

ON THE MAXIMALITY OF PRIME IDEALS IN EXCHANGE RINGS

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ABSTRACT. We investigate the relationship between various generalizations of von Neumann regularity condition and the condition that every prime ideal is maximal in exchange rings.

0. Introduction

Throughout this paper R denotes an associative ring with identity and $J(R)$ always stands for the Jacobson radical of a ring R .

According to Crawley and Jónsson [6], a right R -module M is called to have the *exchange property* if for every right R -module A and any two decompositions of A

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

where $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus \left(\bigoplus_{i \in I} A'_i \right).$$

Warfield [18] called a ring R *exchange* if R_R has the exchange property and proved that this definition is right-left symmetric. The first element-wise characterization of exchange rings was given by Monk [14]: R is an

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exchange ring if and only if for every $a \in R$, there exist $b, c \in R$ such that $bab = b$ and $c(1-a)(1-ba) = 1-ba$. Nicholson [15] gave another characterization of exchange rings: R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents can be lifted modulo $J(R)$ if and only if for any $a \in R$, there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$.

Note that every homomorphic image of exchange rings is exchange. But a subring of exchange rings needs not to be an exchange ring. For example, \mathbb{Q} the set of all rational numbers is an exchange ring but the subring \mathbb{Z} the set of all integers is not exchange. For, if $e^2 = e \in 3\mathbb{Z}$, then $e = 0$. But $1-0 = 1 \notin (1-3)\mathbb{Z} = 2\mathbb{Z}$. Moreover, the following example shows that there exists an exchange ring R which is finitely generated as a right S -module, where S is a subring of R and S is not exchange.

EXAMPLE 0.1. Let $R = \text{Mat}_3(\mathbb{Q})$ be the ring of all 3×3 matrices over the set of all rational numbers \mathbb{Q} . Then R is semisimple Artinian and so R is exchange. Let

$$S = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{bmatrix}.$$

Then S is a subring of R and R is finitely generated as a right S -module.

In fact, we have $R = \sum_{i=1}^4 r_i S$, where $r_1 = 1_S$, $r_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
 $r_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $r_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Now we take $a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then 0 is the only idempotent contained in aS . However $1_S \notin (1_S - a)S$. Therefore S is not exchange.

In general, all idempotents of a ring R need not to be lifted modulo $J(R)$. For example, let $R = \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ is relatively prime to } 6\}$. Then the idempotents 3 and 4 are cannot be lifted modulo $J(R) = 6R$. However, if I is a nil ideal of a ring R , then idempotents can be lifted modulo I , but there exists a non-exchange ring R with $J(R)$ nil, for example the ring \mathbb{Z} of all integers.

The class of exchange rings is quite large. It includes all local rings, all von Neumann regular rings and all semiperfect rings. Moreover, it

properly contains the class of π -regular rings which is one of the generalizations of von Neumann regular rings.

On the other hand, the relationship between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal have been investigated by many authors [3, 4, 7, 8, 11, 17 and 20, etc.]. The first clearly established equivalence between a generalization of von Neumann regularity and the maximality of prime ideals seems to have been made by Storrer [17] in the following result: If R is a commutative ring with identity then R is π -regular if and only if every prime ideal of R is maximal. Storrer's result was extended to PI-rings, right duo rings and bounded weakly right duo rings [7, 8 and 19], respectively. Moreover, Hirano [8] proved that if R is a 2-primal ring, then R is π -regular if and only if every prime ideal of R is a maximal one-sided ideal. Birkenmeier, Kim and Park [3] showed that if R is a 2-primal ring, then every prime ideal of R is maximal if and only if $R/\mathbf{P}(R)$ is right weakly π -regular, where $\mathbf{P}(R)$ is the prime radical of R . These results were mainly explained the relation between the π -regularity and the maximality of prime ideals of rings.

In this paper, we investigate the connection between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal in exchange rings.

1. Preliminaries and Examples

Recall that a ring R is called to be (*strongly*) π -regular if for every $x \in R$ there exists a positive integer n , depending on x , such that $(x^n \in x^{n+1}R) x^n \in x^n R x^n$. Strong π -regularity is right-left symmetric. R is called to be *right (left) weakly π -regular* if for every $x \in R$ there exists a positive integer n , depending on x , such that $x^n \in x^n R x^n R (x^n \in R x^n R x^n)$. R is *weakly π -regular* if it is both right and left weakly π -regular. Note that π -regular rings are weakly π -regular rings but the converse is not true.

