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A NOTE ON q-DIFFERENCE OPERATORS

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ABSTRACT. The object of this paper is to present a q-analogue of the Newton's forward-difference interpolation formula. We relate the q-beta function and the q-gamma function by the q-difference operators.

1. Introduction

K. Conrad (see [2]) used the q-difference operators

$$\triangle_q^n = \begin{cases} I, & n = 0 \\ (E - q^{n-1}) \cdots (E - q)(E - I), & n \geq 1, \end{cases} \tag{1.1}$$

where $(Ef)(x) = f(x+1)$ is the shift operator. He showed how $\triangle_q^n$ was arisen very naturally from the classical finite difference operators. Using this operator, he also examined the q-analogue of the Mahler expansion for continuous functions in p-adic analysis.

The purpose of this paper is to find a new q-analogue of the Newton's forward-difference interpolation formula and the relation between the q-beta function and the q-gamma function by the q-difference operators.

In order to describe our results we introduce some notation. For a complex number q other than 1, set

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \in \mathbb{Z}[q]. \tag{1.2}$$

As $q \to 1$, $[n] \to n$, and this is the hallmark of a q-analogue: the limit as $q \to 1$ recovers the classical object. The expressions $[n]$ is called

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q-numbers. The q-factorial of n is defined by $[0]! = 1$ and

$$[n]! = [1][2]\cdots[n] = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n} \tag{1.3}$$

when $n > 0$. The q-binomial coefficient for nonnegative integers m and n with $m \geq n$ is defined by

$$\binom{m}{n}_q = \frac{[m]!}{[n]![m-n]!} \in \mathbb{Z}[q] \quad \text{(see [2], [5])}. \tag{1.4}$$

In particular, $\binom{m}{n}_1 = \binom{m}{n}$, where $\binom{m}{n}$ is the ordinary binomial coefficient.

2. Some properties of q-difference operators

Now we consider a slightly different finite q-difference operator and q-binomial formula.

If the constant difference between successive values of x is h, so that the general value of x is $x_k = x_0 + kh, k \in \mathbb{Z}$, and the corresponding functional value is $f(x_k) = f(x_0 + kh) = f_k$.

Let E_h be the translating operator defined by (cf. [5])

$$E_h(f_k) = f(x_k + h) = f(x_{k+1}). \tag{2.1}$$

Applying E_h again increases the argument of f by h, i.e., $E_h^2 f(x_k) = E_h(E_h f(x_k)) = f(x_k + 2h) = f(x_{k+2}) = f_{k+2}$, and generally $E_h^r f(x_k) = f_{k+r}$ for $r \in \mathbb{N}$.

The q-difference of f is defined by the formula (cf. [2])

$$\triangle_{q,h}^n := \begin{cases} I, & n = 0 \\ (E_h - q^{n-1})\cdots(E_h - q)(E_h - I), & n \geq 1. \end{cases} \tag{2.2}$$

Thus we have

$$\triangle_{q,h}^1 f_k = (E_h - I)f_k = E_h(f(x_k)) - f(x_k) = f_{k+1} - f_k,$$
$$\triangle_{q,h}^2 f_k = (E_h - q)(f_{k+1} - f_k) = f_{k+2} - [2]f_{k+1} + qf_k.$$

In like manner, we obtain

$$\triangle_{q,h}^3 f_k = f_{k+3} - [3]f_{k+2} + q[3]f_{k+1} - q^3 f_k,$$
$$\triangle_{q,h}^4 f_k = f_{k+4} - [4]f_{k+3} + q\frac{[4][3]}{[2]}f_{k+2} - q^3[4]f_{k+1} + q^6 f_k,$$

and, in general, for positive integral values of n,

$$\text{(2.3)}\qquad \triangle_{q,h}^{n} f_k = (-1)^n q^{\binom{n}{2}} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q f_{k+i} q^{i(i-2n+1)/2}.$$

Specifically, taking $k = 0$ and $n \in \mathbb{N}$, we have

$$\text{(2.4)}\qquad \triangle_{q,h}^{n} f_0 = (-1)^n q^{\binom{n}{2}} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q f_i q^{i(i-2n+1)/2}.$$

Note that the q-difference operator $\triangle_{q,h}^{n}$ has the characteristic properties of a linear operator, for $\triangle_{q,h}^{n}(f_k \pm g_k) = \triangle_{q,h}^{n} f_k \pm \triangle_{q,h}^{n} g_k$ and if c is a constant, $\triangle_{q,h}^{n} c f_k = c \triangle_{q,h}^{n} f_k$.

