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STABLE RANKS OF MULTIPLIER ALGEBRAS OF C^*-ALGEBRAS

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ABSTRACT. We estimate the stable rank, connected stable rank and general stable rank of the multiplier algebras of C^*-algebras under some conditions and prove that the ranks of them are infinite. Moreover, we show that for any σ-unital subhomogeneous C^*-algebra, its stable rank is equal to that of its multiplier algebra.

Introduction

The stable rank of Banach or C^*-algebras was introduced by M. A. Rieffel [18] as an analogy to the covering dimension for spaces, and this rank, the connected stable rank and general stable rank play important roles in the (non-stable) K-theory of C^*-algebras. The stable rank and connected stable rank of group C^*-algebras of some Lie groups have been computed in terms of groups by [20], [29, 30] and [21-27], which partially answers to an interesting question by Rieffel [18, Question 4.14]. Also, he raised another interesting question such that on what condition the stable rank of C^*-algebras is equal to that of their multiplier algebras [18, Question 4.16]. In this paper we compute these ranks of the multipliers algebras of some C^*-algebras including some group C^*-algebras of Lie groups, and obtain that the latter question is affirmative for σ-unital, subhomogeneous C^*-algebras.

Notation. Let $C_0(X)$ be the C^*-algebra of all complex-valued, continuous functions on a locally compact Hausdorff space X vanishing

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at infinity, and $C^b(X)$ the C^*-algebra of all bounded continuous functions on X. When X is compact, we set $C(X) = C_0(X)$. We denote by $\dim X$ the covering dimension of a topological space X. We set $\dim_{\mathbb{C}}(X) = [\dim X/2] + 1$ where $[x]$ means the maximum integer $\leq x$. Let βX be the Stone-Čech compactification of X. We denote by $\mathbb{K}$ and $\mathbb{B}$ respectively the C^*-algebra of all compact (resp. bounded) operators on a separable, infinite dimensional Hilbert space. We denote by $M(\mathfrak{A})$ the multiplier algebra of a C^*-algebra $\mathfrak{A}$. For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), its stable rank, connected stable rank and general stable rank are denoted by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ and $\mathrm{gsr}(\mathfrak{A})$ respectively (cf. [18]). Then we have by [18, Corollary 4.10 and p. 328] that $\mathrm{gsr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$.

1. Stable ranks of multiplier algebras of C^*-algebras

We first prove the following which is a modification of [8, Proposition 1.4]:

PROPOSITION 1.1. *Let $\mathfrak{B}$ be a unital C^*-algebra having a quotient $\mathfrak{D}$ containing n orthogonal isometries $\{S_j\}_{j=1}^n$ such that $\sum_{j=1}^n S_j S_j^* = 1$ for any $n \geq 2$. Then we have that $\mathrm{csr}(\mathfrak{B}) = \infty$ and $\mathrm{gsr}(\mathfrak{B}) = \infty$.*

PROOF. We now suppose that $\mathrm{csr}(\mathfrak{B}) \leq n+1$. Let $q : \mathfrak{B} \to \mathfrak{D}$ be the quotient map. We take $L_j \in \mathfrak{B}$ such that $q(L_j) = S_j$ $(1 \leq j \leq n)$. Put $T = \sum_{j=1}^n L_j L_j^* - 1$ which is in the kernel of q. Then by definition of connected stable rank, there exists an invertible matrix (K_{ij}) of the connected component of $GL_{n+1}(\mathfrak{B})$ with the unit such that

$$(K_{ij})(-T, L_1^*, \cdots, L_n^*)^t = (1, 0, \cdots, 0)^t \in \mathfrak{B}^{n+1},$$

where $(\cdot)^t$ means the transpose. Hence we have that

$$(q(K_{ij}))(0, S_1^*, \cdots, S_n^*)^t = (1, 0, \cdots, 0)^t \in \mathfrak{D}^{n+1}$$

which implies that $q(K_{ij}) = 0$ $(2 \leq i, j \leq n+1)$ since $S_j^* S_i = \delta_{ij}$ $(1 \leq i, j \leq n)$. This is a contradiction to that $(q(K_{ij}))$ is in $GL_{n+1}(\mathfrak{D})$. Hence $\mathrm{csr}(\mathfrak{B}) = \infty$.

