{d/n}(C') \xrightarrow{\pi^*} Z$ is dominant. Hence, $d - 3r - 1 = \dim Z \leq \dim W^r_{d/n}(C') = g' = \frac{d}{n} - r$. Hence, $(n-1)d \leq n(2r+1)$. Since $d \geq 3r + 1$, it follows that $nr \leq 3r + 1$, i.e. $n = 4$ $(r = 1)$ or $n = 3$. If $n = 4$, $r = 1$, then $d = 4$ whence $g' = 0$ which is a contradiction. Let $n = 3$. Since $g' = \frac{d}{3} - r \geq 1$, we have $d \geq 3r + 3$ and $(n-1)(3r+3) \leq d(r-1) = n(2r+1)$. This contradicts $n = 3$.

Assume $g^r_{d/n}$ is special. By H. Martens' theorem [12], we have

$$d - 3r - 1 = \dim Z \leq \dim W^r_{d/n}(C') \leq \frac{d}{n} - 2r.$$

Hence, we have $(n-1)d \leq n(r+1)$ and $d \geq 3r + 1$ which implies $n \leq 2$. This is a contradiction.

Next, we consider the case that $g^r_d(z)$ is simple for a general $z \in Z$, whence $r \geq 2$.

By the Accola-Griffiths-Harris theorem[7, p.73], we have $h^0(2D) \geq d - 1$ for $D \in g^r_d(z)$. Since $g \geq d + 4$, we have $h^1(2D) \geq 2$.

$$\mathrm{Cliff}(C) \leq \mathrm{Cliff}(2D) = 2d - 2(h^0(2D) - 1) \leq 4.$$

If $\mathrm{Cliff}(C) = 0$, then C is hyperelliptic. If $\mathrm{Cliff}(C) = 1$, then C is a smooth plane quintic (because C is of even gonal), whence $g = 6$ and $d \leq 2$ which is absurd. Thus, we have $\mathrm{Cliff}(C) \geq 2$.

If $h^0(2D) = d$, then $\mathrm{Cliff}(C) = 2$ and $2D$ computes the Clifford index. Thus, by the same procedure as in the proof of Theorem 1, we have a contradiction. Note that we need not take care about the case $g - d - 2 = 1$ because $d \leq g - 4$ by the hypothesis.

Hence, we may assume $h^0(2D) = d - 1$.

Let $\mathrm{Cliff}(C) = 4$. Since $2D$ computes the Clifford index, by Lemma 7 we have $2d \geq g$, $\delta = 2g - 2 - 2d \leq 12$, whence $\rho = g - d - 3 \leq 4$. Hence, the Clifford dimension of C, $\mathrm{Cliff}\dim(C) \leq 4$.

If $3 \leq \mathrm{Cliff}\dim(C) \leq 4$, then by Lemma 8, C is of odd Clifford index. If $\mathrm{Cliff}\dim(C) = 2$, then C is birationally equivalent to a smooth plane octic which is of 7-gonal. Thus, $\mathrm{Cliff}\dim(C) = 1$ and $\mathrm{gon}(C) = 6$. We shall consider the cases $\rho = \dim|K_C - 2D| = 1, 2, 3, 4$, separately.

First, assume $\rho = 1$. Since $\mathrm{gon}(C) = 6$, we have $\dim W^1_6(C) \leq 1$. Hence,

$$d - 3r - 1 = \dim W^r_d(C) \leq \dim W^1_6(C) \leq 1.$$

Let $\dim W^1_6(C) = 0$. Then, $d = 3r + 1$, $g = 3r + 5$. Since $\mathrm{Cliff}(C) = 4 \leq d - 2r = r + 1$, we have $r \geq 3$. Let C' be the model of C of degree $3r + 1$ in $\mathbb{P}^r$ induced by g^r_{3r+1}. Since $\pi_1(3r + 1, r) = 3r + 3$, by Lemma 4, C' lies on a surface S of degree $r - 1$ in $\mathbb{P}^r$.

Assume $r = 5$ and S is a Veronese surface. Since $d = 3r + 1 = 16$, we deduce that C' is the image of a plane curve C'' of degree 8 with one ordinary node or cusp under the Veronese embedding.

Let S be a rational normal cone with vertex v over a rational normal curve in $\mathbb{P}^{r-1}$. Let $m \geq 0$ be the multiplicity of C' at v. Let n be the degree of the pencil cut out on C' by the ruling of the cone S. Then, as in the proof of Theorem 1, we have

$$d - m = 3r + 1 - m = n(r - 1),$$

which is impossible because of $n \geq 6$ and $r \geq 3$.