By (2.4), we have $\triangle_{q,h}^{3} f_0 = f_3 - [3] f_2 + q[3] f_1 - q^3 f_0$. Then

$$\begin{aligned} f_3 &= [3](f_2 - q f_1) + q^3 f_0 + \triangle_{q,h}^{3} f_0 \\ &= \triangle_{q,h}^{3}(f_0) + [3](E_h - q)(f_1 - f_0) + [3](f_1 - f_0) + f_0 \end{aligned}$$

since $q^3 f_0 - q[3] f_0 = f_0 - [3] f_0$. Also we obtain

$$f_3 = f_0 + [3]\triangle_{q,h} f_0 + [3]\triangle_{q,h}^{2} f_0 + \triangle_{q,h}^{3} f_0.$$

Using the binomial expansion, in general,

$$\text{(2.5)}\qquad f_k = f_0 + \binom{k}{1}_q \triangle_{q,h} f_0 + \binom{k}{2}_q \triangle_{q,h}^{2} f_0 + \cdots + \triangle_{q,h}^{k} f_0,$$

that is, $f_k = \sum_{i=0}^{k} \binom{k}{i}_q \triangle_{q,h}^{i} f_0$.

We now will give a q-analogue fundamental theorem for the n-th q-difference of the function $f(x)$ on $\mathbb{R}$. By (2.4) and (2.5), we obtain the following results:

THEOREM 1. *Let $\triangle_{q,h}^{n}$ be a difference operator in* (2.2) *for* $n \geq 1$, *which assigns to every function $f : \mathbb{R} \to \mathbb{R}[q]$, with values in the polynomial ring $\mathbb{R}[q]$. Then we have*

(1) $\triangle_{q,h}^{n} f(x) = (-1)^n q^{\binom{n}{2}} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q f(x+ih) q^{i(i-2n+1)/2}$.

(2) $f(x+nh) = f(x) + \binom{n}{1}_q \triangle_{q,h} f(x) + \cdots + \binom{n}{n}_q \triangle_{q,h}^{n} f(x)$, i.e., $f(x+nh) = \sum_{i=0}^{n} \binom{n}{i}_q \triangle_{q,h}^{i} f(x)$.

Let $f_q(x) = q^{2x}$, we may have $\triangle_q^3 q^{2x} = 0$. It may also be noted that the third q-difference is here zero. Generally if $f_q(x)$ is the q-number function of q^{nx} polynomials, then its $(n+1)$th difference is zero.

DEFINITION 1. A positive integer n is called the *degree* of $f_q(x)$ if it is the smallest number that satisfy $\triangle_q^{n+1} f_q(x) = 0$.

The q-factorial polynomial in q is

$$[x]^{(\underline{m})} = \begin{cases} 1, & m = 0 \\ [x][x-1]\cdots[x-m+1], & m \geq 1. \end{cases} \tag{2.6}$$

Let $\triangle_q^n := \triangle_{q,1}^n$ as in (2.2) with $h = 1$. Then for each m we can write

$$\triangle_q [x]^{(\underline{m})} = -q^{x+1}[-m][x]^{(\underline{m-1})}.$$

In this like manner we have

$$\triangle_q^n [x]^{(\underline{m})} = (-1)^n q^{n(x+(n+1)/2)}[-m][-m+1]\cdots[-m+n-1][x]^{(\underline{m-n})}.$$

Hence, $\triangle_q^{m+1}[x]^{(\underline{m})} = 0$, that is, m is the degree of q-factorial polynomials.