The proof for $\mathrm{gsr}(\mathfrak{B}) = \infty$ is the same as above from definition of $\mathrm{gsr}(\cdot)$. □

THEOREM 1.2. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra. Suppose that $\mathfrak{A}$ has $\mathbb{K}$ as a quotient. Then* $\mathrm{sr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{csr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{gsr}(M(\mathfrak{A})) = \infty$.

PROOF. By assumption and (noncommutative) Tietze's extension theorem (cf. [31, Theorem 2.3.9]) we see that $M(\mathfrak{A})$ has $\mathbb{B} \cong M(\mathbb{K})$ as a quotient. By [18, Theorem 4.3 and Proposition 6.5] and Proposition 1.1, we have that

$$\mathrm{sr}(M(\mathfrak{A})) \geq \mathrm{sr}(\mathbb{B}) = \infty, \quad \mathrm{csr}(M(\mathfrak{A})) = \infty, \quad \mathrm{gsr}(M(\mathfrak{A})) = \infty. \quad \square$$

REMARK. We may take $\mathfrak{A}$ as a C^*-algebra of continuous fields on a locally compact Hausdorff space with one of their fibers isomorphic to $\mathbb{K}$, which implies that Theorem 1.2 covers a large number of somewhat interesting examples since arbitrary C^*-algebra could be taken as a fiber (cf. [10]).

COROLLARY 1.3. *Let $\mathfrak{A}$ be a separable, liminal C^*-algebra having $\mathbb{K}$ as a quotient. Then* $\mathrm{sr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{csr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{gsr}(M(\mathfrak{A})) = \infty$.

REMARK. Every separable C^*-algebra is σ-unital (cf. [12, p. 108]). We may take $\mathfrak{A}$ as the group C^*-algebra $C^*(G)$ of G either a connected nilpotent Lie group or a connected semi-simple Lie group (cf. [7]). In particular, we have that $\mathrm{sr}(C^*(G)) = 2 = \mathrm{csr}(C^*(G))$ in the case of G the real 3-dimensional Heisenberg Lie group (cf. [29], [23], [8, Corollary 1.6]). Also, this $C^*(G)$ is regarded as a C^*-algebra of continuous fields on $\mathbb{R}$ with its fibers $\{\mathfrak{A}_t\}_{t\in\mathbb{R}}$ given by $\mathfrak{A}_t = \mathbb{K}$ for $t \in \mathbb{R} \setminus \{0\}$ and $\mathfrak{A}_0 = C_0(\mathbb{R}^2)$ (cf. [10]).

Recall that a C^*-algebra $\mathfrak{A}$ is stable if $\mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{K}$.

THEOREM 1.4. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra. Suppose that $\mathfrak{A}$ has a stable quotient $\mathcal{Q}$. Then* $\mathrm{sr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{csr}(M(\mathfrak{A})) = \infty$ *and* $\mathrm{gsr}(M(\mathfrak{A})) = \infty$.

PROOF. By assumption and [31, Theorem 2.3.9] we see that $M(\mathfrak{A})$ has $M(\mathbb{K} \otimes \mathcal{Q})$ as a quotient, and we have that $\mathbb{B} \otimes M(\mathcal{Q})$ is a C^*-subalgebra of $M(\mathbb{K} \otimes \mathcal{Q})$ (cf. [1]). Since $\mathbb{B} \otimes M(\mathcal{Q})$ has two orthogonal isometries, by [18] and Proposition 1.1, we have that

$$\begin{aligned} &\mathrm{sr}(M(\mathfrak{A})) \geq \mathrm{sr}(M(\mathbb{K} \otimes \mathcal{Q})) = \infty, \\ &\mathrm{csr}(M(\mathfrak{A})) = \infty, \\ &\mathrm{gsr}(M(\mathfrak{A})) = \infty. \end{aligned}$$ □

REMARK. For a connected locally compact group G, its group C^*-algebra $C^*(G)$ has a simple subquotient which is stable or a finite-dimensional matrix algebra (cf. [9]). Thus $C^*(G)$ for G a noncommutative, connected solvable Lie group has a closed ideal such that its multiplier algebra has the stable ranks infinity. However, we have not been successful to compute the ranks of $M(C^*(G))$ in general and even in the case of the real $ax+b$ group.

