Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 535–540

CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BY ORDER STATISTICS AND CONDITIONAL EXPECTATIONS OF RECORD VALUES

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ABSTRACT. Let $X_1, X_2, \cdots, X_n$ be n independent and identically distributed random variables with continuous cumulative distribution function $F(x)$. Let us rearrange the $X's$ in the increasing order $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. We call $X_{k:n}$ the k-th order statistic. Then $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent if and only if $F(x) = 1 - e^{-\frac{x}{c}}$ with some $c > 0$. And X_j is an upper record value of this sequence if $X_j > \max\{X_1, X_2, \cdots, X_{j-1}\}$. We define $u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}$ with $u(1) = 1$. Then $F(x) = 1 - e^{-\frac{x}{c}}$, $x > 0$ if and only if $E[X_{u(n+3)} - X_{u(n)} | X_{u(m)} = y] = 3c$, or $E[X_{u(n+4)} - X_{u(n)} | X_{u(m)} = y] = 4c$, $n \geq m + 1$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d.) random variables with a common continuous distribution function $F(x)$ and probability density function(p.d.f.) $f(x)$. Let the events $\{X_{k:n} < x\}$ mean that at least k of the events $\{X_i < x\}$ occur. Then the probability density function for $X_{k:n}$,

$$g_k(x_{k:n}) = \begin{cases} k \cdot \binom{n}{k} F^{k-1}(x_{k:n})[1 - F(x_{k:n})]^{n-k} f(x_{k:n}), & a < x_{k:n} < b \\ 0, & \text{otherwise.} \end{cases}$$

Received October 4, 2001.

2000 Mathematics Subject Classification: 60E15, 62E10.

Key words and phrases: absolutely continuous distribution, characterization, conditional expectation, order statistic, record value.

This work was supported by Korea Research Foundation Grant (2000-015-DP0049).

And the multinomial form gives the joint p.d.f. for $X_{i:n}$ and $X_{j:n}(i < j)$

$$g_{i,j}(x_{i:n}, x_{j:n}) = \begin{cases} \frac{n!}{(i-1)!(j-i-1)!(n-j)!} F^{i-1}(x_{k:n}) f(x_{i:n}) \\ \times\ [F(x_{j:n}) - F(x_{i:n})]^{j-i-1} f(x_{j:n}) \\ \times\ [1 - F(x_{k:n})]^{n-j} f(x_{k:n}),\ \ a < x_{i:n} < x_{j:n} < b \\ 0, \qquad \text{otherwise.} \end{cases}$$

Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. By convention X_1 is an upper as well as a lower record value. We can transform from upper records to lower records by replacing the original sequence of random variables by $\{-X_j, j \geq 1\}$.

We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{j | j > u(n-1)\ ,\ X_j > X_{u(n-1)}\ ,\ n \geq 2\}.$$

We start with the characterizations of the exponential distribution which was studied by Galambos(1978), Ahsanullah(1995) and Lee(2001) under somewhat different assumption. Ahsanullah(1982) showed that if F belongs to the class C_1 and $E(X_k)$, $k \geq 1$ is finite then $X_k \in \text{EXP}(\lambda)$, if and only if for some fixed n, $n > 1$, $E(X_{u(n+1)} - X_{u(n)}) = E(X_k)$. Also he characterized that let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous cumulative distribution $F(x)$ and F belongs to the class C_1 and $E(X_k)$, $k \geq 1$, is finite. Then $X_k \in \text{EXP}(\lambda)$ if and only if for some fixed n, $E(X_{L_n} - X_{L_{n-1}} | X_{L_{n+1}} = u)$ is independent of u. And Lee(2001) showed that $X_k \in \text{EXP}(\lambda)$ if and only if $E[X_{u(n+1)} - X_{u(n)} | X_{u(m)} = y] = c$ and $E[X_{u(n+2)} - X_{u(n)} | X_{u(m)} = y] = 2c$, $c > 0$, $n \geq m+1$.

