# Note on Robust Stabilization of Uncertain Input-delay Systems by Sliding Mode Control With Delay Compensation

Dong Yue, Sangchul Won and Ohmin Kwon

**Abstract:** In this note, we suggest a new matching condition and extend the stability analysis to the ultimate boundness of state $x(t)$ for uncertain input-delay systems, using a sliding control with delay compensation.

**Keywords**: delay compensation, matching condition, robust stabilization, sliding mode control, uncertain input-delay systems

## I. Introduction

In real physical systems, input delays are frequently encountered because both measurement delays and computational delays are represented by input delay and many researchers have studied the problem of designing a controller for systems with input delay [1][2][5][6].

In [6], a new sliding mode control was proposed for uncertain input-delay systems with nonlinear parameter perturbations. The sliding surface was designed to compensate for the input-delay. Also, system behavior was examined in the sliding mode by focusing on the reduced order dynamics of the transformed delay-free system under the proposed control. However, the assumed matching conditions cannot be used for the delay-free system (20) in [6] and analyzed the stability of the predictive state $\bar{x}(t)$, not the state $x(t)$.

In this note, we suggest a new matching condition for the delay-free system (20) in [6] and extend a sliding mode controller in [6] which can stabilize the state $x(t)$. In order to analyze the stability of the state $x(t)$, we derive a new relationship between state $x(t)$ and predictive state $\bar{x}(t)$ on the sliding surface in [6]. Next, we develop a new sufficient condition for the ultimate boundedness [3] of state $x(t)$ under the control. Finally, we show the ultimate bound of state $x(t)$ and maximum allowable value of the input delay.

## II. Problem statements and controller design

Consider an uncertain system with time delay [6]

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t-\tau)+f_0(t,x(t))+f_1(t,x(t-\tau))\\ x(s)&=\phi(s),\ u(s)=v(s),\ s\in[-2\tau,0],\end{aligned} \quad (1)$$

where $x(t)\in R^n$, $u(t)\in R^m$ and $\tau>0$ are the state vector, the input vector, and the known time delay, respectively. $A$ and $B$ are constant matrices of appropriate dimensions. The nonlinear uncertainty, $f_0(t,x(t))$ and $f_1(t,x(t-\tau))$ are continuous in $t, x(t)$ and $x(t-\tau)$. $\phi(s)$ and $v(s)$ denote the initial condition functions.

In this note, we assume that the pair $(A,B)$ is controllable and that $B$ is of full rank and the uncertainty, $f_0(t,x(t))+f_1(t,x(t-\tau))$, satisfies the following matching condition

$$f_0(t,x(t))+f_1(t,x(t-\tau))=e^{-A\tau}B\{e_0(t,x(t))+e_1(t,x(t-\tau))\}, \quad (2)$$

where $\|e_0(t,x)+e_1(t,x(t-\tau))\|\le \rho_0\|x\|+\rho_1\|x(t-\tau)\|+k$, $\rho_0,\rho_1$ and $k$ are some known constants.

Remark 1 : In [6], the uncertainties, $f_0(t,x(t))$ and $f_1(t,x(t-\tau))$, are assumed to be satisfied in the following relations

$$f_0(t,x(t))=Be_0(t,x(t)),\quad f_1(t,x(t-\tau))=Be(t,x(t-\tau)).$$

However, these relations cannot be the matching conditions for the delay-free system (20) in [6]. Therefore, the stability analysis technique for the state $\bar{x}(t)$ proposed in Section 4 of [6] cannot be used.

Under the new matching condition (2), with the same predictive state as in [6],

$$\bar{x}(t)=e^{A\tau}x(t)+\int_{-\tau}^{0}e^{-A\theta}Bu(t+\theta)d\theta, \quad (3)$$

we have the delay-free system

$$\dot{\bar{x}}(t)=A\bar{x}(t)+Bu(t)+B\{e_0(t,x(t))+e_1(t,x(t-\tau))\} \quad (4)$$

Consider a switching surface for system (4)

$$\sigma(t)=S\bar{x}(t). \quad (5)$$

where $S\in R^{m\times n}$ is chosen, such that $SB$ is nonsingular. Without loss of generality we assume $SB=I$ (identity matrix) [4].

Under the new matching condition (2), we propose a modified variable structure control

$$u(t)=-SA\bar{x}(t)-\delta(t,x(t),x(t-\tau))\,\mathrm{sgn}(\sigma), \quad (6)$$

where

$$\delta(t,x(t),x(t-\tau))=\rho_0\|x\|+\rho_1\|x(t-\tau)\|+qk,$$

$$\mathrm{sgn}(\sigma)=[\mathrm{sgn}(\sigma_1),\ldots,\mathrm{sgn}(\sigma_m)]^T \text{ and } q>1.$$

Theorem 1 : Under the control (6), the sliding mode, $\sigma(t)=0$, is reachable in a finite time.