THEOREM 1.5. *For any C^*-algebra $\mathfrak{A}$, we have* $\mathrm{sr}(M(\mathfrak{A} \otimes \mathbb{K})) = \infty$ *and* $\mathrm{csr}(M(\mathfrak{A} \otimes \mathbb{K})) = \infty$ *and* $\mathrm{gsr}(M(\mathfrak{A} \otimes \mathbb{K})) = \infty$.

PROOF. Note that $M(\mathfrak{A}) \otimes \mathbb{B}$ is a C^*-subalgebra of $M(\mathfrak{A} \otimes \mathbb{K})$. Then we use the same argument as in the proof of Theorem 1.4. □

REMARK. By [18, Theorems 3.6 and 6.4] we have $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \min\{2, \mathrm{sr}(\mathfrak{A})\}$ while $\mathrm{gsr}(\mathfrak{A} \otimes \mathbb{K}) \leq \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq \min\{2, \mathrm{csr}(\mathfrak{A})\}$ by [18, p. 328], [20, Theorem 3.10], [14].

EXAMPLE 1.6. Let $M_5 = \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ be the Mautner group where the action α is defined by $\alpha_t(z, w) = (e^{2\pi i t} z, e^{2\pi i \theta t} w)$ for $z, w \in \mathbb{C}$, $t \in \mathbb{R}$ and θ an irrational number (cf. [2]). Then we have the following isomorphism and quotient of the group C^*-algebra $C^*(M_5)$:

$$C^*(M_5) \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} \mathbb{R} \to C(\mathbb{T}^2) \rtimes_{\hat{\alpha}} \mathbb{R} \to 0,$$

where the quotient is the crossed product associated with the invariant subspace $\mathbb{T}^2$ of $\mathbb{C}^2$ under the action $\hat{\alpha}$ defined by the complex conjugate of α, and we further have $C(\mathbb{T}^2) \rtimes_{\hat{\alpha}} \mathbb{R} \cong \mathbb{K} \otimes (C(\mathbb{T}) \rtimes \mathbb{Z})$ (cf. [5, II.8]). By Theorem 1.4, we get that

$$\begin{aligned} \mathrm{sr}(M(C^*(M_5))) &= \infty, \\ \mathrm{csr}(M(C^*(M_5))) &= \infty, \\ \mathrm{gsr}(M(C^*(M_5))) &= \infty. \end{aligned}$$

On the other hand, $\mathrm{sr}(C^*(M_5)) = 2 = \mathrm{csr}(C^*(M_5)) \geq \mathrm{gsr}(C^*(M_5))$ (cf. [24], [18, p. 328]). Since $C^*(M_5)^+$ is finite, $\mathrm{gsr}(C^*(M_5)) = 1$ ([19, p. 247]).

EXAMPLE 1.7. Let $D_7 = \mathbb{C}^2 \rtimes_\beta H_3$ be the Dixmier group where $\beta_g(z, w) = (e^{ia}z, e^{ib}w)$ for $z, w \in \mathbb{C}$, $g = (c, b, a) \in H_3$, and H_3 is the real 3-dimensional Heisenberg group with $(c, 0, 0)$ in its center (cf. [6, 7]). Then we have the following quotient:

$$C^*(D_7) \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\beta}} H_3 \to C(\mathbb{T}^2) \rtimes_{\hat{\beta}} H_3 \to 0,$$