In this paper, we will show $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent if and only if $F(x) = 1 - e^{-\frac{x}{c}},\ c > 0,\ x > 0$. And we will give characterizations of the exponential distribution by considering conditional expectations of record values.

2. Main results

THEOREM 1. *Let $X_i (i = 1, 2, \cdots, n)$ be i.i.d. random variables with common continuous distribution function $F(x)$. Then $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent if and only if $F(x) = 1 - e^{-\frac{x}{c}},\ c > 0,\ x > 0$.*

PROOF. The joint p.d.f. of $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ is

$$g_{n-1,n}(x_{n-1:n}, x_{n:n}) = \frac{1}{c^2} n(n-1) e^{-\frac{1}{c}(x_{n-1:n}+x_{n:n})}(1 - e^{-\frac{1}{c}x_{n-1:n}})^{n-2}.$$

Consider the function $Z_1 = (X_{n:n} - X_{n-1:n})$, $Z_2 = X_{n-1:n}$ and their inverses $x_{n-1:n} = z_2$, $x_{n:n} = z_1 + z_2$. So the corresponding Jacobian of the one-to-one transformation is $|J| = 1$. Thus, the joint p.d.f. of Z_1 and Z_2 is

$$h(z_1, z_2) = \frac{1}{c^2} n(n-1) e^{-\frac{1}{c}(z_1+2z_2)}(1 - e^{-\frac{1}{c}z_2})^{n-2}.$$

Using the beta function, the marginal p.d.f. of Z_1 is given by

$$h_1(z_1) = \frac{1}{c} e^{-\frac{1}{c}z_1}.$$

And the marginal p.d.f. of Z_2 is denoted by

$$h_2(z_2) = \frac{1}{c} n(n-1) e^{-\frac{2}{c}z_2}(1 - e^{-\frac{1}{c}z_2})^{n-2}.$$

Since $h(z_1, z_2) = h(z_1) \cdot h(z_2)$, Z_1 and Z_2 are independent. That is, $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent for the exponential distribution $F(x) = 1 - e^{-\frac{x}{c}}, x \geq 0, c > 0$.

Next, if $X_{n-1:n}$ and $X_{n:n} - X_{n-1:n}$ are independent then

$$P(X_{n:n} - X_{n-1:n} < x | X_{n-1:n} = z) = P(X_{n:n} - X_{n-1:n} < x) \tag{1}$$

for almost all $z > 0$. Here, $F(x)$ is continuous and strictly increasing for all $x > 0$. On the other hand, by Galambos(1978), it holds the following equation.

$$\begin{aligned} P(X_{n:n} - X_{n-1:n} < x | X_{n-1:n} = z) &= P(X_{n:n} < x + z | X_{n-1:n} = z) \\ &= P(X^*_{1:1} < x + z), \end{aligned}$$

where $X^*_{1:1}$ is the indicated order statistic from a population with parent distribution

$$F^*(x) = \begin{cases} \dfrac{F(x) - F(z)}{1 - F(z)}, & x \geq z \\ 0, & \text{otherwise.} \end{cases}$$

That is, if we denote the right hand side of (1) by $H(x)$, then

$$\frac{F(x+z)-F(z)}{1-F(z)} = H(x)$$

for all $x \geq 0$ and almost all $z > 0$. Let $\lim_{x\to 0} F(x) = H(x)$. Thus, if we write

$$\frac{F(x+z)-F(z)}{1-F(z)} = 1 - \frac{1-F(x+z)}{1-F(z)}$$

then $1 - F(x+z) = [1-F(x)][1-F(z)]$ for all $x \geq 0$ and almost all $z > 0$.

This is an extended form of the lack of memory property.

