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AN INVOLUTION ON THE SHIFTED RIM HOOK TABLEAUX

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ABSTRACT. We construct a sign reversing involution on the shifted rim hook tableaux with some conditions. Using the Stembridge's combinatorial interpretation for Morris' rule, this involution give a combinatorial proof of the orthogonality of the first kind for the spin characters of $\tilde{S}_n$.

1. Introduction

There has been a recent surge of interest in the projective representations of symmetric groups and shifted tableaux. Morris [3] constructed a projective analogue of the Murnaghan–Nakayama character recurrence and Stembridge [5] found a Frobenius-type characteristic map and an analogue of the Littlewood–Richardson rule. Sagan [4] and Worley [7] has developed independently a combinatorial theory of shifted tableaux parallel to the theory of ordinary tableaux. This theory includes shifted versions of the Robinson-Schensted-Knuth correspondence, Green's invariants, Knuth relation, and Schützenberger's jeu de taquin.

In [6] White gives a combinatorial proof of the orthogonality of the first kind for the ordinary characters of S_n. His proof is based on the Murnaghan-Nakayama formula. In this paper we will give a similar kind of proof for the spin characters of $\tilde{S}_n$ which is based on the Stembridge's combinatorial interpretation for Morris' rule. White described a sign reversing involution on triples (S, T, σ), where S and T are different rim hook tableaux and $\sigma \in S_n$, and where content(S) = content(T) = content(σ).

The involution he described proceeds as follows. First, a common "core" Q of S and T is found. That is, if k is the smallest value in a rim

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hook which is different in S and T, then $Q = S|_{\{1,\dots,k-1\}} = T|_{\{1,\dots,k-1\}}$. A key fact is that $\kappa_S\langle k\rangle$ and $\kappa_T\langle k\rangle$ will either overlap ("overlapping case") in a rim hook outside Q or will be disjoint ("disjoint case"), but connected by a rim hook inside Q. The involution reverses these two cases.

Our proof for the spin characters will follow a similar outline. However, certain new difficulties arise. In addition to the overlapping and disjoint cases, there is a third case ("double overlapping"). These cases fall into three groups of two, depending on the parity of certain parameters. Consequently, in addition to showing that only these three cases can occur, they must also be shown to satisfy certain conditions. The involution will then reverse each of the three pairs. Finally, the involution must also account for the more complicated weights.

In section 2, we outline the definitions and notation used in this paper. In section 3, we construct a sign reversing involution on the shifted rim hook tableaux with some conditions.

2. Definitions

In this section we introduce the most basic unit in this paper.

DEFINITION 2.1. A *partition* λ of a nonnegative integer n is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that

1. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
2. $\sum_{i=1}^{\ell} \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$. We say each term λ_i is a *part* of λ. The number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. We sometimes abbreviate the partition λ with the notation $1^{j_1}2^{j_2}\dots$, where j_i is the number of parts of size i. Sizes which do not appear are omitted and if $j_i = 1$, then it is not written. Thus, a partition $(5, 3, 2, 2, 2, 1) \vdash 15$ can be written $12^3 35$.

NOTATION 2.2. We denote

$$\begin{aligned} \mathcal{P}_n &= \{\mu \mid \mu \text{ is a partition of } n\ \} \\ OP_n &= \{\mu \in \mathcal{P}_n \mid \text{every part of } \mu \text{ is odd}\} \\ DP_n &= \{\mu \in \mathcal{P}_n \mid \mu \text{ has all distinct parts}\} \\ DP &= \{\mu \mid \mu \text{ is a partition with all distinct parts}\}. \end{aligned}$$

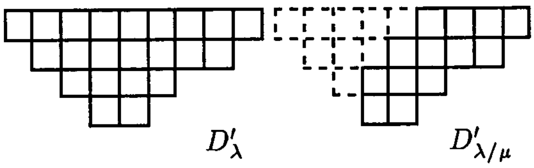


Figure 2.1

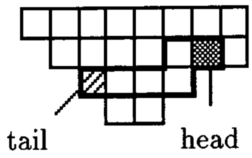


Figure 2.2

DEFINITION 2.3. For each $\lambda \in DP$, a *shifted diagram* D'_λ of shape λ is defined by

$$D'_\lambda = \{(i,j) \in \mathbb{Z}^2 \mid i \leq j \leq \lambda_j + i - 1, 1 \leq i \leq \ell(\lambda)\}.$$

And for $\lambda, \mu \in DP$ with $\mu \subseteq \lambda$, a *shifted skew diagram* $D'_{\lambda/\mu}$ is defined as the set-theoretic difference $D'_\lambda \setminus D'_\mu$. Figure 2.1 shows D'_λ and $D'_{\lambda/\mu}$ when $\lambda = (9,7,4,2)$ and $\mu = (5,3,1)$.

