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DIVISORS OF THE PRODUCTS OF CONSECUTIVE INTEGERS

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ABSTRACT. In this paper, we look at a simple function L assigning to an integer n the smallest positive integer m such that any product of m consecutive numbers is divisible by n. Investigated are the interesting properties of the function. The function $L(n)$ is completely determined by $L(p^k)$, where p^k is a factor of n, and satisfies $L(m \cdot n) \leq L(m) + L(n)$, where the equality holds for infinitely many cases.

1. Introduction

A simple, but interesting function is proposed by P. D. Tiu [**1**, p. 141]. The function L is defined to assign the smallest number m to a given number n such that all of the products of m consecutive integers are divisible by n.

The value of the function turns out to be evaluated based on the prime factorization of the given number. We use some of simple properties of binomial coefficients to simplify the definition of the function, and then we find the properties satisfied by the function. It is not only interesting but also heuristic to find such interesting properties.

We first show that $L(n)$ is completely determined by $L(p^k)$, where p^k is a factor of n. One of the properties of the function L is that it is analogous to the logarithmic function with the equality replaced by the inequality: $L(m \cdot n) \leq L(m) + L(n)$ for positive integers m and n and $L(n^k) \leq kL(n)$ for $k \geq 1$. We also get to know that the equality holds very occasionally: $L(p^k) = kL(p) = kp$ for a prime p and $k = 1, 2, \ldots, p$, and $L(p^k) < kL(p)$ otherwise. We explicitly compute $L(2^{2n})$ in terms of

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$L(2^n)$, showing that $L(n^2) = 2L(n)$ holds for infinitely many n's. Also, the preimages of the function L are computed.

2. Divisors of the products of consecutive integers

An interesting function $L : \mathbf{N} \to \mathbf{N}$ is proposed by Tiu [**1**, p. 141].

DEFINITION. A function $L : \mathbf{N} \to \mathbf{N}$ is defined by assigning to an integer n the smallest positive integer m such that any product of m consecutive numbers is divisible by n. That is,

$$\begin{aligned} L(n) &= m \\ &= \min\{k : n \mid (N+1)(N+2)\cdots(N+k) \ \text{ for all } N \in \mathbf{Z}\}. \end{aligned} \tag{1}$$

For example, it is easy to see that

$$L(1)=1,\ L(2)=2,\ L(3)=3,\ L(4)=4,\ L(5)=5,\ L(6)=3,$$
$$L(7)=7,\ L(8)=6,\ L(9)=6,\ L(10)=5,\ L(11)=11,\ L(12)=4.$$

Notice that it seems that $L(n) \le n$. This can be seen by the following lemma that provides an easier and clearer, but simpler equivalent definition of the function L.

LEMMA 1. *For any* $n \in \mathbf{N}$, $L(n) = m$ *if and only if* $n | m!$ *and* $n \not| (m-1)!$.

PROOF. We first prove the sufficiency. Since $n \not| (m-1)!$, it holds that $L(n) \ge m$. That $n \mid m!$ implies that n also divides any products of m consecutive integers. In fact, for $N \ge 1$,

$$(N+1)(N+2)\cdots(N+m) = \frac{(N+m)!}{N!} = \binom{N+m}{N} \cdot m! \tag{2}$$

so that $m! | (N+1)(N+2)\cdots(N+m)$. So $L(n) \le m$. Therefore, $L(n) = m$.

The necessity is clear from the definition of the function L. In fact, $L(n) = m$ implies that $n|m!$ and $n \not| (m-1)!$. If $n \mid (m-1)!$, then, by (2), n also divides any product of $(m-1)$ consecutive integers, i.e., $L(n) < m$. □

Note that we have considered the products of consecutive *positive* integers only in the above proof. Even when m consecutive integers contains 0 or consists of negative integers only, the divisibility by n of their products and so $L(n)$ are not changed. By virtue of the above lemma, the definition of L can be restated as follows.