Proof : Define a Lyapunov function as

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$$V(t)=\frac{1}{2}\sigma^T\sigma .$$

Then, the time derivative of $V(t)$ yields

$$\begin{aligned}\dot V(t) &= \sigma^T\dot\sigma \\ &= \sigma^T S[A\bar x(t)+Bu(t)+B\{e_0(t,x(t))+e_1(t,x(t-\tau))\}] \\ &= -\sigma^T\delta(t,x)\mathrm{sgn}(\sigma)+\sigma^T\{e_0(t,x(t))+e_1(t,x(t-\tau))\} \\ &\le -\delta(t,x(t),x(t-\tau))\|\sigma\|+(\rho_0\|x\|+\rho_1\|x(t-\tau)\|+k)\|\sigma\| \\ &= -(q-1)k\|\sigma\| .\end{aligned}$$

Therefore, $\sigma(t)=0$ is reachable in a finite time. ■

Using Theorem 1, we know that $\sigma(t)=0$ can be realized after a finite time. Combining (4), (5) and using the similar analysis of [4], it can be shown that appropriately chosen the matrix $S$ can guarantee the sliding mode

$$\dot{\bar x}(t)=A\bar x(t)+Bu(t)+B\{e_0(t,x(t))+e_1(t,x(t-\tau))\},$$
$$\sigma(t)=0,$$

is asymptotically stable.

### III. Stability analysis of x(t)

In [6], the stability analysis is limited on $\bar x(t)$, therefore, in this section, we extend the analysis to the ultimate boundedness of $x(t)$. To accomplish this, we begin the analysis by finding the relationship between $x(t)$ and $\bar x(t)$ on the sliding surface.

We infer from the section above that control (6) can force the state $\bar x(t)$ to reach the sliding surface $\sigma=0$. In the sliding mode, the real control can also be expressed as the following equivalent form, by setting $\dot\sigma=0$,

$$u(t)=-SA\bar x(t)-e_0(t,x)-e_1(t,x(t-\tau)). \tag{7}$$

Substituting (7) into (3) and arranging the terms, we have

$$x(t)=e^{-A\tau}\bar x(t)+ e^{A(t-\tau)}\int_{t-\tau}^{t} e^{-As}B\{SA\bar x(s)+e_0(s,x)+e_1(s,x(s-\tau))\}ds, \tag{8}$$

then

$$\|x(t)\|\le\rho\beta\|B\|\sup_{t-2\tau\le s\le t}\|x(s)\|+\beta\|BSA\|\sup_{t-\tau\le s\le t}\|\bar x(s)\| +\|e^{-A\tau}\bar x(t)\|+k\beta\|B\|, \tag{9}$$

where $\beta=\frac{1}{\|A\|}(e^{\|A\|\tau}-1)$, $\rho=\rho_0+\rho_1$.

**Theorem 2** : Consider the system (1). If there exists $\tau$ which satisfies $\tau<\tau_M=\frac{1}{\|A\|}\ln\left(\frac{\|A\|}{\rho\|B\|}+1\right)$, then the state $x(t)$ of the system (1) is ultimately bounded [3] and the ultimate bound is $\frac{k\beta\|B\|}{1-\rho\beta\|B\|}$ under the control (6).

**Proof**: The proof consists of two parts. In the first part, we show the boundedness of $x(t)$, and, in the second part, we show the ultimate boundedness of $x(t)$ by using the results of the first part.

Part 1: Since the delay free system (4) is asymptotically stable, there exists a constant $M_1$ exists such that

$$\sup_{t-\tau\le s\le t}\|\bar x(s)\|<M_1 \text{ and } \|e^{-A\tau}\bar x(t)\|<M_1,\ t\ge 0 .$$

Hence, from (9), we obtain

$$\|x(t)\|\le\alpha\sup_{t-2\tau\le s\le t}\|x(s)\|+\delta,\ t\ge 0 \tag{10}$$

where $\alpha=\rho\beta\|B\|$, $\delta=(1+\beta\|BSA\|)M_1+k\beta\|B\|$.

Since $\tau<\tau_M=\frac{1}{\|A\|}\ln\left(\frac{\|A\|}{\rho\|B\|}+1\right)$, it is easy to see that $\alpha=\rho\beta\|B\|<1$.

Next, we show that, for any $\xi>0$,

$$\|x(t)\|<\sup_{-2\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma t}+\frac{\delta}{1-\alpha}+\xi,\ t\ge 0 \tag{11}$$

where $0<\varsigma<-\frac{1}{2\tau}\ln\alpha$.

To prove (11), we use the way of contradiction.

Since, for any $\xi>0$, from (10), we know that

$$\|x(0)\|\le\alpha\sup_{-2\tau\le s\le 0}\|\phi(s)\|+\delta<\sup_{-\tau\le s\le 0}\|\phi(s)\|+\frac{\delta}{1-\alpha}+\xi. \tag{12}$$

Therefore, if (11) does not hold, then $\exists\xi>0,\bar t>0$ such that

$$\|x(\bar t)\|=\sup_{-\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma\bar t}+\frac{\delta}{1-\alpha}+\xi \tag{13}$$

and

$$\|x(t)\|<\sup_{-2\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma t}+\frac{\delta}{1-\alpha}+\xi,\ t<\bar t. \tag{14}$$