DEFINITION 2.4. A shifted skew diagram θ is called a *single rim hook* if θ is connected and contains no 2×2 block of cells. If θ is a single rim hook, then its *head* is the upper rightmost cell in θ and its *tail* is the lower leftmost cell in θ. See Figure 2.2.

DEFINITION 2.5. A *double rim hook* is a shifted skew diagram θ formed by the union of two single rim hooks both of whose tails are on the main diagonal. If θ is a double rim hook, we denote by $\mathcal{A}[\theta]$ (resp., $\alpha_1[\theta]$) the set of diagonals of length two (resp., one). Also let $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) be a single rim hook in θ which starts on the upper (resp., lower) of the two main diagonal cells and ends at the head of $\alpha_1[\theta]$. The tail of $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) is called the *first tail* (resp., *second tail*) of θ. Hence we have the following descriptions for a double rim hook θ: $\theta = \mathcal{A}[\theta] \cup \alpha_1[\theta] = \beta_1[\theta] \cup \beta_2[\theta] = \gamma_1[\theta] \cup \gamma_2[\theta]$.

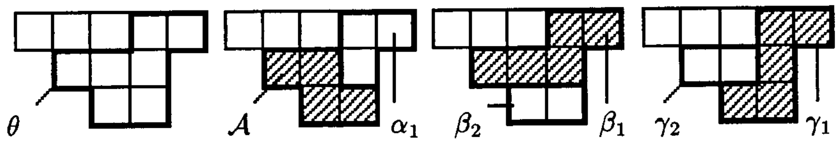


Figure 2.3

Definition 2.5 is illustrated in Figure 2.3.

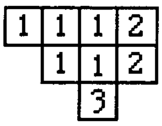
We will use the term *rim hook* to mean a single rim hook or a double rim hook.

DEFINITION 2.6. A *shifted rim hook tableau* of shape $\lambda \in DP$ and content $\rho = (\rho_1, \ldots, \rho_m)$ is defined recursively. If $m = 1$, a rim hook with all 1's and shape λ is a shifted rim hook tableau. Suppose P of shape λ has content $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the cells containing the m's form a rim hook inside λ. If the removal of the m's leaves a shifted rim hook tableau, then P is a shifted rim hook tableau.

DEFINITION 2.7. If θ is a single rim hook then the *rank* $r(\theta)$ is one less than the number of rows it occupies and the *weight* $w(\theta) = (-1)^{r(\theta)}$; if θ is a double rim hook then the *rank* $r(\theta)$ is $|\mathcal{A}[\theta]|/2 + r(\alpha_1[\theta])$ and the *weight* $w(\theta)$ is $2(-1)^{r(\theta)}$.

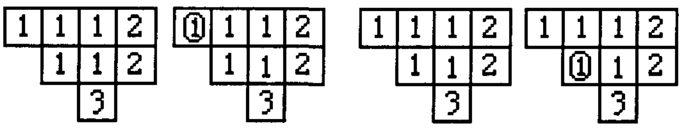
The *weight* of a shifted rim hook tableau P, $w(P)$, is the product of the weights of its rim hooks.

Let P be a shifted rim hook tableau. We write $\kappa_P\langle r\rangle$ or $rh_r(P)$ for a rim hook of P containing r. Figure 2.4 shows an example of a shifted rim hook tableau P of shape $(4,3,1)$ and content $(5,2,1)$. Here $r(\kappa\langle 1\rangle) = 2$, $r(\kappa\langle 2\rangle) = 1$ and $r(\kappa\langle 3\rangle) = 0$. Also $w(\kappa\langle 1\rangle) = 2$, $w(\kappa\langle 2\rangle) = -1$ and $w(\kappa\langle 3\rangle) = 1$. Hence $w(P) = 2 \cdot (-1) \cdot 1 = -2$.



P

Figure 2.4



P^1 P_2

Figure 2.5

DEFINITION 2.8. Suppose P is a shifted rim hook tableau. Then we denote by P^1 (resp., P_2) one of the tableaux obtained from P by circling or not circling the first tail (resp., second tail) of each double rim hook in P. The P^1 (resp., P_2) is called a *first* (resp., *second*) *tail circled rim hook tableau.* Similarly P_2^1 is obtained from P by circling or not circling the first tail and the second tail of each double rim hook in P and is called a *tail circled rim hook tableau.* We use the notation $|\cdot|$ to refer to the uncircled version; e.g., $|P^1| = |P_2| = |P_2^1| = P$.

We now define a new weight function w' for first (resp., second) tail circled rim hook tableaux. If τ is a rim hook of P^1 (resp., P_2), we define $w'(\tau) = (-1)^{r(\tau)}$. The weight $w'(P^1)$ (resp., $w'(P_2)$) is the product of the weights of rim hooks in P^1 (resp., P_2). If P is a shifted rim hook tableau in Figure 2.4, Figure 2.5 shows first and second tail circled rim hook tableaux P^1 and P_2, and Figure 2.6 shows tail circled rim hook tableaux P_2^1.