Let S be a smooth rational normal scroll. Put $C' \sim \alpha H + \beta L$, where H and L are the classes of a hyperplane section of S and a line of the ruling, respectively. If $r = 3$, the existence of g^3_{10} implies $\mathrm{gon}(C) \leq 5$. For $r \geq 4$, since $g = 3r + 5$, $p_a(C') = 3r + 5$ or $3r + 6$. By the adjunction formula we have a system of equations

$$\begin{aligned} p_a(C') &= \frac{(\alpha - 1)(\alpha - 2)}{2}(r - 1) + (r - 2 + \beta)(\alpha - 1), \\ d &= (r - 1)\alpha + \beta. \end{aligned}$$

If $p_a(C') = 3r + 5$, there is an integral solution $r = 5$, $\alpha = 5$ and $\beta = -4$. If $p_a(C') = 3r + 6$, then we have $((r - 1)\alpha - 3(r + 1))(\alpha - 4) = 0$. Thus, C' is 4-gonal.

Let $\dim W^r_d(C) = 1$. Since $\dim W^1_6(C) = 1$, by [10, Corollary 3.3] and our hypothesis, C is a 3-sheeted covering of an elliptic curve E. Let $\varphi : C \to E$ be the covering map. Note that for any pair of points $P, Q \in E$, $|\varphi^*(P+Q)| = g^1_6$ on C. Let $\psi : C \to C' \subset \mathbb{P}^r$ be the morphism induced by g^r_d. Since $Z + Z = 2Z \subset K_C - W^1_6$, $|g^r_d + g^1_6| = g^{r+3}_{d+6}$. Hence, by Lemma 6, we have $\dim |g^r_d - g^1_6| \geq r - 3$. It follows that the image of $\varphi^*(P + Q)$ under ψ lies on a plane in $\mathbb{P}^r$ for any pair of points $P, Q \in E$. Thus, for any point $P \in E$, the image of $\varphi^*(P)$ under ψ lies on a line and the lines $\overline{\psi(\varphi^*(P))} \subset \mathbb{P}^r$ are concurrent, i.e. there exists a point $p_0 \in \mathbb{P}^r$ such that $p_0 \in \overline{\psi(\varphi^*(P))}$ for any $P \in E$. Take general $r - 1$ points $P_1, \dots, P_{r-1} \in E$. Then, $H = \overline{\psi(\varphi^*(P_1 + \cdots + P_{r-1}))}$ is a hyperplane in $\mathbb{P}^r$ and $p_0 \in H$. Thus, for any point p in $H.C - \psi(\varphi^*(P_1 + \cdots + P_{r-1}))$, $\overline{\psi(\varphi^*(\varphi(p)))} \subset H$, whence $\deg H.C'$ is a multiple of 3. This is a contradiction.

Assume $\rho = 2$. If the linear series $g^2_8 = |K_C - 2D|$ is composite, it follows that C is a two-sheeted cover of a curve or 4-gonal which is not our case. Thus, g^2_8 is simple. If $\dim W^2_8(C) = 1$, then by Lemma 2, $g \leq 12$, whence $d = g - 5 \leq 7$. This implies that $\mathrm{Cliff}(C) \leq d - 2r \leq 3$. Hence, we have

$$d - 3r - 1 = \dim W^r_d(C) \leq \dim W^2_8(C) = 0,$$

and $d = 3r + 1$. The existence of a simple g^2_8 implies $g \leq 21$. If $g = 21$, then C is birationally equivalent to a smooth plane octic, which is of 7-gonal, whence $g \leq 20$ and $r \leq 4$. If $r = 4$, then $d = 13$, $g = 18$. Hence C is an extremal curve of degree 13 in $\mathbb{P}^4$, whence $\mathrm{gon}(C) = 4$. If $r = 3$, then the existence of a simple g^3_{10} implies $\mathrm{gon}(C) \leq 5$. If $r = 2$, then $\mathrm{Cliff}(g^2_7) = 3$. Thus we have $\mathrm{Cliff}(C) \neq 4$ for all of these cases.

