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# MAXIMAL COMMUTATIVE SUBALGEBRAS OF MATRIX ALGEBRA WITH $i(m) = 3$

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ABSTRACT. Let $(R, m, k)$ be a maximal commutative $k$-subalgebra of $M_n(k)$ where the index of nilpotency $i(m)$ of $m$ is 3. If the socle of $R$ is of special case, then we can construct some isomorphic maximal commutative subalgebras.

## 1. Introduction

Throughout this paper, $(R, m, k)$ is a maximal commutative subalgebra of $M_n(k)$ where the index of nilpotency $i(m)$ of $m$ is 3. The socle of the algebra $R$ is denoted by $soc(R)$. It is known that $V = k^n$ is a faithful $R$-module. If we consider two subspaces $V_1 = (0) :_V m$ and $V_2 = (0) :_V m^2$ of $V$, then any element $r$ of $m$ can be assumed of the following form:

$$\begin{pmatrix} O_p & O & O \\ A & O_q & O \\ C & B & O_t \end{pmatrix}$$

where $dim_k(V_1) = t, \quad dim_k(V_2) = q + t$, and $p + q + t = n$.

In the Courter's algebra [1], $p = t = 2$, $q = 10$.

In this paper, we assume the socle of $R$ consists of the following matrices:

$$(*) \qquad \begin{pmatrix} O_p & O & O \\ O & O_q & O \\ C & O & O_t \end{pmatrix}.$$

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DEFINITION 1.1. Let $A \in M_{q\times p}(k)$. Then $KER(A)$ and $NS(A)$ are defined as follows:

$$KER(A) = \{u \in M_{1\times q}(k) | uA = 0\}, \quad NS(A) = \{v \in M_{p\times 1}(k) | Av = 0\}.$$

Then, it is well known that the next theorem holds.

THEOREM 1.2. *Let $(R, m, k)$ be a maximal commutative $k$-subalgebra of $M_n(k)$ with index of nilpotency $i(m)$ of $m$ is 3. If $\dim_k(m/soc(R)) = \nu$ and the socle of the algebra $R$ consists of the matrices in $(*)$, then $\cap_{i=1}^{\nu} KER(A_i) = (0)$ and $\cap_{i=1}^{\nu} NS(B_i) = (0)$, where $R = k[r_1, r_2, \ldots, r_\nu] \oplus soc(R)$ and for $i = 1, 2, \ldots, \nu$,*

$$r_i = \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ 0 & B_i & O_t \end{pmatrix}.$$

PROOF. Let $Z = (z_1, z_2, \ldots, z_q) \in \cap_{i=1}^{\nu} KER(A_i)$ and let

$$B = \begin{pmatrix} Z \\ O_{(t-1)\times q} \end{pmatrix}.$$

Then, the following matrix $A$

$$A = \begin{pmatrix} O_p & O & O \\ O & O_q & O \\ O & B & O_t \end{pmatrix}$$

should belong to $soc(R)$. But, the assumption for $soc(R)$ implies that $Z = (0, 0, \cdots, 0)$ and so $\cap_{i=1}^{\nu} KER(A_i) = (0)$. Similarly, we can have $\cap_{i=1}^{\nu} NS(B_i) = (0)$. □

## 2. Main results

Let $(R, m, k)$ be a maximal commutative $k$-subalgebra of $M_n(k)$ with index of nilpotency $i(m)$ of $m$ is 3. Assume $\dim_k(m/soc(R)) = \nu$, then any element in $m$ which is not in $soc(R)$ is spanned by the following form of matrices:

$$\lambda_i = \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ O & B_i & O_t \end{pmatrix}$$

for $i = 1, 2, \ldots, \nu$, where $A_i \in M_{q\times p}(k), B_i \in M_{t\times q}(k)$. Thus, the algebra $R$ is $k[\lambda_1, \lambda_2, \ldots, \lambda_\nu] \oplus soc(R)$. For the brief notation, we will let $R \in \Gamma$ if $R = k[\lambda_1, \lambda_2, \ldots, \lambda_\nu] \oplus soc(R)$.

