Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 459–465

SOME FIXED POINT THEOREMS FOR CONTRACTIVE AND EXPANSIVE MAPS

GUO-JING JIANG AND SHIN MIN KANG

ABSTRACT. In this paper, fixed point theorems for contractive and expansive maps are established, some of which extend a few results of Das and Debata, Edelstein, Fisher, Leader, Shih and Yeh, and Jungck.

1. Introduction

Let f and g be continuous self maps of a compact metric space (X, d) and let N be the set of positive integers. For $x, y \in X$ and $A, B \subset X$, define

$$O(x, f) = \{f^n x \mid n \in N \cup \{0\}\},$$
$$O(x, y, f) = O(x, f) \cup O(y, f),$$
$$O(x, y, f, g) = O(x, y, f) \cup O(x, y, g),$$
$$\delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

Let $\delta(A)$ denote the diameter of A. Define

$$C_f = \{h \mid h : X \to X \text{ and } hf = fh\},$$
$$A_f = \{h \mid h : X \to X \text{ and } h \cap_{n\in N} f^n X = \cap_{n\in N} f^n X\},$$
$$H_f = \{h \mid h : X \to X \text{ and } h \cap_{n\in N} f^n X \subset \cap_{n\in N} f^n X\}.$$

Clearly C_f, A_f and H_f are semigroups under composition. Let $\mathcal{F}$ and $\mathcal{T}$ be families of self maps on X. A point x in X is called a *fixed point* of $\mathcal{F}$ if $fx = x$ for $f \in \mathcal{F}$, a *common fixed point* of $\mathcal{F}$ and $\mathcal{T}$ if $fx = gx = x$ for $f \in \mathcal{F}$ and $g \in \mathcal{T}$.

Edelstein [2] established the existence of a unique fixed point of a self map f of a compact metric space satisfying the inequality $d(fx, fy) <$

Received September 7, 1999.
2000 Mathematics Subject Classification: 54H25.
Key words and phrases: fixed point, contractive map, expansive map, compact metric space, semigroup.

$d(x, y)$. Das and Debata [1], Fisher [3], Leader [7], Shih and Yeh [8] obtained a number of generalizations of this result. Jungck [6] proved two interesting results on fixed point in compact metric spaces, one of which deals with the existence of fixed point of C_{gf} and extends the results of Das and Debata [1], Edelstein [2], Fisher [3], Leader [7], Shih and Yeh [8].

The main purpose of this paper is to extend Jungck's results to a few much wider classes of maps. In section 2, fixed point theorems are proved by considering a few contractive types conditions for H_{gf}, H_f and H_g. In section 3, fixed point theorems are established by considering a few expansive types conditions for H_{gf}, C_f and C_g.

By Proposition 4.1 of Jungck [6] and Proposition 1 of Leader [7], we obtain the following lemmas:

LEMMA 1.1. *Let f be a continuous self map of a compact metric space (X, d). Let $A = \cap_{n \in N} f^n X$. Then*

(i) *A is a nonempty compact subset of X;*
(ii) *$\{f^n | \in N \cup \{0\}\} \subset A_f \cap C_f$;*
(iii) *$C_f \cup A_f \subset H_f$.*

LEMMA 1.2. *Let f and g be self maps of a compact metric space (X, d) such that gf is continuous and $f \in A_{gf}$. Then $g \in A_{gf}$.*

LEMMA 1.3. *Let f and g be commuting self maps of a compact metric space (X, d) such that gf is continuous. Then $f, g \in A_{gf}$.*

2. Fixed point theorems for H_{gf}, H_f and H_g

THEOREM 2.1. *Let f and g be self maps of a compact metric space (X, d) such that gf is continuous and $f \in A_{gf}$. Assume that there exist $S, T \in A_{gf}$ satisfying*

$$d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g)) \tag{2.1}$$

for $Sx \neq Ty$. Then f, g, S and T have a unique common fixed point which is a unique fixed point of H_{gf}.