Assume $\rho = 3$. As in the preceding case, we may assume $g^3_{10} = |K_C - 2D|$ is simple. Hence, by Lemma 1, we have $g \leq 16$ and $d = g - 6 \leq 10$. Since $d - 3r - 1 = \dim W^r_d(C) \leq \dim W^3_{10}(C)$, if $\dim W^3_{10}(C) = 0$, then $d = 3r + 1$, whence $d = 7$ or 10. If $d = 7$, then $\mathrm{Cliff}(g^2_7) = 3$. Thus we have $\mathrm{Cliff}(C) \neq 4$ in both cases. If $d = 10$, then C is an extremal curve of degree 10 in $\mathbb{P}^3$, whence $\mathrm{gon}(C) \leq 5$. If $\dim W^3_{10}(C) = 1$, then by Lemma 2, $g \leq \pi(11, 4)$ and $d \leq 6 < 3r + 1$ for $r \geq 2$. This is absurd.

Assume $\rho = 4$. As in the case $\rho = 2$, we may assume $g^4_{12} = |K_C - 2D|$ is simple. Hence, by Lemma 1, we have $g \leq 15$ and $d = g - 7 \leq 8$. Since $d - 3r - 1 = \dim W^r_d(C) \leq \dim W^4_{12}(C) = 0$ and $r \geq 2$, we have $r = 2$ and $d = 7$. It follows that $\mathrm{Cliff}(C) \neq 4$.

Let $\mathrm{Cliff}(C) = 3$. Since $\mathrm{gon}(C)$ is even, we have $\mathrm{gon}(C) = \mathrm{Cliff}(C) + 3$, whence $\lambda = \mathrm{Cliff\,dim}(C) \geq 2$. If $\lambda \geq 3$, the existence of a very ample $g^\lambda_{2\lambda+3}$ induces $\mathrm{gon}(C) \leq 4$. It follows that C is a smooth plane septic. Since $d \geq 7$, for $D \in g^r_d(z)$, $z \in Z$, we have $\deg |K_C - 2D| \leq 2g - 2 - 14 = 14$. Thus, by Lemma 9, the possibilities of (ρ, δ) for $g^\rho_\delta = |K_C - 2D|$ with $\mathrm{Cliff}(g^\rho_\delta) = 4$ are $(\rho, \delta) = (1, 6), (2, 8)$ or $(5, 14)$. Then, we have $d = 11, 10, 7$, respectively. Since $\dim W^1_6(C) = \dim W^2_8(C) = 1$, we have $r = 3$, in cases $d = 11, 10$. However, by the Noether bound (appeared in the proof of Lemma 9), we have $r \leq 2$ which is a contradiction. Thus, only possible case is $(d, r) = (7, 2)$. In this case, C is a smooth plane septic and $\dim W^2_7(C) = 7 - 3 \cdot 2 - 1 = 0$.

Let $\mathrm{Cliff}(C) = 2$. Since $\mathrm{gon}(C)$ is even, we have $\mathrm{gon}(C) = 4$ and $\mathrm{Cliff\,dim}(C) = 1$. From $\mathrm{Cliff}(C) = 2 \leq 2d - 2h^0(2D) + 2$, we have $h^0(2D) = d - 1$ or d.

Assume $h^0(2D) = d$. Then, $|2D|$ computes the Clifford index. If $2d \leq g - 1$, by Lemma 7, we have $d \leq 4$. This case does not occur. Let

$2d \geq g$. Let $\delta = \deg|K_C - 2D| = 2g - 2 - 2d$ and $\rho = \dim|K_C - 2D| = g-2-d$. Since $\delta \geq 3\rho$, we have $d \geq g-4$, whence $d = g-4$. Thus, $\delta = 6$, $\rho = 2$. If $|K_C - 2D|$ is composite, then C is trigonal. If it is simple, then $g \leq 10$. On the other hand, we have $7 \leq 3r + 1 \leq d = g - 4 \leq 6$ which is a contradiction.

Hence, we may assume $h^0(2D) = d - 1$. Let $\lambda = d - 2 = \dim|2D|$. Then, by Lemma 5, we have

(1) $$\lambda \geq 3r - 1 \geq 5.$$

Applying Lemma 1 to $2g^r_d(z) = g^{\lambda}_{2\lambda+4}$ ($m = 2, \varepsilon = 5$ if $\lambda \geq 7$ and $m = 3, \varepsilon = 6 - \lambda$ if $\lambda = 5, 6$), we have

(2) $$g \leq \lambda + 9 \quad \text{if} \quad \lambda \geq 6 \quad \text{and} \quad g \leq 15 \quad \text{if} \quad \lambda = 5.$$

We now consider $3g^r_d(z) = g^{2\lambda-1+\mu}_{3\lambda+6}$ ($\mu \geq 0$); cf. Lemma 5.