Let $X \in GL_q(k)$ and let for $i = 1, 2, \ldots, \nu$,

$$\delta_i = \begin{pmatrix} O_p & O & O \\ X^{-1}A_i & O_q & O \\ O & B_iX & O_t \end{pmatrix}.$$

If we define an algebra $S$ by $S = k[\delta_1, \delta_2, \ldots, \delta_\nu] \oplus soc(R)$, then the following theorem implies $S$ is a maximal commutative subalgebra of $M_n(k)$ which is isomorphic to the algebra $R$.

THEOREM 2.1 *Let $R \in \Gamma$ be a maximal commutative subalgebra of $M_n(k)$ and let $S = k[\delta_1, \delta_2, \ldots, \delta_\nu] \oplus soc(R)$ be an algebra defined as above. Then, $S$ is a maximal commutative subalgebra of $M_n(k)$.*

PROOF. Obviously, the algebra $S$ is a commutative algebra. Now, let $L \in M_n(k)$ be a matrix in the centralizer of $S$. Then, $LD = DL$ for all $D \in S$. Let $L$ and $D$ be defined as following block matrices:

$$L = \begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{pmatrix}, \quad D = \begin{pmatrix} O_p & O & O \\ X^{-1}A_i & O_q & O \\ C & B_iX & O_t \end{pmatrix},$$

where $L_1 \in M_p(k), L_5 \in M_q(k), L_9 \in M_t$.

Then, from the relation, $LD = DL$, the following equations hold:

(1) $L_2X^{-1}A_i + L_3C = O_p$
(2) $L_3B_iX = O_{p\times q}$
(3) $L_5X^{-1}A_i + L_6C = X^{-1}A_iL_1$
(4) $L_6B_iX = X^{-1}A_iL_2$
(5) $X^{-1}A_iL_3 = O_{q\times t}$
(6) $L_8X^{-1}A_i + L_9C = CL_1 + B_iXL_4$
(7) $L_9B_iX = CL_2 + B_iXL_5$
(8) $CL_3 + B_iXL_6 = O_t$

From the equation (1), by letting $C = O_{t\times p}$, we have $L_2X^{-1}A_i = O_p$ for all $i$. Thus,

$$L_2X^{-1} \in \cap_{i=1}^{\nu} KER(A_i).$$

Since $\cap_{i=1}^{\nu} KER(A_i) = (0)$, $L_2X^{-1} = O_{p\times q}$ and hence $L_2 = O_{p\times q}$. Again, from the equation (1), $L_3C = O_p$ and by letting $C = E_{ij}$, where $E_{ij}$ is the $(i,j)$-th matrix unit in $soc(R)$ for $1 \le i \le t, 1 \le j \le p$, we have $L_3 = O_{p\times t}$. Thus, $B_iXL_6 = O_t$ for all $i$ in the equation (8) and so

$$XL_6 \in \cap_{i=1}^{\nu} NS(B_i).$$

Since $\cap_{i=1}^{\nu} NS(B_i) = (0)$, $XL_6 = O_{q\times t}$ and so $L_6 = O_{q\times t}$.

Now, by letting $C = E_{ij}$ for $1 \leq i \leq t, 1 \leq j \leq p$, we can have the following:

$$L_1 = aI_p,\ L_9 = aI_t$$

for some $a \in k$.

Finally, in the equation (3),

$$L_5X^{-1}A_i = X^{-1}A_iL_1 = X^{-1}A_i(aI_p) = aX^{-1}A_i.$$

This implies, for all $i$,

$$(L_5X^{-1} - aX^{-1})A_i = O_{q\times p}$$

and hence

$$L_5X^{-1} - aX^{-1} \in \cap_{i=1}^{\nu} KER(A_i) = (0).$$

Thus,

$$L_5 = aI_q.$$

Therefore, the matrix $L$ is of the form

$$L = \begin{pmatrix} aI_p & O & O \\ L_4 & aI_q & O \\ L_7 & L_8 & aI_t \end{pmatrix}.$$