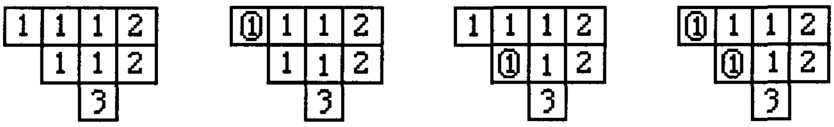


Figure 2.6

THEOREM 2.9. *Let $\lambda \in D$. Let*

$$A = \{\, P_2^1 \mid P \text{ is a shifted rim hook tableau of shape } \lambda \,\},$$

$$B = \{\, (P^1, P_2) \mid P \text{ is a shifted rim hook tableau of shape } \lambda \,\}.$$

Then there is a bijection η from A onto B.

PROOF. See Figure 2.7 for the definition of η. Clearly η is a bijection from A onto B. □

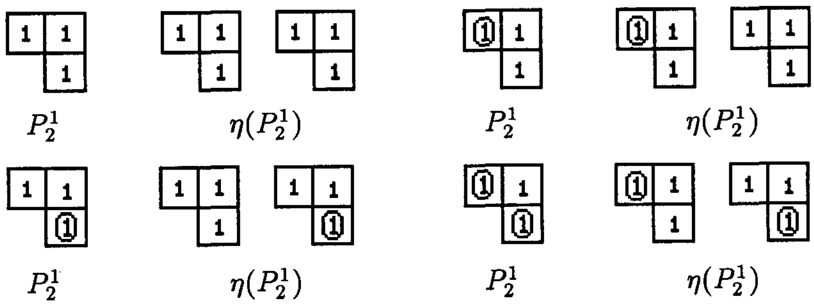


Figure 2.7

3. The sign reversing involution

In this section we describe a sign reversing involution on a certain set which proves the orthogonality relation of the first kind for the spin characters of $\tilde{S}_n$.

DEFINITION 3.1. A rim hook γ is called *the rim hook inside* λ if γ is contained in λ and its removal from λ leaves another legal shape. The shape created by the removal of γ is denoted by $\lambda - \gamma$. If γ is disjoint from λ but its addition to λ creates a new shape, then γ is *a rim hook outside* λ and the new shape formed by its addition to λ is denoted by $\lambda + \gamma$. In Figure 3.1, σ is the rim hook inside λ and τ is the rim hook outside λ, where $\lambda = (7, 4, 2, 1)$.

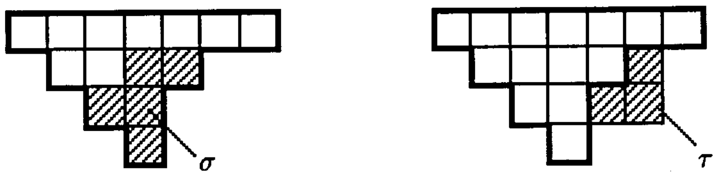


Figure 3.1

It is frequently necessary in discussions involving tableaux and shape to refer to the directions within the shape. Generally speaking, x will be SE of y if the row of x is the same as or below the row of y and the column of x is the same as or to the right of the column of y. Also, x will be *strictly* SE of y if x is SE of y but not in the same row or same column.

Let S_X be the set of permutations on X (if $X = \{1, 2, \dots, n\}$, $S_X = S_n$). Let $\sigma \in S_X$ and write σ in cycle form, $\sigma = \sigma_1 \sigma_2 \dots \sigma_m$, where the cycles σ_i are written in increasing order of the largest in the cycle. Recall that content (σ) is the sequence $\rho = (\rho_1, \rho_2, \dots, \rho_m)$, where $\rho_i = |\sigma_i|$ =length of the cycle σ_i. If $\sigma \in S_X$, then let $\overline{\sigma}$ be a permutation obtained from σ in which each cycle of σ is either barred or unbarred.