(I) If it is non-special, then $g = (3\lambda + 6) - (2\lambda - 1 + \mu) = \lambda + 7 - \mu$ i.e. $\lambda + 6 \leq g \leq \lambda + 7$.

(II) Suppose $3g^r_d(z) = g^{2\lambda-1+\mu}_{3\lambda+6}$ ($\mu \geq 0$) is special. Since $|K_C - 3g^r_d| = g^{g-\lambda-8+\mu}_{2g-3\lambda-8}$ ($\mu \geq 0$), we have $2g - 3\lambda - 8 \geq 0$, i.e. $g \geq \frac{3}{2}\lambda + 4$. Note that if $g - \lambda - 8 \geq 1$, we have $2g - 3\lambda - 8 \geq 4$, because we assumed that $\operatorname{gon}(C) = 4$. Thus, if $g = \lambda + 9$ then $4 \leq 2g - 3\lambda - 8 = 2(\lambda+9) - 3\lambda - 8 = -\lambda + 10$, whence $\lambda = 5$ or 6. Therefore, if $\lambda \geq 7$ then $g \leq \lambda + 8$ by (2). Consequently, $0 \leq 2g - 3\lambda - 8 \leq 2(\lambda + 8) - 3\lambda - 8 = -\lambda + 8$, whence $\lambda \leq 8$. Thus, when $3g^r_d(z)$ is special the only cases we have to study are $\lambda = 5$ with $11 < \frac{3}{2}\lambda + 4 \leq g \leq 15$ and $\lambda = 6$ with $13 \leq g \leq 15$, $\lambda = 7$ with $g = 15$ and $\lambda = 8$ with $g = 16$.

(I-i) First, assume that $3g^r_d$ is non-special and $g = \lambda + 7$, i.e. $|K_C - 2g^r_d(z)| = g^2_8$.

Assume g^2_8 is composite. Since C is not a 2-sheeted covering of a curve, we have $g^2_8 = 2g^1_4$. Thus, $d - 3r - 1 \leq \dim W^r_d(C) \leq \dim W^2_8 = 0$, i.e. $d = 3r + 1$ and $g = 3r + 6$. If $|g^r_{3r+1} + g^1_4| = g^{r+2}_{3r+5}$, then this simple linear series induces an extremal curve of degree $3r + 5$ in $\mathbb{P}^{r+2}$. It is of course 4-gonal. If $|g^r_{3r+1} + g^1_4| = g^{r+1}_{3r+5}$, then by Lemma 6, we have $|g^r_{3r+1} - g^1_4| = g^{r-1}_{3r-3}$. By Theorem 1, this case does not occur if $\operatorname{gon}(C) = 4$.

Assume g^2_8 is simple with base points. Let g^2_n ($n \leq 7$) be base point free. If $n \leq 6$, then by Lemma 1, $g \leq 10$, which contradicts that $g \geq \lambda + 7 \geq 12$. Let $n = 7$. Then, C is a plane curve of degree 7.

By Lemma 1, we have $\dim|g_7^2+g_4^1|\le 4$. Thus, by Lemma 6, we have $\dim|g_7^2-g_4^1|\ge 0$, whence g_4^1 is cut out by a pencil of lines. Thus C is a plane curve of degree 7 with one singular point of multiplicity 3 whence $g=12$. Then, $d=(2g-2-8)/2=7$ and $r=2$. In this case, $|K_C-2g_7^2|=2g_4^1$. Thus $g_8^2=|K_C-2g_7^2|$ is not simple.

Assume g_8^2 is a base point free simple linear series. Since $\operatorname{gon}(C)=4$, we have $g\le\pi(8,2)-1=20$. Hence, $3r-1\le\lambda=g-7\le 13$, i.e. $r\le 4$.

If $\lambda\ge 3r\ge 6$, then $\dim W_8^2\ge d-3r-1=\lambda-3r+1\ge 1$. Thus, using Lemma 2, we have $\lambda+7=g\le 12$, i.e. $\lambda\le 5$, which is an absurdity. Hence, $\lambda=3r-1$.

Let $r=2$. Then, $g=\lambda+7=12$ and $d=(2g-2-8)/2=7$. In this case, $|K_C-2g_7^2|=2g_4^1$.

Let $r=3$. Then, $g=\lambda+7=15$ and $d=(2g-2-8)/2=10$. Since $\operatorname{gon}(C)=4$, this case occurs only if $g_{10}^3=g_4^1+g_6^1$.

Let $r=4$. Then, $g=\lambda+7=18$ and $d=(2g-2-8)/2=13$. Hence, C is an extremal curve of degree 13 in $\mathbb{P}^4$ which is 4-gonal.