which is associated with the invariant subspace $\mathbb{T}^2$ of $\mathbb{C}^2$ under the action $\hat{\beta}$ defined by the complex conjugate of β, and we have $C(\mathbb{T}^2) \rtimes_{\hat{\beta}} H_3 \cong \mathbb{K} \otimes C^*((H_3)_{1_2})$ by [9, Corollary 2.10] since $\hat{\beta}$ is transitive on $\mathbb{T}^2$, where $(H_3)_{1_2}$ means the stabilizer of $1_2 = (1, 1) \in \mathbb{T}^2$ under $\hat{\beta}$. By Theorem 1.4, we get that

$$\mathrm{sr}(M(C^*(D_7))) = \infty, \quad \mathrm{csr}(M(C^*(D_7))) = \infty, \quad \mathrm{gsr}(M(C^*(D_7))) = \infty.$$

On the other hand, $\mathrm{sr}(C^*(D_7)) = 2 = \mathrm{csr}(C^*(D_7)) > \mathrm{gsr}(C^*(D_7)) = 1$ (cf. [25, 26]).

REMARK. The groups M_5 and D_7 are typical and important examples in the unitary representation theory of connected solvable Lie groups of non type I. The above method for computing the ranks of $M(C^*(G))$ for G as more general examples is applicable through the structure of $C^*(G)$.

EXAMPLE 1.8. Let X be a σ-compact, locally compact Hausdorff space. Then $\dim X = \dim \beta X$ since X is normal (cf. [13, Theorem 9-5]), and $M(C_0(X)) = C(\beta X)$. Hence

$$\mathrm{sr}(M(C_0(X))) = \dim_{\mathbb{C}} \beta X = \dim_{\mathbb{C}} X = \mathrm{sr}(C_0(X)),$$

while there exists a locally compact Hausdorff space X with $\dim X = 1$ and $\dim \beta X = 0$ (cf. [16, 4.6 Remarks, p. 234]). On the other hand, we let $X = [0,1]^n \setminus \{(0, \cdots, 0)\}$. Then by [14] and [8],

$$\begin{cases} \mathrm{csr}(C_0(X)) = \mathrm{csr}(C([0,1]^n)) = 1, \\ \mathrm{csr}(M(C_0(X))) \leq [(\dim \beta X + 1)/2] + 1 = [(n+1)/2] + 1. \end{cases}$$

For X a contractible compact space, we get $\mathrm{csr}(C(X)) = 1$. Hence $\mathrm{gsr}(C(X)) = 1$.

EXAMPLE 1.9. If $\mathfrak{A} = \mathbb{K} \oplus \mathbb{B}$, then $M(\mathfrak{A}) = \mathbb{B} \oplus \mathbb{B}$. Hence we have that

$$\begin{aligned} \mathrm{sr}(\mathfrak{A}) &= \infty = \mathrm{sr}(M(\mathfrak{A})), \\ \mathrm{csr}(\mathfrak{A}) &= \infty = \mathrm{csr}(M(\mathfrak{A})), \\ \mathrm{gsr}(\mathfrak{A}) &= \infty = \mathrm{gsr}(M(\mathfrak{A})). \end{aligned}$$

If $\mathfrak{A}$ is a σ-unital, simple C^*-algebra with real rank zero and $M(\mathfrak{A})/\mathfrak{A}$ is simple (cf. [33, Corollary 1.6], [4]), then we have $\mathrm{sr}(M(\mathfrak{A})) = \infty$ and $\mathrm{csr}(M(\mathfrak{A})) = \infty$ and $\mathrm{gsr}(M(\mathfrak{A})) = \infty$.

2. Stable rank of multiplier algebras of subhomogeneous C^*-algebras

Recall that a C^*-algebra $\mathfrak{A}$ is subhomogeneous (in a general sense) if any irreducible representation of $\mathfrak{A}$ is finite dimensional. Denote by $\hat{\mathfrak{A}}_n$ the space of all n-dimensional irreducible representations of $\mathfrak{A}$ up to unitary equivalence (cf. [7, Chapter 3]).