DEFINITION. $L(n) = m$ if $n \mid m!$ and $n \nmid (m-1)!$.

Notice that since $L(n!) = n$ holds, the function L can be seen as the left inverse function of the factorial operation $F : \mathbf{N} \to \mathbf{N}$ which assigns $n!$ to n: $L \circ F = \mathrm{id}$.

The new definition is so simple that $L(n)$ can be easily found by definition when n is a prime or a product of distinct primes.

LEMMA 2. *For primes $p < q$, it holds that $L(pq) = L(q) = q$. More generally, for $k \geq 1$, if $p_1 < p_2 < \cdots < p_k$ are primes, then $L(p_1 p_2 \cdots p_k) = L(p_k) = p_k$.*

PROOF. First of all, it is clear that if p is a prime, then $L(p) = p$. For, $p \mid p!$ but $p \nmid (p-1)!$ if p is a prime.

The proof for the general case is exactly the same as that for the case of pq. It would be enough to see that $L(pq) = q$ for primes $p < q$. Since $p < q$, p divides $q! = 1 \cdot 2 \cdots p \cdots q$. So $pq \mid q!$. However, q does not divide $(q-1)!$, and neither does pq. So by definition, $L(pq) = q = L(q)$. □

THEOREM 3. *For an integer $n = \prod_{i=1}^{k} p_i^{r_i}$ with $p_1, p_2, \cdots, p_k$ distinct primes,*

$$L(n) = L\Big(\prod_{i=1}^{k} p_i^{r_i}\Big) = \max_{1 \leq i \leq k} \{L(p_i^{r_i})\}.$$

PROOF. We will prove for the case when $k = 2$, i.e., we will show that for distinct primes p, q, $L(p^r q^s) = \max\{L(p^r), L(q^s)\}$. Let $L(p^r) = a$ and $L(q^s) = b$ for some positive integers a and b. Assume $a \leq b$. Then $q^s \mid b!$ and $q^s \nmid (b-1)!$. Since $a! \mid b!$ and $p^r \mid a!$, we have $p^r \mid b!$. Since p^r and q^s are relatively primes, $p^r q^s \mid b!$ and $p^r q^s \nmid (b-1)!$, i.e., $L(p^r q^s) = b$. □

COROLLARY 4. *Let m and n be relatively prime integers. Then $L(m \cdot n) = \max\{L(m), L(n)\}$. Generally, if $n_1, n_2, \ldots, n_k$ are relatively prime integers, then $L(\prod_{i=1}^{k} n_i) = \max\{L(n_i) : 1 \leq i \leq k\}$.*

PROOF. Since they are relatively primes, m and n are written $m = \prod_{i=1}^{k} p_i^{r_i}$ and $n = \prod_{j=1}^{l} q_j^{s_j}$, where $p_1, \ldots, p_k, q_1, \ldots, q_l$ are all distinct primes.

By Theorem 3, we obtain

$$\begin{aligned} L(m\cdot n) =& L\Big(\prod_{i=1}^{k} p_i^{r_i}\cdot\prod_{j=1}^{l} q_j^{s_j}\Big)\\ =& \max_{1\le i\le k, 1\le j\le l}\{L(p_i^{r_i}),\, L(q_j^{s_j})\} = \max\{L(m),\, L(n)\}. \qquad \square \end{aligned}$$

3. Properties on $L(p^k)$

From Theorem 3, it is enough to find $L(p^k)$ for each prime p and for all k, in order to know $L(n)$ for an integer n. So we now consider n of the form p^k for a prime p and $k \geq 2$. For example, let $n = 3^4$. Then what will $L(3^4)$ be? We should find the smallest integer m such that $3^4 \mid m!$. The prime factorization of $m!$ is a product of some powers of all the primes less than or equal to m. That is, $m! = 2^{r_1}\cdot 3^{r_2}\cdot 5^{r_3}\cdots p_l^{r_l}$ for some $r_i \geq 1$, $i = 1, 2, \ldots, l$, where $2 < 3 < 5 < \cdots < p_l$ are all the primes less than or equal to m. Notice that r_2 should be at least 4 so that $3^4 \mid m!$. Looking closely at