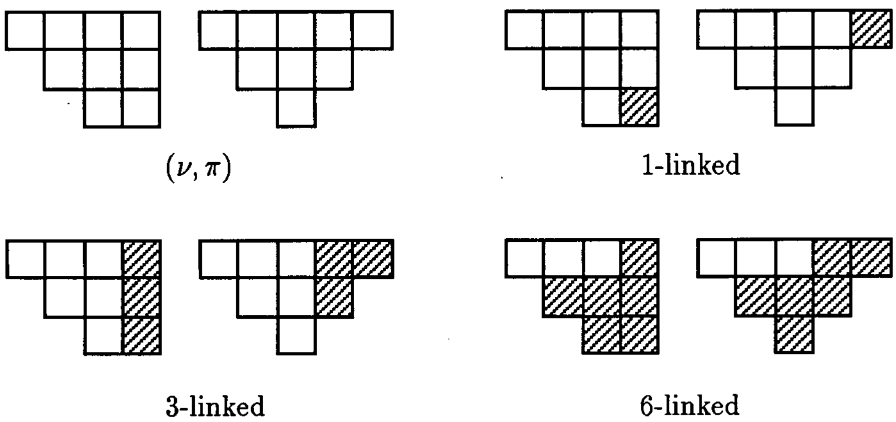


Figure 3.2

If $\sigma = (42)(8371)$, $\overline{\sigma}$ is one of $(42)(8371)$, $\overline{(42)}(8371)$, $(42)\overline{(8371)}$ and $\overline{(42)(8371)}$.

DEFINITION 3.2. Let P be a shifted rim hook tableau of content $\rho = (\rho_1, \dots, \rho_m)$. If $\rho' = (\rho_1, \dots, \rho_k), k < m$, then the *restriction* of P to ρ', denoted by $P|_{\rho'}$, is the shifted rim hook tableau obtained from P by removing all entries greater than k. If P and Q are shifted rim hook tableaux, let $P \cap Q$ denote the largest shifted rim hook tableau R which has the property that $R = P|_{\rho'} = Q|_{\rho'}$.

DEFINITION 3.3. Let $\nu, \pi \in DP_n$ be shifted shapes and $\nu \neq \pi$. (ν, π) is called *k-linked* if there are k-rim hooks δ and ϵ inside ν and π respectively such that $\nu - \delta = \pi - \epsilon$. For example, if $\nu = (4, 3, 2)$ and $\pi = (5, 3, 1)$, then (ν, π) is 1-linked, 3-linked and 6-linked. This example is illustrated in Figure 3.2.

From Definition 3.3 we have the following properties. See [1] for proofs.

PROPOSITION 3.4. *Let $\nu, \pi \in DP_n$ be shifted shapes and $\nu < \pi$. Then*

1. *(ν, π) is k-linked for some k if and only if $\nu/(\nu \cap \pi)$ is a rim hook and $\pi/(\nu \cap \pi)$ is a single rim hook.*
2. *If (ν, π) is k-linked, then there is only one pair δ, ϵ of rim hooks inside ν and π respectively such that $\nu - \delta = \pi - \epsilon$ and $|\delta| = |\epsilon| = k$.*
3. *If $\nu/(\nu \cap \pi)$ is a rim hook inside ν and $\pi/(\nu \cap \pi)$ is a single rim hook inside π with $|\nu/(\nu \cap \pi)| = |\pi/(\nu \cap \pi)| = k$, then (ν, π) is k-linked, l-linked and m-linked for some positive integers l, m with $k < l < m$.*
   Furthermore, (ν, π) is not i-linked except for $i = k, l, m$.

NOTATION 3.5. If (ν, π) is k-linked for some k, let $\mathcal{B}$ be the rim hook inside $\nu \cap \pi$ connecting $\nu/(\nu \cap \pi)$ and $\pi/(\nu \cap \pi)$ and let δ_i , ϵ_i be the rim hooks inside ν and π, respectively, such that $\nu - \delta_i = \pi - \epsilon_i$, $|\delta_i| = |\epsilon_i|$ for $i = 1, 2, 3$ and $|\delta_1| < |\delta_2| < |\delta_3|$.

DEFINITION 3.6. Let (ν, π) be k-linked for some k. We say (ν, π) is in *Class EE* if both $|\nu/(\nu \cap \pi)|$ and $|\mathcal{B}|$ are even. We say (ν, π) is in *Class EO* if $|\nu/(\nu \cap \pi)|$ is even and $|\mathcal{B}|$ is odd. *Class OE* and *Class OO* are defined in a similar way.

Proposition 3.4 and Definition 3.6 give us the following properties. See [1].

PROPOSITION 3.7. *Let ν, $\pi \in DP_n$ with $\nu < \pi$. Let (ν, π) be k-linked for some k. Then we have*

1. *If (ν, π) is in Class EE, then $|\delta_i|$ is even for each i.*

2. *If (ν,π) is in Class OE, then $|\delta_1|$ and $|\delta_2|$ are odd but $|\delta_3|$ is even.*
3. *If (ν,π) is in Class EO, then $|\delta_2|$ and $|\delta_3|$ are odd but $|\delta_1|$ is even.*
4. *If (ν,π) is in Class OO, then $|\delta_1|$ and $|\delta_3|$ are odd but $|\delta_2|$ is even.*

Note that either each $|\delta_i|$ is even or only one of the $|\delta_i|$'s is even.