THEOREM 2.1. *Let $\mathfrak{A}$ be a subhomogeneous C^*-algebra and $M(\mathfrak{A})$ its multiplier algebra. Then we have that*

$$\mathrm{sr}(M(\mathfrak{A})) \leq N \equiv \sup_{1 \leq n < \infty} \mathrm{sr}(C^b(\hat{\mathfrak{A}}_n) \otimes M_n(\mathbb{C})).$$

PROOF. Following the idea of [27] we construct a C^*-subalgebra $\mathfrak{B}$ of the direct product $P_{\mathfrak{A}} \equiv \Pi_{1\leq n<\infty} \Pi_{\pi\in\hat{\mathfrak{A}}_n} C_0(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))$, consisting of all elements $(f_{n,\pi}) \in P_{\mathfrak{A}}$ such that for some $a \in \mathfrak{A}$, $f_{n,\pi}(\pi) = \pi(a)$ for all $1 \leq n < \infty$, $\pi \in \hat{\mathfrak{A}}_n$. Then $\mathfrak{A}$ is a quotient of $\mathfrak{B}$ by the identification of a with $(\pi(a))_{1\leq n<\infty, \pi\in\hat{\mathfrak{A}}_n}$.

Note that $M(\mathfrak{B})$ is a C^*-subalgebra of

$$Q_{\mathfrak{A}} \equiv \Pi_{1\leq n<\infty}\Pi_{\pi\in\hat{\mathfrak{A}}_n} C^b(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))$$

by definition of multiplier algebras (cf. [31, Definition 2.2.2]), since we have $M(C_0(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))) = C^b(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))$ (cf. [1]). Moreover we have that

$$M(C_0(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))) = M(C_0(\hat{\mathfrak{A}}_n) \otimes M_n(\mathbb{C})) \cong C^b(\hat{\mathfrak{A}}_n) \otimes M_n(\mathbb{C})$$

(cf. [31, 2.R, p. 50]). Also, there exists a homomorphism φ from $M(\mathfrak{B})$ to $M(\mathfrak{A})$ (cf. [31, Proposition 2.2.16]). Furthermore, we check that φ is onto. In fact, any element of $M(\mathfrak{A})$ is represented as $(l_{n,\pi})_{1\leq n<\infty,\pi\in\hat{\mathfrak{A}}_n}$ with $l_{n,\pi} \in M(\pi(\mathfrak{A})) = \pi(\mathfrak{A})$ since any element of $\mathfrak{A}$ is represented as $(\pi(a))_{1\leq n<\infty,\pi\in\hat{\mathfrak{A}}_n}$ on the direct sum of Hilbert spaces indexed by $1 \leq n < \infty, \pi \in \hat{\mathfrak{A}}_n$. For any $(l_{n,\pi}) \in M(\mathfrak{A})$, we have $(1_{\hat{\mathfrak{A}}_n} \otimes l_{n,\pi}) \in M(\mathfrak{B})$ where $1_{\hat{\mathfrak{A}}_n}$ is the unit of $C^b(\hat{\mathfrak{A}}_n)$. Then by [18, Theorem 4.3] we have that $\mathrm{sr}(M(\mathfrak{A})) \leq \mathrm{sr}(M(\mathfrak{B}))$.