$$\begin{aligned} m! &= (1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9)\,(10\cdots m)\\ &= (2^7\cdot 3^4\cdot 5\cdot 7)\,(10\cdots m), \end{aligned}$$

we easily see that the smallest $m!$ whose prime factorization contains 3^4 is $9!$. So $L(3^4) = 9$. However, when k is large, it is difficult to find the smallest $m!$ such that p^k is a factor of $m!$. So we look at some properties satisfied by $L(p^k)$.

LEMMA 5. *Let p be a prime. Then $p \mid L(p^k)$ for all $k \geq 1$. And, for $m \in \mathbf{N}$ a multiple of p, there exists a positive integer n such that $L(p^n) = m$.*

PROOF. If $L(p^k) = l$ then $p^k \mid l!$ and $p^k \nmid (l-1)!$, so $p \mid l$.
Since m is a multiple of p, it is written $m! = p^j N$ for some $j \geq 1$ and N with $p \nmid N$. Then from $p \mid m$ and $p^j \mid m!$, $p^j \nmid (m-1)!$. So $L(p^j) = m$. $\square$

Note that the existence of n in Lemma 5 such that $L(p^n) = m$ for given m, a multiple of a prime p, is *not* unique. For example, $L(5^5) = L(5^6) = 25$.

LEMMA 6. *For a prime p, $L(p^k) \leq kp$ for all $k \geq 1$. Equality holds only for $k = 1, 2, \ldots, p$.*

PROOF. For $1 \leq k \leq p-1$, $(kp)! = 1 \cdots p \cdots (2p) \cdots (kp) = p^k N$ where $p \nmid N$. So $p^k \mid (kp)!$ and $p^k \nmid (kp-1)!$, and hence $L(p^k) = kp$.

For $k = p$, consider $p^2! = 1 \cdots p \cdots (2p) \cdots (p-1)p \cdots p^2 = p^{p+1}M$ with $p \nmid M$. So $p^p \mid p^2!$ and $p^p \nmid (p^2-1)!$. Therefore, $L(p^p) = p^2$.

For the case $k > p$, we use mathematical induction on $k = p+i$, $i \geq 1$. When $k = p+1$, the fact $p^2! = p^{p+1}M$ with $p \nmid M$ implies $L(p^{p+1}) = p^2 < (p+1)p$. Suppose $L(p^{p+i}) < (p+i)p$ for $i \geq 1$. Then, since the product of p consecutive integers right after $L(p^{p+i})$ is a multiple of p,

$$L(p^{p+i+1}) \leq p + L(p^{p+i}) < (p+i+1)p.$$

Therefore, for $k > p$, $L(p^k) < kp$. □

Remark that the above lemma implies $L(p^k) = O(k)$.

The following theorem is a generalization of Corollary 4.

THEOREM 7. *For any positive integers m, n, $L(m \cdot n) \leq L(m) + L(n)$. In particular, $L(n^k) \leq kL(n)$ for any $k \geq 1$.*

PROOF. Let $L(m) = a$ and $L(n) = b$. Then $m \mid a!$ and $n \mid b!$. Since $b!$ divides a product of b consecutive integers, $m \cdot n \mid 1 \cdot 2 \cdots a \cdot (a+1) \cdots (a+b)$. So $L(m \cdot n) \leq a + b$. □

COROLLARY 8. *Let $n = \prod_{i=1}^{l} p_i^{k_i}$, for p_i distinct primes and some $k_i \geq 1$. Then*

$$L(n) = \max_{1 \leq i \leq l} \{L(p_i^{k_i})\} \leq \max_{1 \leq i \leq l} \{k_i L(p_i)\} = \max_{1 \leq i \leq l} \{k_i p_i\}.$$