PROPOSITION 3.8. *Let $\nu,\pi \in DP_n$ with $\nu < \pi$. Let (ν,π) be a k-linked for some k and $\mathcal{C} = \delta_3 - \delta_2 = \epsilon_3 - \epsilon_2$. Then*

$$\begin{aligned}
rank\,(\delta_2) &= rank\,(\delta_1) + rank\,(\mathcal{B}) + 1,\\
rank\,(\epsilon_2) &= rank\,(\epsilon_1) + rank\,(\mathcal{B}),\\
rank\,(\delta_3) &= rank\,\ (\delta_2) + |\mathcal{C}| - rank\,(\mathcal{C}) - 1, \quad \text{and}\\
rank\,(\epsilon_3) &= rank\,(\epsilon_2) + rank\,(\mathcal{C}).
\end{aligned}$$

Now let $\lambda,\mu \in DP_n$ and $\Psi_{\lambda\mu}$ denote the set of all triples $(P^1, Q_2\ ,\overline{\sigma})$, where P is a shifted rim hook tableau of shape λ , Q is a shifted rim hook tableau of shape μ , $\sigma \in S_n$ with type$(\sigma) \in OP_n$ and content(P) = content(Q) = content(σ). Let

$$\begin{aligned}
\Psi^+_{\lambda\mu} &= \{\ (P^1, Q_2\ ,\overline{\sigma}) \in \Psi_{\lambda\mu} \mid w'(P^1)w'(Q_2) = 1\ \},\\
\Psi^-_{\lambda\mu} &= \{\ (P^1, Q_2\ ,\overline{\sigma}) \in \Psi_{\lambda\mu} \mid w'(P^1)w'(Q_2) = -1\ \}.
\end{aligned}$$

Then we have the following theorem.

THEOREM 3.9. *Let $\lambda,\mu \in DP_n$ with $\lambda \neq \mu$. Then there is a bijection between $\Psi^+_{\lambda\mu}$ and $\Psi^-_{\lambda\mu}$.*

PROOF. Let $(P^1, Q_2\ ,\overline{\sigma}) \in \Psi_{\lambda\mu}$. Assume we have content$(P)$ = $\rho = (\rho_1,\rho_2,\ldots,\rho_m)$. Let $\overline{\sigma} = \sigma_1\sigma_2\ldots\sigma_m$ denote the cycle decomposition with each cycle barred or unbarred. Thus $|\sigma_i| = \rho_i$. Suppose $P \cap Q$ has shape ψ and content $\rho' = (\rho_1,\rho_2,\cdots,\rho_k)$, $k < m$. Let $\overline{\sigma'} = \sigma_1\sigma_2\ldots\sigma_k$ and let $P^1 \cap Q_2 = \eta^{-1}(P^1|_{(\rho_1,\rho_2,\ldots,\rho_k)}, Q_2|_{(\rho_1,\rho_2,\ldots,\rho_k)})$. Finally suppose $P^1|_{(\rho_1,\rho_2,\ldots,\rho_{k+1})}$ has shape ν and $Q_2|_{(\rho_1,\rho_2,\ldots,\rho_{k+1})}$ has shape π. Then (ν,π) is $|\rho_{k+1}|$-linked. Assume $\nu < \pi$.

Call $(P^1, Q_2\ ,\overline{\sigma})$ *disjoint* if $rh_{k+1}(P^1) \cap rh_{k+1}(Q_2) = \emptyset$. And we call $(P^1, Q_2\ ,\overline{\sigma})$ *overlapping* if $rh_{k+1}(P^1) \cap rh_{k+1}(Q_2) = \mathcal{B}$. If $rh_{k+1}(P^1) \cap rh_{k+1}(Q_2)$ properly contains $\mathcal{B}$, we call $(P^1, Q_2\ ,\overline{\sigma})$ *double overlapping.* Since (ν,π) is $|\rho_{k+1}|$-linked, (ν,π) can only belong to Class OE, Class EO or Class OO. We therefore have six cases:

1. (ν,π) is in Class OE and $(P^1, Q_2\ ,\overline{\sigma})$ is disjoint.
2. (ν,π) is in Class OE and $(P^1, Q_2\ ,\overline{\sigma})$ is overlapping.
3. (ν,π) is in Class EO and $(P^1, Q_2\ ,\overline{\sigma})$ is overlapping.
4. (ν,π) is in Class EO and $(P^1, Q_2\ ,\overline{\sigma})$ is double overlapping.

5. (ν, π) is in Class OO and $(P^1, Q_2, \overline{\sigma})$ is disjoint.
6. (ν, π) is in Class OO and $(P^1, Q_2, \overline{\sigma})$ is double overlapping.

The involution will pair off cases 1 and 2, 3 and 4, and 5 and 6. That is, the result of the involution on a member of case 1 will be a member of case 2.

