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$(n+1)$-DIMENSIONAL CONTACT CR-SUBMANIFOLDS OF $(n-1)$ CONTACT CR-DIMENSION IN A SASAKIAN SPACE FORM

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ABSTRACT. In this paper we study $(n+1)$-dimensional contact CR-submanifolds of $(n-1)$ contact CR-dimension immersed in a Sasakian space form $M^{2m+1}(c)$ $(2m = n+p,\ p > 0)$, and especially determine such submanifolds under additional condition concerning with shape operator.

1. Introduction

Let $M^{2m+1}(c)$ be a $(2m+1)$-dimensional Sasakian space form with Sasakian structure $(\phi, \xi, \eta, \overline{g})$. Then by definition([9]) it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &\overline{g}(\phi X, \phi Y) = \overline{g}(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = \overline{g}(X,\xi) \end{aligned} \tag{1.1}$$

for any vector fields X, Y tangent to $M^{2m+1}(c)$. Denoting by $\overline{\nabla}$ the Levi-Civita connection on $M^{2m+1}(c)$, we have

$$\overline{\nabla}_X \xi = \phi X, \tag{1.2}$$

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(X,Y)\xi + \eta(Y)X. \tag{1.3}$$

Moreover, since $M^{2m+1}(c)$ is of constant ϕ-sectional curvature c, its curvature tensor $\overline{R}$ has the form

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\begin{aligned}
\overline{R}_{XY}Z = \frac{c+3}{4} \{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} - \frac{c-1}{4}\{\eta(Y)\eta(Z)X \\
- \eta(X)\eta(Z)Y + \eta(X)\overline{g}(Y,Z)\xi - \eta(Y)\overline{g}(X,Z)\xi \\
- \overline{g}(\phi Y,Z)\phi X + \overline{g}(\phi X,Z)\phi Y + 2\overline{g}(\phi X,Y)\phi Z\}
\end{aligned} \tag{1.4}
$$

for any vector fields X, Y, Z tangent to $M^{2m+1}(c)$.

Let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension isometrically immersed in $M^{2m+1}(c)$ $(2m = n + p,\ p > 0)$. Then, by definition([7]) it follows that M is tangent to the structure vector field ξ and the ϕ-invariant subspace

$$\mathcal{D}_x := T_xM \cap \phi T_xM$$

of the tangent space T_xM of M at x in M has constant dimension $n-1$ everywhere. So there is a unit tangent vector field U_1 to M, which is orthogonal to ξ and satisfies

$$\mathcal{D}_x^{\perp} := \mathrm{Span}\{\xi, U_1\} \quad {}^{\forall}x \in M,$$

where $\mathcal{D}_x^{\perp}$ is the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. We now put

$$N_1 := \phi U_1. \tag{1.5}$$

Then N_1 is a unit normal vector field to M and

$$\phi T_xM \subset T_xM \oplus \mathrm{Span}\{N_1\}$$

at each point x in M. Hence we have, for any tangent vector field X and for a local orthonormal basis $\{N_1, N_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M, the following decomposition in tangential and normal components :

$$\phi X = FX + u^1(X)N_1, \tag{1.6}$$

$$\phi N_\alpha = -U_\alpha + PN_\alpha, \quad \alpha = 1, \dots, p. \tag{1.7}$$

By means of (1.1) we can easily show that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^{\perp}$ respectively, where $T_xM^{\perp}$ denotes the normal space of M at x in M.

We first notice that

$$\phi N_1 = -U_1, \tag{1.8}$$

which is a direct consequence of (1.1), (1.5) and

$$\eta(U_1) = \overline{g}(\xi, U_1) = 0. \tag{1.9}$$

Thus (1.7) and (1.8) imply

$$PN_1 = 0. \tag{1.10}$$

Since the structure vector field ξ is tangent to M, (1.1), (1.6) and (1.7) imply

$$g(FU_\alpha, X) = -u^1(X)g(N_1, PN_\alpha), \tag{1.11}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(PN_\alpha, PN_\beta), \tag{1.12}$$

where here and in the sequel g denotes the Riemannian metric induced from $\overline{g}$ on M. We can also find

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha}$$

and consequently

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \dots, p. \tag{1.13}$$

Furthermore from (1.1) and (1.6) it is clear that

$$F\xi = 0, \quad u^1(\xi) = 0. \tag{1.14}$$

Also, from (1.5) and (1.6) it follows that

$$FU_1 = 0, \quad u^1(U_1) = 1, \tag{1.15}$$

which are also derived from (1.10) and (1.11) with $\alpha = 1$. The equations (1.10), (1.12) and (1.13) also yield

$$g(U_1, U_1) = 1, \tag{1.16}$$

$$g(PN_\alpha, PN_\beta) = \delta_{\alpha\beta}, \quad 2 \leq \alpha, \beta \leq p. \tag{1.17}$$

Thus, putting

$$PN_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta} N_\beta, \quad \alpha = 2, \dots, p,$$

we have

$$\sum_{\gamma=2}^{p} P_{\alpha\gamma} P_{\gamma\beta} = -\delta_{\alpha\beta}, \quad \alpha, \beta = 2, \dots, p \tag{1.18}$$

and $(P_{\alpha\beta})$ is a skew-symmetric matrix.