However, from (10), (13) and (14), we have

$$\begin{aligned}\|x(\bar t)\| &\le\alpha\sup_{\bar t-2\tau\le s\le\bar t}\|x(s)\|+\delta \\ &\le\alpha e^{2\varsigma\tau}\sup_{-2\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma\bar t}+\frac{\alpha\delta}{1-\alpha}+\alpha\xi+\delta \\ &<\sup_{-2\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma\bar t}+\frac{\delta}{1-\alpha}+\xi,\end{aligned}$$

which contradicts (13). Hence, (11) is true. Since $\xi$ is arbitrary, letting $\xi\to 0$ in (11) we can further obtain

$$\|x(t)\|\le\sup_{-2\tau\le s\le 0}\|\phi(s)\|e^{-\varsigma t}+\frac{\delta}{1-\alpha},\ t\ge 0. \tag{15}$$

Defining $M_2=\sup_{-2\tau\le s\le 0}\|\phi(s)\|+\frac{\delta}{1-\alpha}$, we have

$$\|x(t)\|\le M_2,\ t\ge 0 . \tag{16}$$

Part 2: From the asymptotic stability of $\bar x(t)$, we know that for any $\varepsilon>0$, $\bar t_1>0$ exists such that

$$\sup_{t-\tau \le s \le t} \|\bar{x}(s)\| < \frac{1-\alpha}{(1+\beta\|BSA\|)}\frac{\varepsilon}{2}$$

and

$$\left\|e^{-A\tau}\bar{x}(t)\right\| < \frac{1-\alpha}{(1+\beta\|BSA\|)}\frac{\varepsilon}{2}, \quad t \ge \bar{t}_1 .$$

Next, from (9) we obtain

$$\|x(t)\| \le \alpha \sup_{t-2\tau \le s \le t} \|x(s)\| + \frac{\varepsilon(1-\alpha)}{2} + k\beta\|B\|, \quad t \ge \bar{t}_1 .$$

By the similar analysis method of Part 1 we show that

$$\|x(t)\| \le \sup_{\bar{t}_1-2\tau \le s \le \bar{t}_1} \|x(s)\| e^{-\varsigma(t-\bar{t}_1)} + \frac{\varepsilon}{2} + \frac{k\beta\lambda_{\max}^{1/2}(B^T B)}{1-\alpha}, \quad t \ge \bar{t}_1 . \tag{17}$$

Since $x(t)$ is bounded by $M_2$, therefore, there exists $\bar{t}_2 > \bar{t}_1$, such that

$$M_2 e^{-\varsigma(t-\bar{t})} < \frac{\varepsilon}{2}, \; t > \bar{t}_2. \tag{18}$$

Combining (17) and (18), we have

$$\|x(t)\| < \frac{k\beta\|B\|}{1-\alpha} + \varepsilon, \quad t > \bar{t}_2 .$$

Since $\varepsilon > 0$ is arbitrarily chosen, we can complete our proof by letting $\varepsilon \to 0$. ■

Remark 2 : If $k = 0$ in Assumption 1, then system (1) under control (6) is asymptotically stable.

Remark 3 : $\tau_M = \frac{1}{\|A\|}\ln\left(\frac{\|A\|}{\rho\|B\|}+1\right)$ in Theorem 1 is the maximum allowable value of $\tau$ that guarantees the system (1) is ultimately bounded under control (6).

## IV. Example

Consider the following system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-0.2) + \begin{bmatrix} 1 \\ 0.8187 \end{bmatrix} \|x(t)\|. \tag{19}$$

Obviously, the uncertainty satisfies the assumption (2). Define a new state as

$$\bar{x}(t) = e^{A\tau}x(t) + \int_{-\tau}^{0} e^{-A\theta}Bu(t+\theta)d\theta, \tag{20}$$

Then, we get the delay-free system

$$\dot{\bar{x}}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \|x(t)\|. \tag{21}$$

Consider a switching surface for system (21)

$$\sigma(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \bar{x}(t) . \tag{22}$$

Using (22), we design a variable structure controller of form

$$u(t) = -\begin{bmatrix} 2 & 2 \end{bmatrix} \bar{x}(t) - \|x(t)\| \operatorname{sgn}(\sigma). \tag{23}$$

Fig.1 are the simulation results of system (19) under control (23).

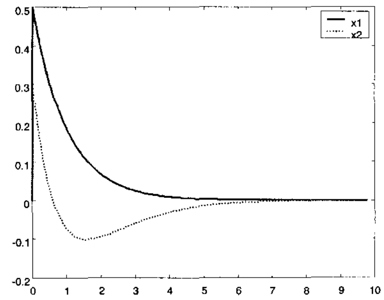


Fig.1 Dynamical performance of state $x_1$ and $x_2$.

## V. Conclusions

In this note, it was pointed out that, under the assumption on the uncertainty in [6], the matching conditions for the delay-free system (20) in [6] cannot be guaranteed and therefore the stability analysis techniques for the state $\bar{x}(t)$ cannot be used. Then, a new matching condition on the uncertainty was given in this note. Under the given assumption, we proposed a variable structure control, which can force the state $x(t)$ onto the sliding mode and then go to zero along the sliding mode.

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