We will describe the involution algorithmically. The algorithm description is similar in all six cases, so we will describe the algorithm once, with case-by-case modifications where necessary. In particular, cases 1, 3 and 5 are very similar. We call these three cases *expanding.* Also, cases 2, 4 and 6 are similar and we call these *contracting.* See [2] for the definitions of ϕ, π_ψ and Γ_ψ.

Step 1. Construct a permutation tableau T. In all cases, use the bijection ϕ to construct $T \in \pi_\psi(\sigma_1 \cup \cdots \cup \sigma_k)$ from $(P^1 \cap Q_2, \sigma_1 \ldots \sigma_k)$.

Step 2. Determine section of cycle to add or delete. Let $\gamma = \mathcal{B}$ if (ν, π) is in Class OE. Let $\gamma = \delta_3 - \delta_2 = \epsilon_3 - \epsilon_2$ if (ν, π) is in Class EO. Let $\gamma = \delta_3 - \delta_1 = \epsilon_3 - \epsilon_1$ if (ν, π) is in Class OO. Note that γ is a rim hook inside ψ in the expanding cases and is a rim hook outside ψ in the contracting cases. Let $r = |\gamma|$ and $t = \rho_{k+1}$. Let $|\sigma_{k+1}| = (a_1 \ldots a_n)$ where a_1 is the largest entry in $|\sigma_{k+1}|$. Finally, in the expanding cases, let $b_1, \ldots, b_r$ be the entries in T in γ, read from tail to head (or from tail to head in $\beta_1[\gamma]$ and then from tail to head in $\beta_2[\gamma]$, if γ is a double rim hook). In the contracting cases, let $|\sigma_{k+1}| = (a_1 \ldots a_{t-r} b_1 \ldots b_r)$.

Step 3. Modify T to form a new permutation tableau T'. In the expanding cases, T' is T with γ removed. In the contracting cases, T' is T with γ added and $b_1, \ldots, b_r$ inserted into γ from tail to head (filling $\beta_1[\gamma]$ from tail to head, then filling $\beta_2[\gamma]$ from tail to head if γ is a double rim hook). γ in T' will have circles in the same positions that γ has circles in P^1 or Q_2.

Step 4. Construct a rim hook tableau R and permutation $\overline{\tau}$ from T', using ϕ^{-1}. In the expanding cases, $(R, \overline{\tau}) \in \Gamma_{\psi-\gamma}(\sigma_1 \cup \cdots \cup \sigma_k - \{b_1, \ldots, b_r\})$, while in the contracting cases, $(R, \overline{\tau}) \in \Gamma_{\psi\cup\gamma}(\sigma_1 \cup \cdots \cup \sigma_k \cup \{b_1, \ldots, b_r\})$. Let $\theta = (\theta_1, \ldots, \theta_l)$ be the content of R and $\overline{\tau}$.

Step 5. Construct P'^1 and Q'_2 as follows:

$$R = \eta\,(P'^1|_\theta\,, Q'_2|_\theta),$$
$$rh_{l+i}(P'^1) = rh_{k+i}(P^1) \quad \text{for } i > 1,$$
$$rh_{l+i}(Q'_2) = rh_{k+i}(Q_2) \quad \text{for } i > 1.$$

In the expanding cases,

$$rh_{l+1}(P'^1) = rh_{k+1}(P^1) + \gamma,$$
$$rh_{l+1}(Q'_2) = rh_{k+1}(Q_2) + \gamma,$$

while in the contracting cases,

$$rh_{l+1}(P'^1) = rh_{k+1}(P^1) - \gamma,$$
$$rh_{l+1}(Q'_2) = rh_{k+1}(Q_2) - \gamma.$$

Step 6. Determine circles. In the expanding cases, $rh_{l+1}(P'^1)$ will have a circle on its first tail if either $rh_{k+1}(P^1)$ has a circle on its first tail or, in expanding Cases EO or OO, T has a circle on the cell which corresponds to the first tail of $rh_{l+1}(P'^1)$. Also $rh_{l+1}(Q'_2)$ will have a circle on its second tail if T has a circle on the cell which corresponds to the second tail of $rh_{l+1}(Q'_2)$. In the contracting cases, $rh_{l+1}(P'^1)$ will have a circle on its first tail α if $rh_{k+1}(P^1)$ has a circle on its first tail α and, in the contracting Cases EO or OO, the circled cell α is not in γ.