Applying ϕ to (1.6) and using (1.1), (1.6) itself and (1.8), we have

$$F^2 X = -X + \eta(X)\xi + u^1(X)U_1, \tag{1.19}$$

$$u^1(FX) = 0. \tag{1.20}$$

The above results (1.9), (1.13)-(1.16) and (1.19)-(1.20) tell us that M admits the so-called (f, g, u, v, λ)-structure with $f = F$, $u = u^1$, $v = \eta$ and $\lambda = 0$(for the definition of (f, g, u, v, λ)-structure, see [11]). Hence $dimM$ is even. Recently Kwon and Pak [7] studied the submanifold M with normal (f, g, u, v, λ)-structure ($f = F$, $u = u^1$, $v = \eta$, $\lambda = 0$) when the ambient manifold $M^{2m+1}(c)$ is a unit $(2m+1)$-sphere S^{2m+1} and proved

THEOREM K-P. *Let M be an $(n+1)$-dimensional contact CR- submanifold of $(n-1)$ contact CR-dimension isometrically immersed in S^{2m+1} ($2m = n+p$, $p > 0$) and let the normal field N_1 be parallel with respect to the normal connection induced from the Levi-Civita connection of S^{2m+1} on the normal bundle of M. If $A_1 F = F A_1$ on M, then M is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres and A_1 is the shape operator corresponding to N_1.*

In this paper we shall study $(n+1)$-dimensional contact CR- submanifolds of $(n-1)$ contact CR-dimension isometrically immersed in $M^{2m+1}(c)$ and prove the following theorems as improvements of Theorem K-P :

THEOREM 1. *Let M be an $(n+1)$-dimensional contact CR- submanifold of $(n-1)$ contact CR-dimension isometrically immersed in S^{2m+1}*

$(2m = n+p,\ p > 0)$ and let the normal field N_1 be parallel with respect to the normal connection. If $\mathcal{L}_{U_1} H = 0$ on M and $g(A_1U_1, U_1) \neq 0$ at a point of M, then M is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres and $\mathcal{L}_{U_1}$ denotes the Lie derivation in the direction of U_1, H being defined by $H(X,Y) := g(A_1X, Y)$.

THEOREM 2. *Let M be as in Theorem 1. If $\mathcal{L}_{U_1} A_1 = 0$ on M, then M is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres.*

2. Preliminaries

We first let M be as in section 1 and use the same notation as shown in that section. Denoting by ∇ the Levi-Civita connection induced from $\overline{\nabla}$ on M, the Gauss and Weingartan equations are of the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + D_X N_\alpha, \quad \alpha = 1, \dots, p, \tag{2.2}$$

respectively. Here D denotes the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M, and h and A_α the second fundamental form and the shape operator corresponding to N_α, respectively. It is clear that h and A_α are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y) N_\alpha. \tag{2.3}$$

Especially we put

$$D_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X) N_\beta, \tag{2.4}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D.

Differentiating (1.6) and (1.7) covariantly and using (1.2), (1.3), (1.6)-(1.7) themselves and (2.1)-(2.2), we can easily obtain

$$\begin{aligned}(\nabla_X F)Y = & -g(X,Y)\xi + \eta(Y)X \\ & - g(A_1X, Y)U_1 + u^1(Y)A_1X,\end{aligned} \tag{2.5}$$

(2.6) $$(\nabla_X u^1)Y = g(FA_1X, Y),$$

(2.7) $$\nabla_X U_1 = FA_1X,$$

(2.8) $$g(A_\alpha U_1, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P_{\beta\alpha}, \ \alpha = 2, \dots, p.$$

Since the structure vector field ξ is tangent to M, (1.2), (1.6), (2.1) and (2.3) imply

(2.9) $$\nabla_X \xi = FX,$$

(2.10) $$g(A_1\xi, X) = u^1(X), \text{ that is, } A_1\xi = U_1,$$

(2.11) $$A_\alpha \xi = 0, \quad \alpha = 2, \dots, p.$$

In the rest of this paper *we assume that the normal field N_1 is parallel with respect to the normal connection D*. Hence (2.4) yields

(2.12) $$s_{\alpha 1} = 0, \quad \alpha = 2, \dots, p,$$

from which together with (2.8), it follows that

(2.13) $$A_\alpha U_1 = 0, \quad \alpha = 2, \dots, p.$$

On the other hand, the ambient manifold $M^{2m+1}(c)$ is of constant ϕ-sectional curvature c and so it follows from (1.4) that the equation of Codazzi is of the form