To get an idea to find $L(p^k)$ for all $k \geq 1$, we try $L(2^k)$ for $k \geq 1$. We first find some of $L(2^k)$'s:

$$\begin{array}{llll} L(2^1) = 2, & L(2^5) = 8, & L(2^9) = 12, & L(2^{13}) = 16, \\ L(2^2) = 4, & L(2^6) = 8, & L(2^{10}) = 12, & L(2^{14}) = 16, \\ L(2^3) = 4, & L(2^7) = 8, & L(2^{11}) = 14, & L(2^{15}) = 16, \\ L(2^4) = 6, & L(2^8) = 10, & L(2^{12}) = 16, & L(2^{16}) = 18. \end{array}$$

By Theorem 7, if $L(2^n) = m$, then $L(2^{2n}) \leq 2m$ for all $n \geq 1$. From $L(2^1) = 2$, $L(2^2) = 4$ and $L(2^4) = 6$, we see that the inequality is sometimes equal and sometimes strict. We investigate when the equality holds.

LEMMA 9. *Given an even integer m,*

$$|\{n : L(2^n) = m\}| = \max\{l : 2^l \mid m\}.$$

PROOF. Write $m! = 2^r P$ and $(m-1)! = 2^s Q$, where $2 \nmid P$ and $2 \nmid Q$. Since m is even, $r > s$. Let $k = r - s \geq 1$. Then $k = \max\{l : 2^l \mid m\}$. We have $2^{s+i} \nmid (m-1)!$ and $2^{s+i} \mid m!$, and so $L(2^{s+i}) = m$ when and only when $i = 1, 2, \dots, k$. □

LEMMA 10. *Let $L(2^k) = m$ and $m \equiv 2 \pmod 4$. Then $L(2^{k-2}) = L(2^{k-1}) = m - 2$ and $L(2^{k+1}) = L(2^{k+2}) = m + 2$.*

PROOF. Write $m! = 2^r P$ and $(m-1)! = 2^s Q$ with $2 \nmid P$ and $2 \nmid Q$. Since $m \equiv 2 \pmod 4$, $r = s + 1$. By Lemma 9, k is the unique integer such that $L(2^k) = m$. So $L(2^{k-1}) = m - 2$ and $L(2^{k+1}) = m + 2$. Since $4 \mid (m \pm 2)$, we have $2^{k-2} \mid (m-2)!$ and $2^{k-2} \nmid (m-3)!$ and $2^{k+2} \mid (m+2)!$ and $2^{k+2} \nmid (m+1)!$. Hence, $L(2^{k-2}) = m - 2$ and $L(2^{k+2}) = m + 2$. □

LEMMA 11. *For all $m \geq 1$, $\binom{2m}{m}$ is even. In particular, for all $n \geq 1$, $\binom{2 \cdot 2^n}{2^n} \equiv 2 \ (mod 4)$.*

PROOF. First of all, it is clear that $\binom{2m}{m}$ is even:

$$\binom{2m}{m} = \frac{(2m)(2m-1)!}{m(m-1)!m!} = 2\binom{2m-1}{m} \quad \text{for } m \geq 1.$$

For the second part, write

$$\binom{2^{n+1}}{2^n} = \prod_{i=1}^{2^n} \frac{2^n + i}{i} = \Bigg(\prod_{i=\text{odd}} \frac{2^n + i}{i} \prod_{i=\text{even}<2^n} \frac{2^{n-r_i} + P_i}{P_i} \Bigg) 2,$$

where for $i < 2^n$ even, i is while the last component is $\frac{2^{n+1}}{2^n} = 2$.

No even integer appears in both of the numerator and denominator of the product in the parenthesis. So $\binom{2^{n+1}}{2^n} \equiv 2 \pmod 4$. □

THEOREM 12. *There are infinitely many n's such that $L(2^{2n}) = 2L(2^n)$ holds.*

To see this, we first investigate, case by case, how $L(2^n)$ inductively determines $L(2^{2n})$. Suppose $L(2^n) = m$ is given. We look at $L(2^{n+1})$ and $L(2^{2n+2})$. From Lemma 11, it can be written $(2m)! = 2^k (m!)^2 P$ for some $k \geq 1$ and an odd integer P.