Step 7. Construct the new permutation $\overline{\sigma'}$.
In the expanding cases,

$$\overline{\sigma'} = \begin{cases} \overline{\tau} \cdot (a_1 \dots a_t b_1 \dots b_r) \cdot \sigma_{k+2} \dots \sigma_m & \text{if } \sigma_{k+1} \text{ has no bar on it,} \\ \overline{\tau} \cdot \overline{(a_1 \dots a_t b_1 \dots b_r)} \cdot \sigma_{k+2} \dots \sigma_m & \text{if } \sigma_{k+1} \text{ has a bar on it.} \end{cases}$$

In the contracting cases,

$$\overline{\sigma'} = \begin{cases} \overline{\tau} \cdot (a_1 \dots a_{t-r}) \cdot \sigma_{k+2} \dots \sigma_m & \text{if } \sigma_{k+1} \text{ has no bar on it,} \\ \overline{\tau} \cdot \overline{(a_1 \dots a_{t-r})} \cdot \sigma_{k+2} \dots \sigma_m & \text{if } \sigma_{k+1} \text{ has a bar on it.} \end{cases}$$

Note that Proposition 3.8 implies that

$$w'(rh_{k+1}(P^1) + \gamma)w'(rh_{k+1}(Q_2) + \gamma) = -w'(rh_{k+1}(P^1))w'(rh_{k+1}(Q_2))$$

in the expanding cases and

$$w'(rh_{k+1}(P^1) - \gamma)w'(rh_{k+1}(Q_2) - \gamma) = -w'(rh_{k+1}(P^1))w'(rh_{k+1}(Q_2))$$

in the contracting cases. Therefore, we have

$$w'(P'^1)w'(Q'_2) = -w'(P^1)w'(Q_2).$$

□

Figure 3.3–Figure 3.4 show us examples of the involution described in the above proof for each case. Here we use the alphabet $1 < \cdots < 9 < a < b < c < d$. In Figure 3.3 disjoint $(P^1, Q_2, \overline{\sigma})$ is given. Note that (ν, π) is in Class OE, where $\nu = \text{shape}(P^1)$ and $\pi = \text{shape}(Q_2)$. Next

$(P^1 \cap Q_2, \sigma_1 \ldots \sigma_k)$ is illustrated. Then T is constructed using ϕ (Step 1). γ is constructed using $\gamma = \epsilon_2 - \epsilon_1 = \delta_2 - \delta_1$ (Step 2) and T' is formed

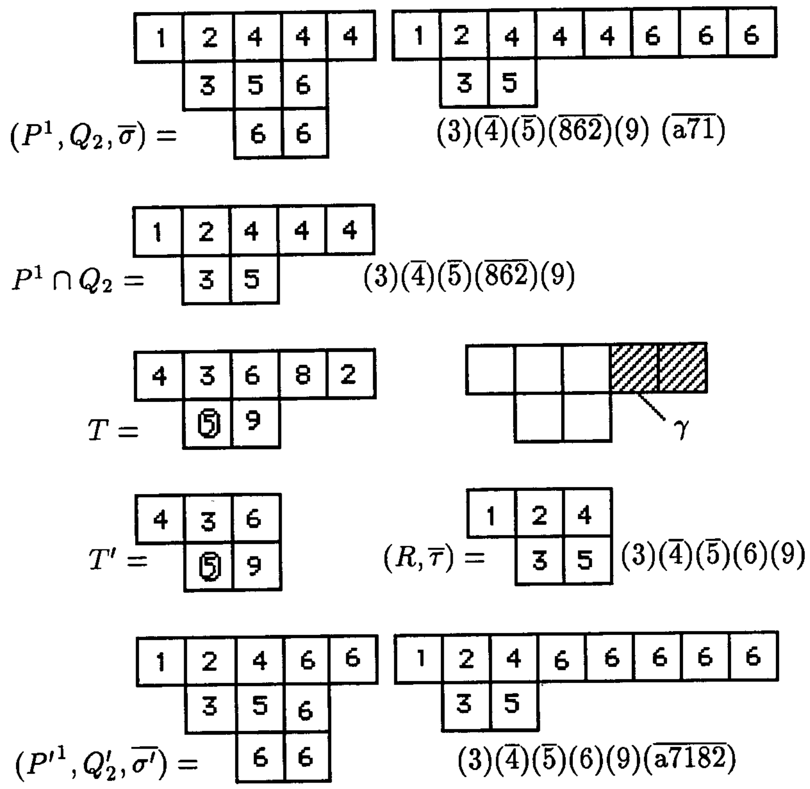


Figure 3.3

by removing the entries in γ from T (Step 3). The $(R, \overline{\tau})$ is constructed using ϕ^{-1} on T' (Step 4). Finally $(P'^1, Q'_2, \overline{\sigma'})$ is built from $(R, \overline{\tau}), \sigma_{k+1}$, the entries in γ, and the rest of P^1, Q_2 and $\overline{\sigma}$ (Steps 5 and 6).