PROOF. Let $A = \cap_{n \in N} (gf)^n X$. It follows from (i) of Lemma 1.1 that A is a nonempty compact subset of X. Thus there exist $a, b \in A$ such that $\delta(A) = d(a, b)$. Since $S, T \in A_{gf}$, we can find $x, y \in A$ such that $Sx = a$ and $Ty = b$. By Lemma 1.2 we have $g \in A_{gf}$. Note that

$f \in A_{gf}$. Then $O(x, y, f, g) \subset A$. We assert that A is a singleton. If not, then $\delta(A) > 0$. Using (2.1),

$$\begin{aligned} d(Sx, Ty) &< \delta(\cup_{h \in H_{gf}} h\, O(x, y, f, g)) \\ &\leq \delta(\cup_{h \in H_{gf}} hA) \\ &\leq \delta(A), \end{aligned}$$

which implies that

$$0 < \delta(A) = d(Sx, Ty) < \delta(A),$$

which is impossible. Hence A is a singleton, i.e., $A = \{w\}$ for some w in X. This implies that w is a fixed point of H_{gf}, in particular, w is a common fixed point of f, g, S and T.

If v is another common fixed point of f, g, S and T, then $(gf)^n = v$ for all n in N. This implies $v \in A$ and $v = w$. Hence f, g, S and T have a unique common fixed point w. Note that f, g, S and $T \in A_{gf} \subset H_{gf}$. Therefore H_{gf} has a unique fixed point w. This completes the proof. □

COROLLARY 2.1. *Let f and g be self maps of a compact metric space (X, d) such that gf is continuous and $f \in A_{gf}$. If $fx \neq gy$ implies*

$$d(fx, gy) < \delta(\cup_{h \in H_{gf}} h\, O(x, y, f, g)),$$

then f and g have a unique common fixed point which is a unique fixed point of H_{gf}.

PROOF. Corollary 2.1 follows from Lemma 1.2 and Theorem 2.1. □

REMARK 2.1. By (iii) of Lemma 1.1 and Lemma 1.3 and Example 3.1 in section 3, it follows that Corollary 2.1 generalizes properly Theorem 4.2 of Jungck [6].

COROLLARY 2.2. *Let f be a continuous self map of a compact metric space (X, d). Assume that there exist $S, T \in A_f$ satisfying*

$$d(Sx, Ty) < \delta(\cup_{h \in H_f} h\, O(x, y, f))$$

for $Sx \neq Ty$. Then f has a uniformly contractive fixed point which is a unique fixed point of H_f.

PROOF. Take $g = i_X$ (: the identity map) in Theorem 2.1. By (ii) of Lemma 1.1, $f \in A_f$. Note that $O(x, y, f, i_X) = O(x, y, f)$. It follows from Theorem 2.1 that $\cap_{n\in N} f^n X = \{w\}$ and w is a unique fixed point of H_f. By Theorem 1 of Leader [7], we conclude that f has a uniformly contractive fixed point w. This completes the proof. □

REMARK 2.2. Corollary 4.3 of Jungck [6] is a special case of Corollary 2.2.

THEOREM 2.2. *Let f and g be self maps of a compact metric space (X, d) such that gf is continuous. Assume that there exist $S, T \in A_{gf}$ satisfying*

$$d(Sx, Ty) < \delta(\cup_{h\in H_{gf}}\{hx, hy\})$$

for $Sx \neq Ty$. Then H_{gf} has a unique fixed point.

PROOF. It follows from (ii) of Lemma 1.1 that $gf \in H_{gf}$. The remaining portion of the proof can be derived as in Theorem 2.1. □

THEOREM 2.3. *Let f and g be self maps of a compact metric space (X, d) such that gf is continuous. Assume that for every compact subset Y of X which contains more than one element and is mapped into itself by gf, there exist $S, T \in A_{gf}$ satisfying*

$$d(Sx, Ty) < \delta(Y) \tag{2.2}$$

for all x, y in Y. Then H_{gf} has a unique fixed point.