By definition of multiplier algebras, $M(\mathfrak{B})$ contains all the elements $(g_{n,\pi}) \in Q_{\mathfrak{A}}$ such that $(g_{n,\pi}(\pi))^{\cdot} = (l_{n,\pi})$ for $(l_{n,\pi}) \in M(\mathfrak{A})$. Therefore, the same methods for the stable rank bound in [27, 28] are applicable to $M(\mathfrak{B})$. In fact, we suppose that the number N defined above is finite. Then we show that $\mathrm{sr}(M(\mathfrak{B})) \leq N$ by checking the definition of stable rank precisely, and using the perturbation along the diagonal elements $(g_{n,\pi}(\pi))$ for $\pi \in \hat{\mathfrak{A}}_n$. In fact, for any $(g^j_{n,\pi})^N_{j=1} \in M(\mathfrak{B})^N$ and $\varepsilon > 0$, we take $(h^j_{n,\pi})^N_{j=1} \in M(\mathfrak{B})^N$ such that $\|g^j_{n,\pi} - h^j_{n,\pi}\| < \varepsilon(n, \pi, j) < \varepsilon$ and $h'(n, \pi) \equiv \sum^N_{j=1}(h^j_{n,\pi})^* h^j_{n,\pi}$ is invertible in $C^b(\hat{\mathfrak{A}}_n, \pi(\mathfrak{A}))$. Note that the restriction of $g^j_{n,\pi}$ to $\hat{\mathfrak{A}}_n$ belongs to the C^*-algebra $\mathfrak{A}_n$ of continuous fields on $\hat{\mathfrak{A}}_n$ with its fibers $\pi(\mathfrak{A})$. In particular, the map $\pi \mapsto \|g^j_{n,\pi}\|$ is continuous on $\hat{\mathfrak{A}}_n$. Thus, for any $\pi \in \hat{\mathfrak{A}}_n$ there is an open neighborhood U of π, and an element $h^j_{n,\pi}|_U$ of the restriction $\mathfrak{A}_n|_U$ of $\mathfrak{A}_n$ to U such that $g^j_{n,\pi}|_U$ is approximated by $h^j_{n,\pi}|_U$ and $\sum^N_{j=1}(h^j_{n,\pi}|_U)^*(h^j_{n,\pi}|_U)$ is invertible in $\mathfrak{A}_n|_U$, which is deduced from a direct computation by using the norm continuity on fibers to show that if $\sum^N_{j=1}(h^j_{n,\pi}|_U)^*(h^j_{n,\pi}|_U)(\pi)$

is invertible, then $\sum_{j=1}^{N}(h_{n,\pi}^{j}|_U)^*(h_{n,\pi}^{j}|_U)(\rho)$ for $\rho \in U$ is invertible. Therefore, our remaining task will be to use this process inductively for a suitable open covering of $\hat{\mathfrak{A}}_n$.

Moreover, if necessary, taking $\varepsilon(n,\pi,j)$ small enough and replacing $h_{n,\pi}^{j}$ with its suitable perturbation, we can assume that $h'(n,\pi)$ is bounded away from zero. In fact, in general, for a C^*-algebra $\mathcal{A}$ we have a continuous map Φ from $L_n(\mathcal{A}) = \{(a_j) \in \mathcal{A}^n \mid \sum_{j=1}^{n} a_j^* a_j \in \mathcal{A}^{-1}\}$ to the positive part $\mathcal{A}_+$ of $\mathcal{A}$ by $(a_j) \mapsto \sum_{j=1}^{n} a_j^* a_j$. Let $\mathcal{S} = \{b \in \mathcal{A}_+ \mid \|\sum_{j=1}^{n} a_j^* a_j - b\| < \eta, \text{ and } b > \sum_{j=1}^{n} a_j^* a_j + \eta' 1\}$ for some $\eta, \eta' > 0$. Then $\mathcal{S}$ is open in $\mathcal{A}_+$ since for $b' \in \mathcal{A}_+$ with $\|b - b'\|$ small, we can make the distance of their spectrums small. Taking η, η' suitably, we make the distance between $\sum_{j=1}^{n} a_j^* a_j$ and $\mathcal{S}$ small enough. Then we can find a small open neighborhood of (a_j) such that its image under Φ has the nonzero intersection with $\mathcal{S}$. □

REMARK. We have proved implicitly in [27] by the similar way as above that for any subhomogeneous C^*-algebra $\mathfrak{A}$,

$$\mathrm{sr}(\mathfrak{A}) = N' \equiv \sup_{1 \le n < \infty} \mathrm{sr}(C_0(\hat{\mathfrak{A}}_n) \otimes M_n(\mathbb{C})).$$

However, we do not know whether $M(\mathfrak{A})$ is subhomogeneous or not in general. In fact, we have that for $\mathfrak{A} = \oplus_{n\in\mathbb{N}} M_n(\mathbb{C})$ the direct sum of $M_n(\mathbb{C})$ for $n \in \mathbb{N}$,

$$M(\mathfrak{A}) \cong \Pi_{n\in\mathbb{N}} M(M_n(\mathbb{C})) \cong \Pi_{n\in\mathbb{N}} M_n(\mathbb{C})$$

(cf. [11, Proposition 7.1.9]). Then $M(\mathfrak{A})/\mathfrak{A}$ may have an infinite dimensional irreducible representation, and $\mathrm{sr}(\mathfrak{A}) = 1 = \mathrm{sr}(M(\mathfrak{A}))$ (cf. [28]).