Hence, $F(x)$ is exponential. □

THEOREM 2. *If $F(x)$ is absolutely continuous with $F(x) < 1$ for all x then*

$$E[X_{u(n+3)} - X_{u(n)} | X_{u(m)} = y] = 3c, \quad c > 0, \quad n \geq m+1 \tag{2}$$

if and only if $F(x) = 1 - e^{-\frac{x}{c}}, \quad x > 0.$

PROOF. If $F(x) = 1 - e^{-\frac{x}{c}}$, $x > 0$, $c > 0$, then $E[X_{u(n)} |\ X_{u(m)} = y] = y + (n-m)c$. Hence (2) holds.

Conversely, suppose (2) holds. Using Ahsanullah formula(1995) it follows the following equation.

$$\begin{aligned} &\frac{1}{1-F(y)} \int_y^\infty \frac{1}{(n-m+2)!} \left(\ln \frac{1-F(y)}{1-F(x)} \right)^{n-m+2} x f(x) dx \\ &- \frac{1}{1-F(y)} \int_y^\infty \frac{1}{(n-m-1)!} \left(\ln \frac{1-F(y)}{1-F(x)} \right)^{n-m-1} x f(x) dx = 3c, \\ &\qquad c > 0, \quad n \geq m+1. \end{aligned}$$

i.e.,

(3)

$$\begin{aligned} &\frac{1}{(n-m+2)!} \int_y^\infty \left(\ln \frac{1-F(y)}{1-F(x)} \right)^{n-m+2} x f(x) dx \\ &- \frac{1}{(n-m-1)!} \int_y^\infty \left(\ln \frac{1-F(y)}{1-F(x)} \right)^{n-m-1} x f(x) dx = 3c(1-F(y)). \end{aligned}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+3)$ times both sides of (3) with respect to y and simplify, then we obtain the following equation

$$7[1-F(y)]-3cf(y)+\frac{6f'(y)[1-F(y)]^2}{f^2(y)}-\frac{f''(y)[1-F(y)]^3}{f^3(y)}+\frac{3[f'(y)]^2[1-F(y)]^3}{f^4(y)}=0.$$

Let $y=F(y), y'=f, y''=f', y'''=f''$. Then it expressed the following form

$$(4)\quad 7(1-y)-3cy'+\frac{6y''(1-y)^2}{y'^2}-\frac{y'''(1-y)^3}{y'^3}+\frac{3y''^2(1-y)^3}{y'^4}=0$$

i.e., $y'''=f(x,y,y',y'')$. Therefore there exists an unique solution of the differential equation (4) that satisfies the prescribed initial conditions $y(0)=0,\ y'(0)=\frac{1}{c},\ y''(0)=-\frac{1}{c^2}$.

By the existence and uniqueness Theorem, we get $F(x)=1-e^{-\frac{x}{c}}$. This completes the proof. □

THEOREM 3. *If $F(x)$ is absolutely continuous with $F(x)<1$ for all x then*

$$E[X_{u(n+4)}-X_{u(n)}|X_{u(m)}=y]=4c,\ \ c>0,\ \ n\geq m+1$$

if and only if $F(x)=1-e^{-\frac{x}{c}},\ \ x>0$.

PROOF. In the same manner as Theorem 2, we obtain the following differential equation.

(5)

$$15(1-y)-4cy'+\frac{25y''(1-y)^2}{y'^2}-\frac{10y'''(1-y)^3}{y'^3}+\frac{30y''^2(1-y)^3}{y'^4}+\frac{y''''(1-y)^4}{y'^4}-\frac{10y'y''(1-y)^4}{y'^5}+\frac{15y''^3(1-y)^4}{y'^6}=0$$

i.e., $y''''=f(x,y,y',y'',y''')$. Therefore there exists an unique solution of the differential equation (5) that satisfies the prescribed initial conditions $y(0)=0,\ y'(0)=\frac{1}{c},\ y''(0)=-\frac{1}{c^2},\ y'''(0)=\frac{1}{c^3}$.

By the existence and uniqueness Theorem, we get $F(x)=1-e^{-\frac{x}{c}}$. This completes the proof. □

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