Note that, by letting $C = O_{t\times p}$ in the equation (6),

$$L_8X^{-1}A_i = B_iXL_4$$

which implies

$$\begin{pmatrix} O_p & O & O \\ XL_4 & O_q & O \\ O & L_8X^{-1} & O_t \end{pmatrix} \in R.$$

Thus,

$$\begin{pmatrix} O_p & O & O \\ L_4 & O_q & O \\ O & L_8 & O_t \end{pmatrix} = \begin{pmatrix} O_p & O & O \\ X^{-1}(XL_4) & O_q & O \\ O & (L_8X^{-1})X & O_t \end{pmatrix} \in S.$$

Now, we conclude that

$$L = \begin{pmatrix} aI_p & O & O \\ L_4 & aI_q & O \\ L_7 & L_8 & aI_t \end{pmatrix} \in S$$

and so the algebra $S$ is a maximal commutative subalgebra of $M_n(k)$. □

Furthermore, it can be proved that the two algebras $R$ and $S$ are isomorphic algebras.

THEOREM 2.2. *The algebra $S$ is isomorphic to $R$ in Theorem 2.1.*

PROOF. Define a map $\phi : R \longrightarrow S$ as follows:

$$\phi(\lambda_i) = \delta_i, i = 1, 2, \ldots, \nu, \ \phi(I_n) = I_n, \ \phi(r) = r, r \in soc(R).$$

Then, we can easily show that the map $\phi$ is an isomorphism. □

For a maximal commutative subalgebra $R \in \Gamma$ mentioned before, let

$$\gamma_i = \begin{pmatrix} O_p & O & O \\ A_i X & O_q & O \\ O & B_i & O_t \end{pmatrix}$$

for $i = 1, 2, \ldots, \nu$, where $X \in GL_p(k)$. If we define an algebra $T$ by $T = k[\gamma_1, \gamma_2, \ldots, \gamma_\nu] \oplus soc(R)$, then by a similar proof of Theorem 2.1, it is easily proved that $T$ is a maximal commutative subalgebra of $M_n(k)$ which is isomorphic to the algebra $R$.

THEOREM 2.3. *Let $R \in \Gamma$ be a maximal commutative subalgebra of $M_n(k)$ and let $T = k[\gamma_1, \gamma_2, \ldots, \gamma_\nu] \oplus soc(R)$ be an algebra defined as above. Then, $T$ is a maximal commutative subalgebra of $M_n(k)$.*

THEOREM 2.4. *Let $R \in \Gamma$ be a maximal commutative subalgebra of $M_n(k)$ and let $T = k[\gamma_1, \gamma_2, \ldots, \gamma_\nu] \oplus soc(R)$ be an algebra defined as above. Then, $T$ is isomorphic to $R$.*

PROOF. Define a map $\sigma : R \longrightarrow T$ by

$$\sigma \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ C & B_i & O_t \end{pmatrix} = \begin{pmatrix} O_p & O & O \\ A_i X & O_q & O \\ CX & B_i & O_t \end{pmatrix}$$

for $i = 1, 2, \ldots, \nu$ and $\sigma(I_n) = I_n$. Then, obviously $\sigma$ is a homomorphism. Let

$$\begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ C & B_i & O_t \end{pmatrix} \in ker(\sigma).$$

Then,

$$A_i X = O_{q \times p}, CX = O_{t \times p}, B_i = O_{t \times q}$$

for all $i = 1, 2, \ldots, \nu$. Since $X$ is invertible,

$$A_i = O_{q \times p}, C = O_{t \times p}, B_i = O_{t \times q}$$

which implies the algebra homomorphism $\sigma$ is a monomorphism. Note that

$$\sigma \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ CX^{-1} & B_i & O_t \end{pmatrix} = \begin{pmatrix} O_p & O & O \\ A_i X & O_q & O \\ C & B_i & O_t \end{pmatrix}$$

and so the algebra homomorphism $\sigma$ is an isomorphism. $\square$

For a maximal commutative subalgebra $R \in \Gamma$ mentioned before, let

$$\eta_i = \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ O & XB_i & O_t \end{pmatrix}$$

for $i = 1, 2, \ldots, \nu$, where $X \in GL_t(k)$. If we define an algebra $T$ by $T = k[\eta_1, \eta_2, \ldots, \eta_\nu] \oplus soc(R)$, then by a similar proof of Theorem 2.1, it is easily proved that $T$ is isomorphic to the algebra $R$.

THEOREM 2.5. *Let $R \in \Gamma$ be a maximal commutative subalgebra of $M_n(k)$ and let $T = k[\eta_1, \eta_2, \ldots, \eta_\nu] \oplus soc(R)$ be an algebra defined as above. Then, $T$ is a maximal commutative subalgebra of $M_n(k)$.*

THEOREM 2.6. *Let $R \in \Gamma$ be a maximal commutative subalgebra of $M_n(k)$ and let $T = k[\eta_1, \eta_2, \ldots, \eta_\nu] \oplus soc(R)$ be an algebra defined as above. Then, $T$ is isomorphic to $R$.*

PROOF. Define a map $\psi : R \longrightarrow T$ by

$$\psi \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ C & B_i & O_t \end{pmatrix} = \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ XC & XB_i & O_t \end{pmatrix}$$

for $i = 1, 2, \ldots, \nu$ and $\psi(I_n) = I_n$. Then, obviously $\psi$ is a homomorphism. Let

$$\begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ C & B_i & O_t \end{pmatrix} \in ker(\psi).$$

Then,

$$A_i = O_{q\times p}, \ \ XC = O_{t\times p}, \ \ XB_i = O_{t\times q}$$

for all $i = 1, 2, \ldots, \nu$. Since $X$ is invertible,

$$A_i = O_{q\times p}, \ \ C = O_{t\times p}, \ \ B_i = O_{t\times q}$$

which implies the algebra homomorphism $\psi$ is a monomorphism. Note that

$$\psi \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ X^{-1}C & B_i & O_t \end{pmatrix} = \begin{pmatrix} O_p & O & O \\ A_i & O_q & O \\ C & XB_i & O_t \end{pmatrix}$$

and so the algebra homomorphism $\psi$ is an isomorphism. $\square$

With these results, we can simplify the matrices $A_i$ and $B_i$ in considering the form of the matrices in the maximal commutative subalgebra of matrix algebra of index of nilpotency 3.

## References

[1] W. C. Brown and F. W. Call, *Maximal Commutative Subalgebras of* $n \times n$ *Matrices*, Communications in Algebra **59** (1993), no. 12, 4439–4460.
[2] W. C. Brown, *Two Constructions of Maximal Commutative Subalgebras of* $n \times n$ *Matrices*, Communications in Algebra **22** (1994), no. 10, 4051–4066.
[3] ———, *Constructing Maximal Commutative Subalgebras of Matrix Rings in Small Dimensions*, Communications in Algebra **25** (1997), no. 12, 3923–3946.
[4] R. C. Courter, *The Dimension of Maximal Commutative Subalgebras of* $K_n$, Duke Mathematical Journal **32** (1965), 225–232.
[5] M. Gerstenhaber, *On Dominance and Varieties of Commuting Matrices*, Annals of Mathematics **73** (1961), no. 2, 324–348.
[6] N. Jacobson, *Schur's Theorem on Commutative Matrices*, Bulletin of the American Mathematical Society **50** (1944), 431–436.
[7] D. A. Suprunenko and R. I. Tyshkevich, *Commutative Matrices*, Academic Press, 1968.
[8] Youngkwon Song, *On the Maximal Commutative Subalgebras of* 14 *by* 14 *Matrices*, Communications in Algebra **25** (1997), no. 12, 3823–3840.
[9] ———, *Maximal Commutative Subalgebras of Matrix Algebras*, Communications in Algebra **27** (1999), no. 4, 1649–1663.
[10] ———, *Notes on the Constructions of Maximal Commutative Subalgebra of* $M_n(k)$, Communications in Algebra **29** (2001), no. 10, 4333–4339.

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