(2.14) $$(\nabla_X A_1)Y - (\nabla_Y A_1)X = \frac{c-1}{4}\{u^1(X)FY - u^1(Y)FX - 2g(FX, Y)U_1\},$$

(2.14)′ $$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = \sum_{\beta=2}^{p}\{s_{\beta\alpha}(Y)A_\beta X - s_{\beta\alpha}(X)A_\beta Y\}, \quad \alpha = 2, \dots, p.$$

From now on we prepare some algebraic identities for later use. We set

(2.15) $$V := \nabla_{U_1} U_1,$$

which is equivalent to

$$V = FA_1U_1 \tag{2.15)'}$$

because of (2.7). Then, from (1.14), (1.15), (1.19) and (2.10) it follows that

$$g(V,\xi) = 0,\ g(V,U_1) = 0,\ g(V,V) = \beta - 1 - \alpha^2, \tag{2.16}$$

$$FV = -A_1U_1 + \xi + \alpha U_1, \tag{2.17}$$

where we put

$$\alpha = g(A_1U_1, U_1) = u^1(A_1U_1), \quad \beta = g(A_1^2U_1, U_1). \tag{2.18}$$

Moreover, by using (2.10), (2.16) and (2.17) we can show that

$$g(A_1V, U_1) = 0, \quad g(A_1V, \xi) = 0. \tag{2.19}$$

Also (1.13), (1.19), (2.7), (2.9), (2.16) and (2.17) yield

$$u^1(\nabla_X V) = u^1(X) + \alpha u^1(A_1X) - u^1(A_1^2X), \tag{2.20}$$

$$\eta(\nabla_X V) = \eta(X) + \alpha u^1(X) - u^1(A_1X). \tag{2.21}$$

On the other hand it is clear from (2.5), (2.7) and (2.10) that

$$\begin{aligned}\nabla_Y\nabla_X U_1 = &- g(A_1X,Y)\xi + u^1(X)Y - g(A_1^2X,Y)U_1 \\ &+ u^1(A_1X)A_1Y + F(\nabla_Y A_1)X + FA_1\nabla_Y X,\end{aligned}$$

from which, putting $X = U_1$ and making use of (1.15), (2.7), (2.15) and (2.18), we obtain

$$\begin{aligned}\nabla_Y V = &- u^1(A_1Y)\xi + Y - u^1(A_1^2Y)U_1 + \alpha A_1Y \\ &+ F(\nabla_Y A_1)U_1 + FA_1FA_1Y.\end{aligned} \tag{2.22}$$

We now take an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ of tangent vectors to M as follow :

$$e_1 := \xi,\ e_2 := U_1,\ e_{k+1} := Fe_3,\ \dots,\ e_{n+1} := Fe_k,$$

where $k = 2 + (n-1)/2$. Then we can easily verify that

$$\begin{aligned}\mathrm{div}V &= \sum_{i=1}^{n+1} g(\nabla_{e_i} V, e_i) \\ &= 1 - \beta^2 + \alpha^2 + 2(k-2) + \alpha(\mathrm{tr}A_1 - \alpha) \\ &+ \sum_{i=3}^{k} g((\nabla_{Fe_i} A_1)e_i - (\nabla_{e_i} A_1)Fe_i, U_1) + 2\sum_{i=3}^{k} g(FA_1FA_1e_1, e_i)\end{aligned}$$

with the help of (1.9), (1.14), (1.15), (1.19), (2.10), (2.18) and (2.22) By the way, as a direct consequence of (2.14), we have

$$\sum_{i=3}^{k} g((\nabla_{Fe_i} A_1)e_i - (\nabla_{e_i} A_1)Fe_i, U_1) = \frac{k-2}{2}(c-1)$$

and consequently

$$\begin{aligned}\mathrm{div}V =& 2k - 3 + \frac{c-1}{2}(k-2) - \beta + \alpha(\mathrm{tr}A_1) \\ &+ 2\sum_{i=3}^{k} g(A_1FA_1Fe_i, e_i).\end{aligned} \tag{2.23}$$

On the other hand, a simple computation by using (2.7) implies

$$\begin{aligned}\|\mathcal{L}_{U_1} g\|^2 &= \|FA_1 - A_1F\|^2 \\ &= \sum_{i=1}^{n+1} g((FA_1 - A_1F)(FA_1 - A_1F)^t e_i, e_i) \\ &= 4\sum_{i=3}^{k} g(A_1FA_1Fe_i, e_i) + 2(\mathrm{tr}A_1^2 - 1 - \beta),\end{aligned}$$

from which and (2.21) it follows that

$$\begin{aligned}\mathrm{div}V =& \frac{1}{2}\|FA_1 - A_1F\|^2 + \frac{c-1}{2}(k-2) \\ &+ 2(k-1) + \alpha(\mathrm{tr}A_1) - \mathrm{tr}A_1^2.\end{aligned} \tag{2.24}$$