In view of formula (2.6), it is a matter of some interest to be able to express an arbitrary polynomial $f_q(x)$ of degree m of q-factorial polynomial, that is,

$$f_q(x) = a_0 + a_1[x]^{(\underline{1})} + a_2[x]^{(\underline{2})} + \cdots + a_m[x]^{(\underline{m})}.$$

We have $a_0 = f_q(0)$ and by (2.6)

$$\triangle_q f_q(x) = -q^{x+1}([-1]a_1 + [-2]a_2[x]^{(\underline{1})} + \cdots + [-m]a_m[x]^{(\underline{m-1})}).$$

If we set $x = 0$ in this expression, $[1]a_1 = -q[-1]a_1 = \triangle_q f_q(0)$. Similarly we obtain

$$a_2 = \frac{\triangle_q^2 f_q(0)}{[2]!},$$

and generally

$$a_j = \frac{\triangle_q^j f_q(0)}{[j]!}, \quad j = 0, 1, 2, \cdots, n. \tag{2.7}$$

Therefore, we have the following:

THEOREM 2. *Let $f_q(x)$ be an arbitrary polynomial of degree m in q-factorial polynomials, then we have*

$$f_q(x) = f_q(0) + \triangle_q f_q(0)[x]^{(\underline{1})} + \frac{\triangle_q^2 f_q(0)}{[2]!}[x]^{(\underline{2})} + \cdots + \frac{\triangle_q^n f_q(0)}{[m]!}[x]^{(\underline{m})},$$

which obviously resembles the q-analogue Maclaurin's expansion.

Suppose $f_q(x)$ is a polynomial of degree m in q-factorial polynomial, say

$$(2.8)\quad f_q(x) = a_0 + a_1[x - x_0] + \cdots + a_m[x - x_0][x - x_1]\cdots[x - x_{m-1}]$$

and $f_q(x_0) = f_{0,q}, f_q(x_1) = f_{1,q}, \cdots, f_q(x_m) = f_{m,q}$. Then we have

$$f_q(x_i) = f_{i,q} = a_0 + a_1[ih] + a_2[ih][(i-1)h] + \cdots + a_i[ih]\cdots[h]$$

for $0 \le i \le m$ and $x_k = x_0 + kh, k \in \mathbb{Z}$. Therefore we have

$$(2.9)\qquad a_i = \frac{1}{[ih]\cdots[h]}\triangle_{q^h,h}^i f_{0,q} \quad \text{for } 0 \le i \le m.$$

Hence, substituting for (2.8),

$$f_q(x) = f_{0,q} + \frac{\triangle_{q^h,h} f_{0,q}}{[h]}[x - x_0] + \cdots + \frac{\triangle_{q^h,h}^m f_{0,q}}{[mh]\cdots[h]}[x - x_0]\cdots[x - x_{m-1}].$$

In this equation, we set $h = 1$, i.e., $x_k = x_0 + k, k \in \mathbb{Z}$ and $[x - x_0] = [u]$. Then we have the following:

THEOREM 3. (q-analogue of Newton's forward-difference interpolation formula) *Let $f_q(x)$ be an arbitrary polynomial of degree m in q-factorial polynomial, then we have*

$$f_q(x) = f_{0,q} + \frac{\triangle_{q^h,h} f_{0,q}}{[h]}[u] + \cdots + \frac{\triangle_{q^h,h}^m f_{0,q}}{[mh]\cdots[h]}[u][u-1]\cdots[u-m+1].$$

In particular, if we set $x_0 = 0$, then we have Theorem 2, i.e.,

$$f_q(x) = f_q(0) + \triangle_q f_q(0)[x]^{(\underline{1})} + \frac{\triangle_q^2 f_q(0)}{[2]!}[x]^{(\underline{2})} + \cdots + \frac{\triangle_q^n f_q(0)}{[m]!}[x]^{(\underline{m})}.$$

By the definition (2.2) of q-difference operators, we easily see that

$$\triangle_q^n \frac{1}{[m]} = \frac{(-1)^n q^{mn+\binom{n}{2}}[n]!}{[m][m+1]\cdots[m+n]} \quad \text{for } n, m \in \mathbb{N}. \tag{2.10}$$

The q-analogue of the binomial formula was introduced by many authors (see [1–4], [6], etc.).

Let $n \in \mathbb{N}$. By the q-analogue of binomial formula one means the relation

$$(1-x)_q^n = (1-x)(1-q^{-1}x)\cdots(1-q^{-n+1}x) = \prod_{i=0}^{n-1}(1-q^{-i}x).$$

(In the literature, the notation $(1-x)_q^n$ may denote a slightly different formula.) Hence we have

$$(1-x)_q^n = \sum_{r=0}^{n} \binom{n}{r}_q (-1)^r x^r q^{r(r-2n+1)/2}. \tag{2.11}$$

Fix $0 < |q| < 1$. For a function f on $\mathbb{R}$, the Jackson integral is defined by (see [3], [6])

$$\int_0^x f(t)\, d_q t := (1-q)\sum_{k=0}^{\infty} f(q^k x) q^k x, \tag{2.12}$$

where $x \in \mathbb{R}$, provided the sum on the right hand side converges absolutely, for instance if f is bounded near zero.