When $L(2^n) = m$, $L(2^{n+1})$ is either m or $m+2$ depending on whether $m \equiv 0$ or $2 \pmod 4$. So we have two cases. And when $L(2^{n+1}) = m$

holds, we further divide the case into three cases according to what $L(2^{n+2})$ is.

CASE 1. Let $L(2^n) = m$ and $m \equiv 2 \pmod 4$. Then we have $L(2^{n+1}) = L(2^{n+2}) = m + 2$. Since $2^{n+2} \mid (m+2)!$ and 2^n divides a product of m consecutive integers, $L(2^{2n+2}) \leq 2m+2 < 2(m+2) = 2L(2^{n+1})$. So the inequality is strict in this case.

To be precise, write $m! = 2^n N$ for some N with $2 \not| N$. Then $(2m)! = 2^k (m!)^2 P = 2^{2n+k} Q$, where $2 \not| P$ and $2 \not| Q$. So $L(2^{2n+2}) = 2m + 2$ if $k = 1$, $L(2^{2n+2}) = 2m$ if $k = 2$, and $L(2^{2n+2}) \leq 2m$ if $k \geq 3$.

CASE 2. Let $m \equiv 0 \pmod 4$, $L(2^n) = m$ and $L(2^{n+1}) = m + 2$. Then $2^n \mid m!$ and $2^{n+1} \not| m!$. From $(2m)! = 2^{2n+k} Q$ for some odd Q and $2m + 2 \equiv 2 \pmod 4$, we have $L(2^{2n+1+k}) = 2m + 2$.

So $L(2^{2n+2}) \leq L(2^{2n+1+k}) = 2m + 2 < 2(m+2) = 2L(2^{n+1})$. In this case the inequality is strict.

CASE 3. Let $m \equiv 0 \pmod 4$, $L(2^n) = L(2^{n+1}) = m$ and $L(2^{n+2}) = m + 2$. Then $m! = 2^{n+1} N$ for some odd N and $(2m)! = 2^{2n+2+k} M$ for some odd M. So $L(2^{2n+2}) \leq L(2^{2n+2+k}) = 2m = 2L(2^{n+1})$.

Since $8 \mid 2m$, $(2m-2)!$ has at most 2^{2n+k-1} as the power of 2 in its prime factorization. So if $k = 1$ or 2, then $2^{2n+2} \not| (2m-2)!$ and $2^{2n+2} \not| (2m-1)!$. Therefore, $L(2^{2n+2}) = 2m$ if $k = 1$ or 2 and $L(2^{2n+2}) \leq 2m - 2$ if $k \geq 3$. In other words, when $k = 1$ or 2, the equality holds. Otherwise, the inequality is strict.

CASE 4. Let $m \equiv 0 \pmod 4$ and $L(2^n) = L(2^{n+1}) = L(2^{n+2}) = m$. Since $2^{n+2} \mid m!$, $2^{2n+4+k} \mid (2m)!$. So $L(2^{2n+2}) \leq L(2^{2n+4+k}) \leq 2m = 2L(2^{n+1})$.

Write $m! = 2^{n+l} N$ for some $l \geq 2$ and odd N. Then $(2m)! = 2^{2n+2l+k} M$ for some odd M. Since $|\{i : L(2^i) = m\}| \geq 3$, m is written $m = 2^j P$ for some $j \geq 3$ and odd P. From $2m = 2^{j+1} P$, $(2m-2)! = 2^{2n+2l-j-1+k} Q$ for some odd Q. So $2^{2n+2} \mid (2m-2)!$ if and only if $2n + 2 \leq 2n + 2l - j - 1 + k$, i.e., $2l + k - j \geq 3$. Therefore, $L(2^{2n+2}) = 2m = 2L(2^{n+1})$ occurs when and only when $2l + k - j = 1$ or 2. A strict inequality holds otherwise.