Differentiating (2.17) covariantly and taking account of (2.5), (2.7), (2.9) and (2.16), we can easily show that

$$\begin{aligned}(2.25)\qquad & g(X,V)\xi + g(A_1X,V)U_1 - F\nabla_X V \\ &= (\nabla_X A_1)U_1 + A_1FA_1X - FX - (X\alpha)U_1 - \alpha FA_1X\end{aligned}$$

and consequently

$$(2.26)\qquad g((\nabla_X A_1)U_1,\xi) = g(X,V),$$

$$(2.27)\qquad g((\nabla_X A_1)U_1,U_1) = 2g(A_1X,V) + X\alpha.$$

3. Proof of Theorem 1

In this section we shall give the proof of Theorem 1 stated in section 1. We first suppose that

$$(3.1)\qquad \mathcal{L}_{U_1}H = 0,$$

where H is a tensor field of type (0,2) defined by

$$H(X,Y) := g(A_1X,Y).$$

We notice that

$$(\mathcal{L}_{U_1}H)(X,Y) = g((\nabla_{U_1}A_1)X,Y)$$

because of (2.7). Therefore the condition (3.1) is equivalent to $\nabla_{U_1}A_1 = 0$, from which and (2.14) with $Y = U_1$ we have

$$(3.2)\qquad (\nabla_X A_1)U_1 = -\frac{c-1}{4}FX.$$

Substituting (3.2) into (2.26) and using (1.14), we have $g(X,V) = 0$, that is, $V = 0$, which together with (2.27) and (3.2) yields α=constant. Combining those results with (2.25), we can find that $\alpha(FA_1 - A_1F) = 0$ on M.

Thus we have the following Proposition 1, which together with Theorem K-P implies Theorem 1.

PROPOSITION 1. *Let M be an $(n+1)$-dimensional contact CR- submanifold of $(n-1)$ contact CR-dimension isometrically immersed in a Sasakian space form $M^{2m+1}(c)$ and let the normal field N_1 be parallel with respect to the normal connection. If $\mathcal{L}_{U_1}H = 0$ on M and $g(A_1U_1,U_1) \neq 0$ at a point of M, then the vector field U_1 is a Killing one, or equivalently $FA_1 = A_1F$ on M.*

REMARK 1. Let M be as in Proposition 1. Then it can be easily shown that $\mathcal{L}_\xi H = 0$ identically on M.

4. Proof of Theorem 2

In order to give the proof of Theorem 2 stated in section 1, we assume that

$$\mathcal{L}_{U_1} A_1 = 0. \tag{4.1}$$

Since (2.7) and (2.14) with $Y = U_1$ imply

$$(\mathcal{L}_{U_1} A_1)X = (\nabla_X A_1)U_1 + \frac{c-1}{4} FX - FA_1^2 X + A_1 F A_1 X,$$

(4.1) is equivalent to

$$(\nabla_X A_1)U_1 = -\frac{c-1}{4} FX + FA_1^2 X - A_1 F A_1 X, \tag{4.2}$$

which together with (1.14), (1.15), (2.10) and (2.15)′ gives

$$g((\nabla_X A_1)U_1, U_1) = g(A_1 X, V), \tag{4.3}$$

$$g((\nabla_X A_1)U_1, \xi) = 0. \tag{4.4}$$

Hence it follows from (2.24) and (4.4) that $V = 0$ and consequently α=constant. Substituting (4.2) into (2.25) and taking account of those results, we have

$$\frac{c+3}{4} FX - FA_1^2 X + \alpha F A_1 X = 0,$$

from which, applying F and using (2.20)-(2.21) with $V = 0$,

$$\begin{aligned}\frac{c+3}{4}\{X - u^1(X)U_1 - \eta(X)\xi\} + u^1(X)U_1 + \eta(X)\xi \\ - A_1^2 X + \alpha A_1 X = 0.\end{aligned}$$

Considering the adapted orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ as shown in section 2 and taking the trace of the last equation, we can easily see that

$$2(k-1) + \frac{c-1}{2}(k-2) + \alpha(\mathrm{tr} A_1) - \mathrm{tr} A_1^2 = 0,$$

which and (2.24) with $V = 0$ give $\|FA_1 - A_1 F\|^2 = 0$.

Thus we have the following Proposition 2, which together with Theorem K-P implies Theorem 2.

PROPOSITION 2. *Let M be as in Proposition 1. If $\mathcal{L}_{U_1} A_1 = 0$ on M, then the vector field U_1 is a Killing one.*

REMARK 2. On the submanifold M as in Proposition 1, it holds identically that $\mathcal{L}_\xi A_1 = 0$.

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