Now, using the definition (2.2) of q-difference operators and the Jackson integral (2.12), it is clear that

$$\int_0^1 t^{m-1} d_q t = \frac{1}{[m]} \quad \text{for } m \in \mathbb{N},$$

$$\triangle_q \frac{1}{[m]} = \int_0^1 (t^m - t^{m-1}) d_q t = -\int_0^1 t^{m-1}(1-t)_q d_q t,$$

and

$$\triangle_q^{n-1} \frac{1}{[m]} = (-1)^{n-1} q^{\binom{n-1}{2}} \int_0^1 t^{m-1}(1-t)_q^{n-1} d_q t. \tag{2.13}$$

Hence by (2.10) and (2.13), we have

$$\int_0^1 t^{m-1}(1-t)_q^{n-1} d_q t = q^{(n-1)m} \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)},$$

where $\Gamma_q(n+1) = [1][2]\cdots[n] = [n]!$ and $m, n \in \mathbb{N}$.

Experience with q-difference operators of the Jackson integral for $\frac{1}{[m]}$ now make us to consider a beta function

$$(2.14) \quad B_q(x,y) := \int_0^1 t^{x-1}(1-t)_q^{y-1} d_q t \quad \text{for } x, y \in \mathbb{R} \text{ and } x, y > 0.$$

The definition (2.14) of the beta function $B_q(x,y)$ is a slightly different from N. Ja. Vilenkin and A. U. Klimyk's beta function (see [6]).

Hence we have the following:

THEOREM 4. *Let B_q be defined as in* (2.14) *and let* $\Gamma_q(n+1) = [n]!$. *Then we have*

$$B_q(m,n) = q^{(n-1)m} \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}$$

for all $m, n \in \mathbb{N}$.

The theorem 4 was proved by N. Ja. Vilenkin and A. U. Klimyk in [6]. On the other hand, we have given a different proof of Theorem 4 using the q-difference operators (2.2).

Now we write some values of $B_q(m,n)$ as follows:

$$B_q(1,1) = \frac{1}{\Gamma_q(2)} = 1, \ B_q(1,2) = q\frac{1}{\Gamma_q(3)} = \frac{q}{[2][1]},$$
$$B_q(2,1) = \frac{1}{\Gamma_q(3)} = \frac{1}{[2][1]}, \ B_q(3,1) = \frac{\Gamma_q(3)}{\Gamma_q(4)} = \frac{1}{[3]},$$
$$B_q(1,3) = q^2\frac{\Gamma_q(3)}{\Gamma_q(4)} = \frac{q^2}{[3]}, \ B_q(2,3) = q^4\frac{\Gamma_q(3)}{\Gamma_q(5)} = \frac{q^4}{[4][3]},$$
$$B_q(3,2) = q^3\frac{\Gamma_q(3)}{\Gamma_q(5)} = \frac{q^3}{[4][3]}, \ \cdots .$$

REMARK. (1) G. Gasper and M. Rahman (see [3]) used $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ for $|q|<1$. They defined the q-gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q^{1-x}), \quad 0<q<1.$$

When $x = n+1$ with $n \geq 0$, this definition reduces to $\Gamma_q(n+1) = [1][2]\cdots[n] = [n]!$, which clearly approaches $n!$ as $q \to 1^-$.

(2) Recall (see [6]) that for $0 < |q| < 1$ the q-gamma function is represented by the integral formula

$$\Gamma_q(x) = q^{x(x-1)/2} \int_0^\infty t^{x-1} E_q(-t)\, d_q t, \quad x > 0,$$

where $E_q(t)$ is defined by $E_q(t) = \sum_{r=0}^{\infty} \frac{t^r}{[r]!}$, which is the q-analogue of the ordinary exponential function. When $0 < |q| < 1$ this series converges uniformly for $|x| < \frac{1}{1-q}$.

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