In Figure 3.4, an overlapping $(P^1, Q_2\ , \overline{\sigma})$ is given. Here (ν, π) is in Class OE, where $\nu = \text{shape}(P^1)$ and $\pi = \text{shape}(Q_2)$. The $(P^1 \cap Q_2, \sigma_1 \ldots \sigma_k)$ is illustrated and T is formed using ϕ (Step 1). Next γ is constructed (Step 2). T' is formed by inserting part of σ_{k+1} onto T (Step 3). $(R, \overline{\tau})$ is constructed from T' using ϕ^{-1} (Step 4). Finally $(P'^1, Q'_2, \overline{\sigma'})$ can be constructed by putting back the rest of σ_{k+1} and the rest of P^1, Q_2 and $\overline{\sigma}$ (Steps 5 and 6). The other cases are shown in a similar way.

Together with the main theorem in [2], Theorem 3.9 gives us the following identity.

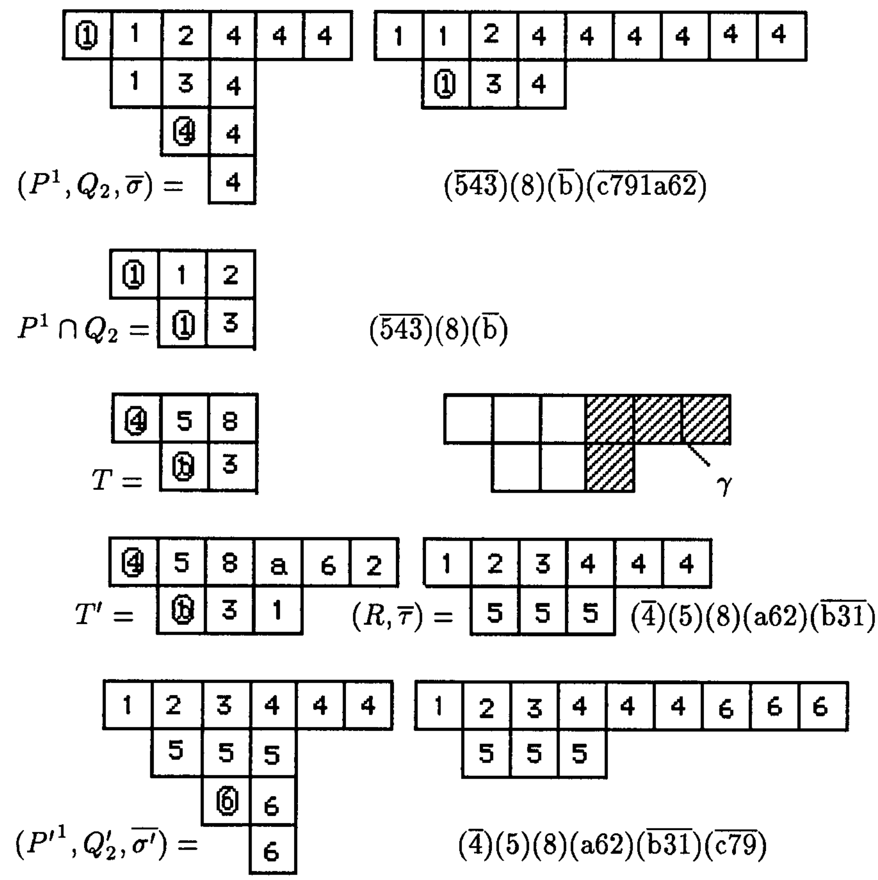


Figure 3.4

THEOREM 3.10. *Let λ , $\mu \in DP_n$. Then*

$$\sum 2^{\ell(type(\sigma))} w(P)w(Q) = \delta_{\lambda\mu} 2^{\ell(\lambda)} n!,$$

where the sum is over triples (P, Q, σ), with P a shifted rim hook tableau of shape λ, Q a shifted rim hook tableau of shape μ and $\sigma \in S_n$, which satisfy $type(\sigma) \in OP_n$, $content(P)$=$content(Q)$=$content(\sigma)$.

DEFINITION 3.11. For $n > 1$ let $\tilde{S}_n$ be the group generated by $t_1, t_2, \ldots, t_{n-1}, -1$ subject to relations

- $t_i^2 = -1$ for $i = 1, 2, \ldots, n-1$,

- $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i = 1, 2, \ldots, n-2$,
- $t_i t_j = -t_j t_i$ for $|i-j| > 1$ $(i, j = 1, 2, \ldots, n-1)$.

If we use the recurrence formula for the irreducible spin characters of the $\tilde{S_n}$, we obtain the following theorem from Theorem 3.10.

THEOREM 3.12. (Orthogonality formula of the first kind)
Let φ and ψ be irreducible spin characters of $\tilde{S_n}$. Then

$$\sum_{\sigma \in \tilde{S_n}} \varphi(\sigma)\psi(\sigma^{-1}) = \begin{cases} 2n! & \text{if } \varphi = \psi, \\ 0 & \text{if } \varphi \neq \psi. \end{cases}$$

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