PROOF. Let $A = \cap_{n\in N}(gf)^n X$. By (i) and (ii) of Lemma 1.1, A is a nonempty compact subset of X and $gf \in A_{gf}$. Suppose that $\delta(A) > 0$. Then there exist $a, b \in A$ such that $\delta(A) = d(a, b)$. Since $SA = A = TA$, we can find $x, y \in A$ such that $Sx = a$ and $Ty = b$. By (2.2), we have

$$0 < \delta(A) = d(Sx, Ty) < \delta(A),$$

which is a contradiction. Hence $\delta(A) = 0$, i.e., A is a singleton. The remaining portion of the proof can be derived as in Theorem 2.1. This completes the proof. □

THEOREM 2.4. *Let f and g be continuous self maps of a compact metric space (X, d). Assume that there exist $S \in A_f$ and $T \in A_g$ satisfying*

$$d(Sx, Ty) < \delta(\cup_{u\in H_f} u\,O(x, f),\ \cup_{v\in H_g} v\,O(y, g)) \tag{2.3}$$

for $Sx \neq Ty$. Then f, g, S and T have a unique common fixed point which is a unique common fixed point of H_f and H_g.

PROOF. Let $A = \cap_{n\in N} f^n X$ and $B = \cap_{n\in N} g^n X$. By (i) and (ii) of Lemma 1.1, A and B are nonempty compact subsets of X and $fA = A$, $gB = B$. Thus there exists $a \in A$ and $b \in B$ such that $\delta(A, B) = d(a, b)$. Note that $SA = A$ and $TB = B$. Then there exist $x \in A$ and $y \in B$ such that $Sx = a$ and $Ty = b$. Suppose that $a \neq b$. Then by (2.3),

$$\begin{aligned} d(a, b) &= d(Sx, Ty) \\ &< \delta(\cup_{u\in H_f} u\, O(x, f),\ \cup_{v\in H_g} v\, O(y, g)) \\ &\leq \delta(\cup_{u\in H_f} u\, A,\ \cup_{v\in H_g} v\, B) \\ &\leq \delta(A, B) = d(a, b), \end{aligned}$$

which is a contradiction. Therefore $a = b$ and $\delta(A, B) = 0$. This implies $A = B =$ a singleton $= \{w\}$, say. Clearly w is a common fixed point of H_f and H_g. Since every common fixed point of f and S belongs to $A = \{w\}$ and $fw = Sw = w$, so f and S have a unique common fixed point w. Similarly w is also a unique common fixed point of g and T. Thus w is a unique common fixed point of H_f and H_g. This completes the proof. □

3. Nonunique fixed points

THEOREM 3.1. *Let f and g be continuous self maps of a compact metric space (X, d) satisfying $f \in A_{gf}$. If $fx \neq gy$ implies*

$$(3.1)\qquad \begin{aligned} d(fx, gy) > \inf\{&d(ux, fux), d(uy, fuy), d(ux, gux), \\ &d(uy, guy), d(hx, hy) \mid u \in H_{gf},\ h \in C_f \cap C_g\}, \end{aligned}$$

then at least one of f or g has a fixed point.

PROOF. Let $A = \cap_{n\in N}(gf)^n X$. By (i) of Lemma 1.1, A is a nonempty compact subset of X. It follows from Lemma 1.2 that $g \in A_{gf}$. By the continuity of f and g and compactness of A, there exist $a, b \in A$ such that

$$(3.2)\qquad d(a, fa) \leq d(x, fx) \quad \text{and} \quad d(b, gb) \leq d(x, gx)$$

for all $x \in A$. We assume without loss of generality that

$$(3.3)\qquad d(a, fa) \leq d(b, gb)$$

Note that $gA = A$. Then there exists a point $w \in A$ such that $gw = a$. Suppose that $a \neq fa$, i.e., $fa \neq gw$. By (3.1), (3.2) and (3.3) we have