(I-ii) Next, assume that $3g_d^r$ is non-special and $g=\lambda+6$, i.e. $|K_C-2g_d^r(z)|=g_6^1$. Since $\operatorname{gon}(C)=4$, we have $\dim W_6^1=2$, whence $\dim W_d^r(C)\le 2$

Let $\dim W_d^r(C)=0$. Then, $d=3r+1$, $\lambda=3r-1$ and $g=3r+5$. If $\dim|g_{3r+1}^r-g_4^1|=r-1$, it would satisfy the hypothesis of Theorem 1. However, we could not find the case $g=3(r-1)+8$ in the list of the conclusion in Theorem 1. Hence, this case does not occur.

If $\dim|g_{3r+1}^r-g_4^1|\le r-2$, then, by Lemma 6, we have $\dim|g_{3r+1}^r+g_4^1|\ge r+2$, whence $|K_C-(g_{3r+1}^r+g_4^1)|=g_{3r+3}^{r+1}$ and equality hold in the above inequality. Hence, this case corresponds to Theorem 1 iv), i.e. C lies on a normal scroll in $\mathbb{P}^{r+1}$ with $p_a(C)=3r+6$.

Let $\dim W_d^r(C)=1$. Then, $d=3r+2$, $\lambda=3r$ and $g=3r+6$. If $\dim|g_{3r+2}^r(z)-g_4^1|\doteq r-1$ for a general $z\in Z$, we would have $\dim W_{3r-2}^{r-1}(C)=1$ which would satisfy the hypothesis of Theorem 1. However, we could not find this case in the list of the conclusion in Theorem 1. Hence, this case does not occur. If $\dim|g_{3r+2}^r(z)-g_4^1|\le r-2$ for a general $z\in Z$, then as in the case $\dim W_d^r(C)=0$, we would have $\dim W_{3r+4}^{r+1}(C)=1$ which would not occur, again.

Let $\dim W_d^r(C)=2$. Then, $d=3r+3$, $\lambda=3r+1$ and $g=3r+7$. As in the previous case, we have $\dim W_{3r-1}^{r-1}(C)=2$ or $\dim W_{3r+5}^{r+1}(C)=2$. Both cases do not occur.

(II) Next, assume $3g_d^r(z)=g_{3\lambda+6}^{2\lambda-1+\mu}$ $(\mu\ge 0)$ is special.

We already know that only cases we have to consider are $\lambda = 5$, $12 \leq g \leq 15$; $\lambda = 6$, $13 \leq g \leq 15$; $\lambda = 7$, $g = 15$ and $\lambda = 8$, $g = 16$. We will treat these cases separately.

(II-i) Case $\lambda = 5, d = 7, r = 2, 12 \leq g \leq 15$.

If $g = 15$, then C is a smooth plane septic, whence $\mathrm{Cliff}(C) \neq 2$. If $12 \leq g \leq 14$, C has a singular plane model of degree 7 with one singular point of multiplicity 3, whence $g = 12$ and $\mathrm{gon}(C) = 4$.

(II-ii) Case $\lambda = 6, d = 8, r = 2, 13 \leq g \leq 15$ and $\dim W^2_8(C) = 8 - 3 \cdot 2 - 1 = 1$.

This case does not occur by Lemma 2.

(II-iii) Case $\lambda = 7, d = 9, r = 2, g = 15$ and $\dim W^2_9(C) = 9-3\cdot 2-1 = 2$.

Since $|K_C - 2g^2_9(z)| = g^3_{10}$, we have $2 = \dim W^2_9(C) \leq \dim W^3_{10}(C) \leq 10 - 3 \cdot 3 = 1$ which is a contradiction, and hence this case does not occur.

(II-iv) Case $\lambda = 8, d = 10, r = 2, g = 16$ and $\dim W^2_{10}(C) = 3$.

For a general $z \in W^2_{10}(C)$, $3g^2_{10}(z)$ is special and since $3d = 2g-2$ one must have $3g^2_{10}(z) = K$. Noting the fact that there exist only finitely many line bundles on C whose triple is the canonical bundle, one sees that $\dim W^2_{10}(C) = 3$ is a contradiction. Therefore this case does not occur.

(II-v) Case $\lambda = 8, d = 10, r = 3, g = 16$ and $\dim W^3_{10}(C) = 0$. This is the case of extremal curve of degree 10 in $\mathbb{P}^3$, which is of odd gonality 5.

This completes the proof. □

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