THEOREM 2.2. *Let $\mathfrak{A}$ be a σ-unital subhomogeneous C^*-algebra and $M(\mathfrak{A})$ its multiplier algebra. Then we have that*

$$\mathrm{sr}(\mathfrak{A}) = \sup_{1 \le n < \infty} (\{([\dim \hat{\mathfrak{A}}_n^+/2])/n\} + 1) = \mathrm{sr}(M(\mathfrak{A})),$$

where $\hat{\mathfrak{A}}_n^+$ is the one-point compactification of $\hat{\mathfrak{A}}_n$, and $\{x\}$ is the least integer $\ge x$.

PROOF. For X any σ-compact, locally compact Hausdorff space, we have $\dim X = \dim \beta X$ since X is normal (cf. [13, Theorem 9-5]). Hence $\dim \hat{\mathfrak{A}}_n = \dim \beta\hat{\mathfrak{A}}_n$, and $C^b(\hat{\mathfrak{A}}_n) \cong C(\beta\hat{\mathfrak{A}}_n)$. Then for N in Theorem 2.1 and N' in the above remark, we have

$$\begin{aligned} N' &= \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(M(\mathfrak{A})) \leq N, \\ N' &= \sup_{1\leq n<\infty} (\{([\dim \hat{\mathfrak{A}}_n^+/2])/n\} + 1) = N \end{aligned}$$

by [18, Proposition 1.7 and Theorem 6.1]. □

EXAMPLE 2.3. Let $\mathfrak{A} = C_0([0,1)) \otimes M_n(\mathbb{C})$. Then we have that

$$M(C_0([0,1)) \otimes M_n(\mathbb{C})) \cong M(C_0([0,1))) \otimes M_n(\mathbb{C}) \cong C^b([0,1)) \otimes M_n(\mathbb{C}),$$

where the first isomorphism is deduced from definition of multiplier algebras (cf. [1, Corollary 3.4]). Moreover, we have that $C^b([0,1)) \cong C(\beta[0,1))$. Then

$$\begin{cases} \mathrm{sr}(\mathfrak{A}) = 1 = \mathrm{sr}(M(\mathfrak{A})), \\ \mathrm{csr}(C_0([0,1))) = \mathrm{csr}(C([0,1])) = 1, \\ \mathrm{csr}(M(\mathfrak{A})) \leq \{(\mathrm{csr}(C(\beta[0,1))) - 1)/n\} + 1 \\ \qquad\qquad\qquad \leq \{\mathrm{sr}(C(\beta[0,1)))/n\} + 1 \leq 2 \end{cases}$$

by using [18, Proposition 1.7, Corollary 4.10 and Theorem 6.1], [19, Theorem 4.7] and [8, Corollary 2.12] (or [14]), where $\{x\}$ means the least integer $\geq x$. Therefore, since $M(\mathfrak{A})$ is finite we obtain that (cf. [19, Propositions 5.2 and 5.3])

$$\begin{aligned} &\mathrm{sr}(\mathfrak{A}) = \mathrm{sr}(M(\mathfrak{A})) = 1, \\ &\mathrm{csr}(\mathfrak{A}) = 1 \leq \mathrm{csr}(M(\mathfrak{A})) \leq 2, \\ &\mathrm{gsr}(\mathfrak{A}) = \mathrm{gsr}(M(\mathfrak{A})) = 1. \end{aligned}$$

If $\beta[0,1)$ is contractible, we have $\mathrm{csr}(\mathfrak{A}) = \mathrm{csr}(M(\mathfrak{A}))$.

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