$$\begin{aligned}
&d(fa, gw)\\
&> \inf\{d(ua, fua), d(uw, fuw), d(ua, gua),\\
&\qquad d(uw, guw), d(ha, hw) \mid u \in H_{gf},\ h \in C_f \cap C_g\}\\
&\geq \inf\{d(a, fa), d(b, gb), d(hgw, hw) \mid u \in H_{gf},\ h \in C_f \cap C_g\}\\
&= \inf\{d(a, fa), d(ghw, hw) \mid u \in H_{gf},\ h \in C_f \cap C_g\}\\
&= d(a, fa),
\end{aligned}$$

which implies that

$$d(a, fa) = d(fa, gw) > d(a, fa),$$

which is impossible. Hence $a = fa$. This completes the proof. □

REMARK 3.1. The following example demonstrates that Theorem 3.1 is more general than Theorem 4.4 of Jungck [6].

EXAMPLE 3.1. Let $X = \{1, 2, 5\}$ and $d(x, y) = |x - y|$. Define $f, g : X \to X$ by

$$f1 = f2 = g1 = 1 \quad \text{and} \quad f5 = g2 = g5 = 2.$$

Then f and g are self maps of a compact metric space (X, d) such that gf is continuous and $\cap_{n\in N}(gf)^n X = \{1\} = f \cap_{n\in N} (gf)^n X$. It is now a simple matter to show that

$$\begin{aligned}
0 &= \inf\{d(ux, fux), d(uy, fuy), d(ux, gux),\\
&\qquad d(uy, guy), d(hx, hy) \mid u \in H_{gf},\ h \in C_f \cap C_g\}\\
&< d(fx, gy) = 1\\
&< \delta(\cup_{h\in H_{gf}} h\, O(x, y, f, g)) = 4
\end{aligned}$$

for $fx \neq gy$. Thus the conditions of the above Corollary 2.1 and Theorem 3.1 are satisfied but Theorems 4.2 and 4.4 of Jungck [6] are not applicable since $fg5 = 1 \neq 2 = gf5$.

REMARK 3.2. Example 4.4 of Jungck [6] shows that not both f and g of the above Theorem 3.1 need have a fixed point and that the fixed point may not be unique.

The proof of the following result goes in a similar fashion as that of Theorem 3.1, so we omit the proof.

THEOREM 3.2. *Let f and g be self maps of a compact metric space (X, d) satisfying gf is continuous. Assume that there exist $S, T \in A_{gf}$ such that S and T are continuous and*

$$d(Sx, Ty) > \inf\{d(ux, Sux), d(uy, Suy), d(ux, Tux), d(uy, Tuy), d(x, y) \mid u \in H_{gf}\}$$

for $Sx \neq Ty$. Then at least one of S or T has a fixed point.

References

[1] G. Das and J. P. Debata, *Corrigendum to: A note on fixed points of commuting mappings of contractive type*, Indian J. Math. **30** (1988), 249–250.

[2] M. Edlstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

[3] B. Fisher, *On three fixed point mappings for compact metric spaces*, Indan J. Pure Appl. Math. **11** (1980), 268–270.

[4] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

[5] ______, *Periodic and fixed points and commuting mappings*, Proc. Amer. Math. Soc. **76** (1979), 333–338.

[6] ______, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc. **103** (1988), 977–983.

[7] S. Leader, *Uniformly contractive fixed point in compact metric space*, Proc. Amer. Math. Soc. **86** (1982), 153–158.

[8] M. H. Shih and C. C. Yeh, *On fixed point theorems of contractive type*, Proc. Amer. Math. Soc. **85** (1982), 465–468.

Guo-Jing Jiang
Dalian Management Cadre's College
Dalian, Liaoning 116031
People's Republic of China

Shin Min Kang
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr