Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 431–437

THE DOUBLE-COMPLETE PARTITIONS OF INTEGERS

HoKyu Lee and SeungKyung Park

Abstract. Representing a positive integer in terms of a sum of smaller numbers with certain conditions has been studied since MacMahon [5] pioneered perfect partitions. The complete partitions is in this category and studied by the second author [6]. In this paper, we study complete partitions with more specified completeness, which we call the double-complete partitions.

1. Introduction

Many mathematicians studied the unique representations of positive integers by some sequences with given properties. For example, Zeckendorf found that every integer can be uniquely represented as a sum of inconsecutive terms of Fibonacci sequences. MacMahon [5] studied perfect partitions of n which are partitions of n such that every integer m with $1 \leq m \leq n$ is uniquely represented in one and only one way. We [3, 4] generalized Zeckendorf theorem and MacMahon's results on perfect partitions. In 1960, Hoggatt [2] considered sequences such that every positive integer can be represented as a sum of some terms of the sequences and Brown [1] studied such sequences and named complete, which are defined as sequences $(s_1, s_2, \cdots)$ such that every integer can be represented as $\sum_{i=1}^{\infty} \alpha_i s_i$, where $\alpha_i \in S = \{0, 1\}$. A partition which is complete was studied in [6]. This was also generalized [7] by replacing the set $S = \{0, 1\}$ by the set $S = \{0, 1, \ldots, r\}$. In this paper, we study complete partitions with at least two different representations.

Received May 4, 2001.
2000 Mathematics Subject Classification: 05A17.
Key words and phrases: complete partition, double complete partition.
The first author was supported by Brain Korea 21 Project and the second author was supported in part BK21 and in part by Yonsei Research Fund.

2. The double-complete partitions of integers

We begin with the definition of partitions of positive integers.

DEFINITION 2.1. A *partition* $\lambda = (\lambda_1 \cdots \lambda_l)$ of a positive integer n is a finite non-decreasing sequence with $\sum_{i=1}^{l} \lambda_i = n$ and $\lambda_i > 0$ for all i. The λ_i are called the *parts* of the partition and the number l is called the *length* of the partition.

We also write partitions in the form $(\lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_t^{m_t})$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_t$, $m_i \geq 1$, and $l = m_1 + m_2 + \cdots + m_t$, which means there are exactly m_i parts equal to λ_i in the partition λ. The positive number m_i is called the *multiplicity* of λ_i.

DEFINITION 2.2. A *double-complete partition* of an integer n is a partition $\lambda = (\lambda_1^{m_1} \cdots \lambda_l^{m_l})$ of n such that each integer m with $2 \leq m \leq n-2$ can be represented at least two different ways as a sum $\sum_{i=1}^{l} \alpha_i \lambda_i$ with $\alpha_i \in \{0, 1, \ldots, m_i\}$.

EXAMPLE 2.3. A partition $(1^3\ 2^2)$ of 7 is a double-complete partitions of 7 because each of the numbers 2, 3, 4, and 5 can be represented as

$$\begin{aligned} 2 &= 2 \cdot 1 + 0 \cdot 2 = 0 \cdot 1 + 1 \cdot 2, \\ 3 &= 3 \cdot 1 + 0 \cdot 2 = 1 \cdot 1 + 1 \cdot 2, \\ 4 &= 2 \cdot 1 + 1 \cdot 2 = 0 \cdot 1 + 2 \cdot 2, \\ 5 &= 3 \cdot 1 + 1 \cdot 2 = 1 \cdot 1 + 2 \cdot 2. \end{aligned}$$

Note that the double-complete partitions of any positive integer n must have 1, 1, and 2 as its parts to be able to represent integer 2 in two ways. So $n \geq 1+1+2 = 4$ and therefore double-complete partitions are considered for only $n \geq 4$. For $n \geq 5$, the double-complete partitions of n have to represent the number 3 at least twice. Thus, it must have 1, 1, 1 and 2, or 1, 1, 2 and 3 as parts. If $\lambda = (\lambda_1^{m_1}\ \lambda_2^{m_2} \cdots \lambda_l^{m_l})$ is a double-complete partition then $\lambda_1 = 1$ and $\lambda_2 = 2$.

THEOREM 2.4. *A partition $\lambda = (\lambda_1^{m_1}\ \lambda_2^{m_2} \cdots \lambda_l^{m_l})$ of a positive integer $n \geq 5$ is a double-complete partition if and only if $\lambda_{i+1} \leq \sum_{j=1}^{i} m_j \lambda_j - 1$ for $i \geq 2$ and λ should have at least three 1's and one 2, or two 1's, one 2 and one 3 as parts.*

PROOF. ($\Rightarrow$) Suppose $\lambda_{i+1} \geq \sum_{j=1}^{i} m_j \lambda_j$ for some $i \geq 2$. Then $\sum_{j=1}^{i} m_j \lambda_j - 1$ can be represented only once, which is a contradiction. If it does not have $1, 1, 1$ and 2 as parts or $1, 1, 2$ and 3, then the number 3 cannot be represented twice.

($\Leftarrow$) We use induction on l. First, we show that the partition $(\lambda_1^{m_1}\ \lambda_2^{m_2}\ \lambda_3^{m_3}) = (1^{m_1}\ 2^{m_2}\ \lambda_3^{m_3})$ with $m_1 \geq 2$, $m_2 \geq 1$, and $\lambda_3 \leq m_1 + 2m_2 - 1$ is a double-complete partition of the integer $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$. It is obvious that $2 \leq k \leq \lambda_3 - 1 \leq k_1 \cdot 1 + m_2 \cdot 2 - 2$ can be represented at least twice using the parts 1 and 2. We can represent integers less than or equal to $m_1 \cdot 1 + m_2 \cdot 2$ by using the parts 1 and 2. Also, $1 \cdot \lambda_3$ and $1 \cdot 1 + 1 \cdot \lambda_3$ are representations of λ_3 and $\lambda_3 + 1$, respectively. Therefore λ_3 and $\lambda_3 + 1$ have two representations. If $\lambda_3 + 1 < k \leq m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 - 2$ then there exist q and r such that $k = r + q \cdot \lambda_3$ with $1 \leq q$ and $0 \leq r < \lambda_3$. If $2 \leq r < \lambda_3$ then r can be represented at least twice, so k does. If r is 0 or 1 then $k = q \cdot \lambda_3 = \lambda_3 + (q-1) \cdot \lambda_3$ or $k = 1 + q \cdot \lambda_3 = 1 + \lambda_3 + (q-1) \cdot \lambda_3$. Since we have two representations of λ_3 and $\lambda_3 + 1$, so k does. Thus $(\lambda_1^{m_1}\ \lambda_2^{m_2}\ \lambda_3^{m_3})$ is a double-complete partition.
Suppose $(\lambda_1^{m_1}\ \lambda_2^{m_2}\ \cdots\ \lambda_i^{m_i})$ be a double-complete partition of n. It is enough to show that $(\lambda_1^{m_1}\ \cdots\ \lambda_i^{m_i}\ \lambda_{i+1}^{m_{i+1}})$ with $\lambda_{i+1} \leq \sum_{j=1}^{i} m_j\lambda_j - 1$ is a double-complete partition of $n + m_{i+1} \cdot \lambda_{i+1}$. Since $(\lambda_1^{m_1}\ \lambda_2^{m_2}\ \cdots\ \lambda_i^{m_i})$ is a double-complete partition, every integer k with $2 \leq k \leq \sum_{j=1}^{i} m_j\lambda_j - 2$ is represented at least twice by using the parts $\lambda_1, \ldots, \lambda_i$. Let $\sum_{j=1}^{i} m_j\lambda_j - 1 + (s-1)\lambda_{i+1} \leq k \leq \sum_{j=1}^{i} m_j\lambda_j - 2 + s\lambda_{i+1}$ for some $s = 1, \ldots, m_{i+1}$. Then $0 \leq \sum_{j=1}^{i} m_j\lambda_j - 1 - \lambda_{i+1} \leq k - s\lambda_{i+1} \leq \sum_{j=1}^{i} m_j\lambda_j - 2$.
If $k - s\lambda_{i+1} \geq 2$, $k - s\lambda_{i+1}$ can be represented at least twice by using the parts $\lambda_1, \ldots, \lambda_i$. So k is represented at least twice. If $k - s\lambda_{i+1} \leq 1$, then $\sum_{j=1}^{i} m_j\lambda_j - 1 - \lambda_{i+1}$ is 0 or 1. So, $\lambda_{i+1} = \sum_{j=1}^{i} m_j\lambda_j - 1$ or $\lambda_{i+1} = \sum_{j=1}^{i} m_j\lambda_j - 2$. Therefore, k has at least two representations ; $s\lambda_{i+1}$, $(m_1 - 1)\lambda_1 + \sum_{j=2}^{i} m_j\lambda_j + (s-1)\lambda_{i+1}$, and $\lambda_1 + s\lambda_{i+1}$, $(m_1 - 1)\lambda_1 + \sum_{j=2}^{i} m_j\lambda_j + (s-1)\lambda_{i+1}$, respectively. □

Using the previous theorem, one can find an upper bound of each part.

COROLLARY 2.5. *Let $\lambda = (\lambda_1\ \lambda_2 \cdots\ \lambda_l)$ be a double-complete partition of a positive integer n. Then $\lambda_i \leq 3 \cdot 2^{i-4}$ for each $i = 4, \ldots, l$.*

PROOF. We know that $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 \leq 2$. By Theorem 2.4, $\lambda_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} - 1 \leq 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{i-2} - 2 \leq \cdots \leq 2^{i-4}\lambda_1 + 2^{i-4}\lambda_2 + 2^{i-4}\lambda_3 - 2^{i-4} \leq 3 \cdot 2^{i-4}$. □

We now investigate the length and the largest part of double-complete partitions. We use $\lceil x \rceil$ and $\lfloor x \rfloor$ respectively for the least integer greater than equal to x and the greatest integer less than or equal to x .

PROPOSITION 2.6. *Let $\lambda = (\lambda_1\ \lambda_2 \cdots\ \lambda_l)$ be a double-complete partition of a positive integer $n \geq 5$. Then $l \geq 3 + \lceil \log_2 \frac{n-1}{3} \rceil$.*

PROOF. Let $\lambda = (\lambda_1\ \lambda_2 \cdots\ \lambda_l)$ be a double-complete partition of a positive integer n. Since $n \leq 4 + \sum_{i=4}^{l} \lambda_i \leq 4 + 3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{l-4} = 1 + 3 \cdot 2^{l-3}$, $n - 1 \leq 3 \cdot 2^{l-3}$ and it gives the minimum possible length $3 + \lceil \log_2 \frac{n-1}{3} \rceil$. In fact, $\lambda = (\lambda_1\ \lambda_2 \cdots\ \lambda_l)$ with $\lambda_i = 3 \cdot 2^{i-4}$ $(i = 4, \ldots, l)$ is a double-complete partition with such length. □

PROPOSITION 2.7. *The largest part of a double-complete partition of n is less than or equal to $\lfloor \frac{n-1}{2} \rfloor$ for $n \geq 5$.*

PROOF. Let $\lambda = (\lambda_1\ \lambda_2 \cdots \lambda_l)$ be a double-complete partition of n. Then by Theorem 2.4, λ_l is less than or equal to $\sum_{i=1}^{l-1} \lambda_i - 1 = n - \lambda_l - 1$. Thus, $2\lambda_l \leq n - 1$ and we obtain the result. □

Now we count double-complete partitions by the largest part greater than 4. We consider double-complete partitions $\lambda = (\lambda_1\ \lambda_2 \cdots\ \lambda_l)$ of a positive integer n with k as its largest part. Then there are two cases : one for partitions with one k and the other for the rest of them. If $n \leq 2k$, then $k = \lambda_l \geq n - k = n - \lambda_l > \sum_{i=1}^{l-1} \lambda_i - 1$. In this case, there do not exist double-complete partitions of n. If $2k + 1 \leq n \leq 3k$ and if a double-complete partition λ has at least two k's as parts, then $\sum_{i=1}^{l-2} \lambda_i - 1 = n - 2k - 1 < k = \lambda_{l-1}$. It cannot be a double-complete partition by Theorem 2.4. So double-complete partitions should have only one k and it corresponds to the partition $(\lambda_1\ \lambda_2 \cdots \lambda_l - 1)$ of $n - 1$ with the largest part $k - 1$. If $n \geq 3k + 1$, then double-complete partition λ may have many k's and it corresponds to the partition $(\lambda_1\ \lambda_2 \cdots\ \lambda_{l-1})$ of $n - k$ with the largest part k. We summarize this.

THEOREM 2.8. *Let $D_k(n)$ be the number of double-complete partitions of a positive integer n with largest part k. Then for $k \geq 5$,*

$$D_k(n) = \begin{cases} 0 & \text{if } n \leq 2k \\ D_{k-1}(n-1) & \text{if } 2k+1 \leq n \leq 3k \\ D_{k-1}(n-1) + D_k(n-k) & \text{if } n \geq 3k+1. \end{cases}$$

Now we find the generating function of the numbers $D_k(n)$.

THEOREM 2.9. *Let* $d_k(q) = \sum_{n=0}^{\infty} D_k(n)q^n$. *Then we have*

$$
\begin{aligned}
d_2(q) &= \frac{q^5}{(1-q^2)(1-q)} + q^4, \\
d_3(q) &= \frac{q^7}{(1-q^3)(1-q^2)(1-q)}, \\
d_4(q) &= \frac{q^9 + q^{11} - q^{12}}{(1-q^4)(1-q^3)(1-q^2)(1-q)}, \\
d_k(q) &= \frac{q^{k+5} + q^{k+7} - q^{k+8}}{(1-q^k)(1-q^{k-1})\cdots(1-q^2)(1-q)} \\
&\quad - \left[\frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} + \frac{D_{k-2}(2k-3)q^{2k-1}}{(1-q^k)(1-q^{k-1})} + \cdots \right. \\
&\quad \left. + \frac{D_4(9)q^{k+5}}{(1-q^k)(1-q^{k-1})\cdots(1-q^5)}\right] \quad \text{for } k \geq 5.
\end{aligned}
$$

PROOF. One can easily find the generating function $d_2(q)$ for the double-complete partitions since any partition having only 1's and 2's as parts is a double-complete partition for $n \geq 5$, and the only double-complete partition of the number 4 is (1 1 2). Similarly, we can find $d_k(q)$ for $k = 3$. If $k = 4$, then the double complete partitions may have 3's as their parts or may not. So the double complete partitions have the form $(1^{m_1}\ 2^{m_2}\ 4^{m_4})$ with $m_1 \geq 3$, $m_2 \geq 1$, and $m_4 \geq 1$ or $(1^{m_1}\ 2^{m_2}\ 3^{m_3}\ 4^{m_4})$ with $m_1 \geq 2$ and $m_i \geq 1$ for $i = 2, 3, 4$. By adding the corresponding generating functions of each form, we obtain the generating function $d_4(q)$. Now, we find the generating function $d_k(q)$ for $k \geq 5$. Let $d_k(q) = \sum_{n=0}^{\infty} D_k(n)q^n$. Then

$$
\begin{aligned}
d_k(q) &= \sum_{n=0}^{\infty} D_k(n)q^n = \sum_{n=2k+1}^{\infty} D_k(n)q^n \\
&= \sum_{n=2k+1}^{3k} D_{k-1}(n-1)q^n + \sum_{n=3k+1}^{\infty} \left[D_{k-1}(n-1) + D_k(n-k)\right] q^n \\
&= \sum_{n=2k+1}^{\infty} D_{k-1}(n-1)q^n + \sum_{n=3k+1}^{\infty} D_k(n-k)q^n
\end{aligned}
$$

$$
\begin{aligned}
&= q\sum_{n=2k+1}^{\infty} D_{k-1}(n-1)q^{n-1} + q^k \sum_{n=3k+1}^{\infty} D_k(n-k)q^{n-k} \\
&= q[d_{k-1}(q) - D_{k-1}(2k-1)q^{2k-1}] + q^k d_k(q).
\end{aligned}
$$

Thus we have

$$
d_k(q) = \frac{q}{1-q^k} d_{k-1}(q) - \frac{D_{k-1}(2k-1)q^{2k}}{1-q^k}.
$$

By continuing iteration,

$$
\begin{aligned}
d_k(q) &= \frac{q}{1-q^k} d_{k-1}(q) - \frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} \\
&= \frac{q}{1-q^k}\left[\frac{q}{1-q^{k-1}} d_{k-2}(q) - \frac{D_{k-2}(2k-3)q^{2k-4}}{1-q^{k-1}}\right] \\
&\quad - \frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} \\
&= \frac{q^2}{(1-q^k)(1-q^{k-1})} d_{k-2}(q) \\
&\quad - \left[\frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} + \frac{D_{k-2}(2k-3)q^{2k-1}}{(1-q^k)(1-q^{k-1})}\right] \\
&= \cdots \\
&= \frac{q^{k-4}}{(1-q^k)(1-q^{k-1})\cdots(1-q^5)} d_4(q) \\
&\quad - \left[\frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} + \frac{D_{k-2}(2k-3)q^{2k-1}}{(1-q^k)(1-q^{k-1})} + \cdots \right. \\
&\quad \left. + \frac{D_4(9)q^{k+5}}{(1-q^k)(1-q^{k-1})\cdots(1-q^5)}\right] \\
&= \frac{q^{k+5}+q^{k+7}-q^{k+8}}{(1-q^k)(1-q^{k-1})\cdots(1-q^2)(1-q)} \\
&\quad - \left[\frac{D_{k-1}(2k-1)q^{2k}}{1-q^k} + \frac{D_{k-2}(2k-3)q^{2k-1}}{(1-q^k)(1-q^{k-1})} + \cdots \right. \\
&\quad \left. + \frac{D_4(9)q^{k+5}}{(1-q^k)(1-q^{k-1})\cdots(1-q^5)}\right].
\end{aligned}
$$

□

References

[1] J. L. Brown, *Note on complete sequences of integers*, Amer. Math. Monthly **68** (1961), 557–560.
[2] V. E. Hoggatt and C. King, *Problem E* 1424, Amer. Math. Monthly **67** (1960), p. 593.
[3] H. Lee and S. Park, *The r-perfect Partitions*, preprint.
[4] ———, *The generalizations of Zeckendorf Theorem*, preprint.
[5] P. A. MacMahon, *Combinatory Analysis*, Vol. I and II, Cambridge University Press, 1915, 1916 (reprinted, Chelsea, 1960).
[6] S. Park, *Complete Partitions*, Fibonacci Quart. **36** (1998), 354–360.
[7] ———, *The r-complete Partitions*, Discrete Math. **183** (1998), 293–297.

HoKyu Lee
Department of Mathematics
Ewha Womans University
Seoul 120-750, Korea
E-mail: hokyu@dreamwiz.com

SeungKyung Park
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: sparky@yonsei.ac.kr