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A NOTE ON WEAKLY FIRST COUNTABLE SPACES

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Abstract. We prove that the star operator and the sequential closure operator on a weakly first countable space are the same and show that the Fréchetness is a sufficient condition for a weakly first countable space to be first countable.

1. Introduction

In the short time since A. V. Arhangel'skii[1] introduced weakly first countable spaces in his study of generalized metrizable spaces and metrizations it has become apparent that these spaces occupy a basic position in that study. The class of weakly first countable spaces includes both first countable spaces and symmetrizable spaces.

Weakly first countable spaces have been studied by many authors, too numerous to mention here.

In particular, F. Siwiec[6] proved the following:

(1) The Fréchetness is a sufficient condition that weakly first countable spaces be first countable.

(2) The Fréchetness is a sufficient condition that symmetrizable spaces be semi-metrizable.

In [3], G. Gruenhage also showed the result (2) of F. Siwiec.

Recently, R. E. Hodel[4] introduced the star operator on a weakly first countable space and also showed the result (1) of F. Siwiec using the star operator.

In this paper, we prove that the star operator and the sequential closure operator on a weakly first countable space are the same and get the result (1) of F. Siwiec and R. E. Hodel by a different method from R.

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E. Hodel's. Also we obtain the result (2) of F. Siwiec and G. Gruenhage as a corollary.

2. Results

All spaces in this paper are assumed to be Hausdorff.

We begin by defining weakly first countable spaces. We denote $\mathbb{N}$ by the set of all natural numbers.

DEFINITION 2.1. (1) A topological space X is called *weakly first countable*[1, 4 and 6] if for each $x \in X$, there exists a family $\{N(x,n) | n \in \mathbb{N}\}$ of subsets of X such that the following are true:

(a) $x \in N(x, n+1) \subset N(x,n)$ for each $n \in \mathbb{N}$.

(b) A subset U of X is open in the space X if and only if for each $x \in U$ there exists $n \in \mathbb{N}$ such that $N(x,n) \subset U$.

Such a family $\{N(x,n) | n \in \mathbb{N}\}$ is called a *weak base at* x in X.

(2) A topological space X is called *symmetrizable*[5] if there exists a symmetric(=a metrc except for the triangle inequality) d on X such that the following is true: a subset U of X is open in the space X if and only if for each $x \in U$ there exists a positive real number ϵ such that $B(x,\epsilon) \subset U$, where $B(x,\epsilon) = \{y \in X | d(x,y) < \epsilon\}$.

(3) A topological space X is called *semi-metrizable*[4] if there exists a symmetric d on X such that for each $x \in X$, the family $\{B(x,\epsilon) | \epsilon > 0\}$ forms a (not necessarily open) neighborhood base at x.

Obviously, the following diagram exhibits the relationships among the spaces mentioned above. It is well known that no arrows may be reversed.

$$\begin{array}{ccc} \text{first countable} & \Rightarrow & \text{weakly first countable} \\ \Uparrow & & \Uparrow \\ \text{semi-metrizable} & \Rightarrow & \text{symmetrizable} \end{array}$$

Let X be a weakly first countable space and for each $x \in X$, $\{N(x,n) | n \in \mathbb{N}\}$ a weak base at x in the space X. We denote $P(X)$ by the power set of X.

The function $(.)^* : P(X) \to P(X)$ defined by for each subset A of X,

$$(A)^* = \{x \in X | A \cap N(x,n) \neq \emptyset \text{ for each } n \in \mathbb{N}\}$$

is called *the star operator*[4] on the space X.

The function $[\cdot] : P(X) \to P(X)$ defined by for each subset A of X,

$$[A] = \{x \in X | (x_n) \text{ converges to } x \text{ in } X \text{ for some sequence } (x_n) \text{ in } A\}$$

is called *the sequential closure operator*[2 and 5] on the space X.

It is well known that for each subset A of X, $[A] \subset c(A)$ and $(A)^* \subset c(A)$, where $c(A)$ is the closure of A in X.

THEOREM 2.2. *Let X be a weakly first countable space. Then for each subset A of X, $(A)^* = [A]$.*

PROOF. Let $p \in (A)^*$ and $\{N(p,n)|n \in \mathbb{N}\}$ a weak base at p in X. Then clearly $A \cap N(p,n) \neq \emptyset$ for each $n \in \mathbb{N}$ and so there exists a sequence (x_n) of points in A such that $x_n \in A \cap N(p,n)$ for each $n \in \mathbb{N}$. It is clear that (x_n) converges to p in X. Hence, $p \in [A]$.

Conversely, suppose on the contrary that it is not. Then there exists a point p in X such that $p \in [A]$ but $p \notin (A)^*$. Since $p \notin (A)^*$, there is $n \in N$ such that $A \cap N(p,n) = \emptyset$. And since $p \in [A]$, there exists a sequence (x_n) of points in A such that (x_n) converges to p in X. Now let $U = X - \{x_n|n \in \mathbb{N}\}$, where $\{x_n|n \in \mathbb{N}\}$ is the range of the sequence (x_n). Then, since $U = (X - (\{x_n|n \in \mathbb{N}\} \cup \{p\})) \cup \{p\}$ and $X - (\{x_n|n \in \mathbb{N}\} \cup \{p\})$ is open in X, by weak first countability of X, U is open in X. Thus, we have that there is an open set in X which does not meet the set $\{x_n|n \in \mathbb{N}\}$ and contains the point p, which is a contradiction. □

Recall a definition that a topological space X is *Fréchet*[2] if and only if for each subset A of X, $[A] = c(A)$. Then, by Theorem 2.2, we have directly the following corollary and hence omit the proof.

COROLLARY 2.3. *In a weakly first countable and Fréchet space X for each subset A of X, $(A)^* = c(A)$.*

COROLLARY 2.4. ([6, Theorem 1.10] and [4, Corollary 5.5]) *Every weakly first countable and Fréchet space is first countable.*

PROOF. Let X be a weakly first countable and Fréchet space and for each $x \in X$, $\{N(x,n)|\ n \in \mathbb{N}\}$ a weak base at x in the space X. Then it is sufficient to prove that for each $x \in X$ and $n \in \mathbb{N}$, $N(x,n)$ is a neighborhood of x in X. Suppose that it is not. Then there exists $p \in X$ and $n \in \mathbb{N}$ such that $N(p,n)$ is not a neighborhood of p. Clearly, $p \notin int(N(p,n))$, where $int(N(p,n))$ is the interior of $N(p,n)$ in X, and so

$p \in c(X - N(p,n))$. By Corollary 2.3, it follows that $p \in (X - N(p,n))^*$. This is a contradiction. □

Immediately we have the following corollary.

COROLLARY 2.5. ([6, Theorem 1.10] and [3, Theorem 9.6]) *The following statements are equivalent.*
(1) X *is semi-metrizable.*
(2) X *is symmetrizable and first countable.*
(3) X *is symmetrizable and Fréchet.*

PROOF. By definitions, (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. Since every symmetrizable space is weakly first countable, by Corollary 2.4, we have (3) $\Rightarrow$ (2). □

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