PROOF OF THEOREM 12. For all m of the form $m = 2^l$ with $l \geq 2$, $\binom{2m}{m} = 2Q$ for some odd Q by Lemma 11. Since $|\{j | L(2^j) = m\}| = l \geq 2$, we choose n so that $L(2^n) = m$ and $L(2^{n+1}) = m + 2$. For such m and n, we have Case 3 with $k = 1$. So, for each $l \geq 2$, there exists at least one n which satisfies $L(2^{2n}) = 2L(2^n)$. Hence, we have infinitely many n's such that $L(2^{2n}) = 2L(2^n)$. □

REMARK. From Theorem 12, we know that there are infinitely many m and n such that the equality holds in $L(m \cdot n) \le L(m) + L(n)$.

4. Preimages under L

Now we consider the preimages of m under L.

DEFINITION. $$P(m) = \{n : L(n) = m\} = \{n : n \,|\, m!, \text{ and } n \nmid (m-1)!\}.$$

To find $P(m)$, we first suppose that m be a prime. For $m = 2$, $P(2) = \{2\}$ by definition. Now let $m > 2$. The prime factorization of $m!$ is given as

$$(3) \qquad m! = 2^{r_1} 3^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k} \cdot m \quad \text{for some } r_i \ge 1, i = 1, 2, \dots, k,$$

where $2 < 3 < \cdots < p_{k-1} < p_k$ are all the primes less than m.

An integer n dividing $m!$ is of the form

$$n = 2^{s_1} 3^{s_2} \cdots p_k^{s_k} m^{s_{k+1}}, \quad 0 \le s_i \le r_i \text{ for } i = 1, 2, \dots, k, \ s_{k+1} = 0 \text{ or } 1.$$

If $s_{k+1} = 0$, then $n | (m-1)!$. So $L(n) < m$, i.e., $n \notin P(m)$. Hence an integer n in the set $P(m)$, for a prime m, should be of the form

$$n = 2^{s_1} 3^{s_2} \cdots p_k^{s_k} m, \qquad 0 \le s_i \le r_i \text{ for } i = 1, 2, \dots, k.$$

Such integer n divides $m!$ but not $(m-1)!$.

We summarize the above argument as the following theorem:

THEOREM 13. *For a prime $m > 2$ with the factorization of $m!$ given by* (3),

$$P(m) = \{2^{s_1} 3^{s_2} \cdots p_k^{s_k} \cdot m \,:\, 0 \le s_i \le r_i \text{ for } i = 1, 2, \dots, k\}.$$

In fact, the set $P(m)$ is the set of all divisors of $m!$ which have m as a factor, and the number of elements in the set is

$$|P(m)| = \prod_{i=1}^{k} (1 + r_i).$$

Now we consider the set $P(m)$ for composite number m. Let $m = pq$ for primes $p < q$. Then the prime factorization is given by

$$(4) \qquad m! = 2^{r_1}3^{r_2}\cdots p_i^{r_i}\cdots p_j^{r_j}\cdots p_k^{r_k} \quad \text{for some } r_i \geq 1,$$

where $2 < 3 < \cdots < p_k$ are all the primes less than m and $p_i = p$, $p_j = q$.

Any n dividing $m!$ should be of the form

$$n = 2^{s_1}3^{s_2}\cdots p_i^{s_i}\cdots p_j^{s_j}\cdots p_k^{s_k}, \quad 0 \leq s_i \leq r_i \quad \text{for } i = 1, 2, \ldots k.$$

If both $s_i < r_i$ and $s_j < r_j$ hold, then n divides

$$(m-1)! = 2^{r_1}3^{r_2}\cdots p_i^{r_i-1}\cdots p_j^{r_j-1}\cdots p_k^{r_k}.$$

This means $L(n) < m$, i.e., $n \notin P(m)$. So we conclude that at least one of $s_i = r_i$ and $s_j = r_j$ should be satisfied.

THEOREM 14. *For an integer $m = pq$, with primes $p < q$, let $m!$ is factorized as in* (4), *where $p = p_i$ and $q = p_j$, then*

$$\begin{aligned} P(pq) &= \left\{2^{s_1}3^{s_2}\cdots p_i^{r_i}\cdots p_j^{s_j}\cdots p_k^{s_k} \ : \ 0 \leq s_l \leq r_l \text{ for } l \neq i\right\} \\ &\quad \cup \left\{2^{s_1}3^{s_2}\cdots p_i^{s_i}\cdots p_j^{r_j}\cdots p_k^{s_k} \ : \ 0 \leq s_l \leq r_l \text{ for } l \neq j\right\} \end{aligned}$$

and

$$|P(pq)| = \prod_{l\neq i}(r_l+1) + \prod_{l\neq j}(r_l+1) - \prod_{l\neq i,j}(r_l+1).$$

EXAMPLE. Consider the set $P(14)$. By the definition of $P(m)$,

$$P(14) = \{n \ : \ n \mid 14!, \ n \nmid 13!\}.$$

Since $14! = 2^{11}3^5 5^2 7^2 \cdot 11 \cdot 13$, any integer n dividing $14!$ is of the form

$$n = 2^{s_1}3^{s_2}5^{s_3}7^{s_4}11^{s_5}13^{s_6}, \quad 0 \leq s_i \leq r_i \quad \text{for } i = 1, 2, \ldots, 6,$$

where $r_1 = 11$, $r_2 = 5$, $r_3 = r_4 = 2$ and $r_5 = r_6 = 1$.

If $s_1 < 11$ and $s_4 < 2$, then $n \mid 13!$. For 14 contributes one to each of the powers of 2 and 7 in the prime factorization of $14!$. In fact, $L(2^{11}) = L(7^2) = 14$ and $L(p_i^{s_i}) < 14$ for $p_i = 3, 5, 11, 13$, $0 \leq s_i \leq r_i$. So

$$\begin{aligned} P(14) &= \left\{2^{11}3^{s_2}5^{s_3}7^{s_4}11^{s_5}13^{s_6} \ : 0 \leq s_i \leq r_i,\ i = 2,3,4,5,6\right\} \\ &\quad \cup \left\{2^{s_1}3^{s_2}5^{s_3}7^{2}11^{s_5}13^{s_6} \ : 0 \leq s_i \leq r_i,\ i = 1,2,3,5,6\right\}. \end{aligned}$$

Note that $P(14)$ is the set of all divisors of $14!$ containing 2^{11} or 7^2 as their factor. So by inclusion-exclusion principle,

$$\begin{aligned} |P(14)| &= 6\cdot 3\cdot 3\cdot 2\cdot 2 + 12\cdot 6\cdot 3\cdot 2\cdot 2 - 6\cdot 3\cdot 2\cdot 2 \\ &= 1008. \end{aligned}$$

The largest and smallest elements in $P(14)$ are $14!$ and 7^2.

THEOREM 15. *For an integer* $m = \prod_{i=1}^{k} p_i^{a_i}$ *with* $m! = \prod_{j=1}^{l} q_j^{r_j}$, *where* p_i, $i = 1, 2, \ldots, k$, *and* q_j, $j = 1, 2, \ldots, l$, *are primes,*

$$P(m) = \left\{ \prod_{j=1}^{l} q_j^{s_j} \; : \; 0 \le s_j \le r_j \text{ for } j = 1, 2, \ldots, l, \right.$$
$$\left. \text{and } s_J = r_J \text{ for at least one } J \text{ with } q_J = p_i \text{ for some } i \right\}.$$

Remark that the size of the set $P(m)$ is calculated by the inclusion-exclusion principle.

References

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