In general, π -regular rings are exchange rings [16, Example 2.3]. However, the following example shows that the converse does not hold (also see [15, Example 1.7]).

EXAMPLE 1.1. [21, Example 3.5] Let p be a prime and let $R = \mathbb{Z}_{(p)}$, the localization of integers at (p) . Then R is a commutative exchange ring. But R is not (right weakly) π -regular. For, if R is a (right weakly)

π -regular ring, then $J(R)$ is nil. However $J(\mathbb{Z}_{(p)})$ is not nil. Therefore R is not (right weakly) π -regular.

Recall that a ring R is called to be *right* (resp. *left*) *quasi-duo* if every maximal right (resp. left) ideal of R is an ideal. R is called to be *abelian* if every idempotent of R is central.

The class of right quasi-duo rings and the class of exchange rings do not depend on each other. For example, the ring \mathbb{Z} of all integers is a (right) quasi-duo ring, but it is not exchange. Conversely, we consider the ring of all 2×2 matrices, $\text{Mat}_2(F)$, over a field F . Then it is an exchange ring, but it is not right quasi-duo. However, every abelian exchange ring is a right quasi-duo ring by [5, Theorem 3.1(ii)(b)].

Observe that the ring R , in Example 1.1, is abelian exchange but it is not right weakly π -regular. The following example shows that there exists an abelian right weakly π -regular ring which is not exchange. Thus even if the class of exchange rings properly contains the class of π -regular rings, it does not contain the class of (abelian) right weakly π -regular rings.

EXAMPLE 1.2. Let D be a simple domain which is not a division ring. Consider a ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in D \right\}.$$

Then it can be easily checked that R is abelian. Also, R is right weakly π -regular. For, if $x \in \mathbf{P}(R)$, where $\mathbf{P}(R)$ is the prime radical of R , then $x^2 = 0$ and hence $x^2 \in x^2Rx^2R$. Suppose $x \notin \mathbf{P}(R)$. Then since $R/\mathbf{P}(R) \cong D$ is simple, $RxR + \mathbf{P}(R) = R$. Thus $\alpha + \beta = 1$ for some $\alpha \in RxR$ and $\beta \in \mathbf{P}(R)$. But $\beta^2 = 0$ and so $1 = (\alpha + \beta)^2 \in RxR$. Hence $x \in xRxR$ and so R is right weakly π -regular.

Note that a ring is right quasi-duo if and only if every factor ring modulo a right primitive ideal is a division ring. Now,

$$R / \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \cong D$$

is a simple domain, but it is not a division ring. So R is not a right quasi-duo ring. Moreover, R is not exchange because R is abelian.

The following example shows that there exists an abelian exchange ring R (and so $R/J(R)$ is exchange) with $J(R)$ nil. However $R/J(R)$ is not right weakly π -regular.

EXAMPLE 1.3. Let \mathbb{Q} denote the rational numbers and L the ring of all rational numbers with odd denominators. Define

$$R = \{(x_1, x_2, \dots, x_n, l, l, \dots) \mid 1 \leq n, x_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n, l \in L\}$$

with componentwise operations. Then R is a commutative exchange ring with zero Jacobson radical. However $R/J(R)$ is not right weakly π -regular. For, let $\alpha = (\frac{2}{3}, \dots, \frac{2}{3}, \dots)$, then $\alpha^n \notin \alpha^n R \alpha^n R$ for any positive integer n .

Stock [16] called a ring R *right* (resp. *left*) *P-exchange* if every projective right (resp. left) R -module has the exchange property. Observe that *P-exchange* rings are exchange rings. Recently, Birkenmeier, Kim and Park [4] showed that if R is an abelian right *P-exchange* ring, then R is weakly π -regular.

Recall that a ring R is called to be *weakly right duo* if for each $a \in R$ there is a positive integer $n = n(a)$, depending on a , such that $a^n R$ is an ideal of R [19]. Note that every weakly right duo ring is an abelian right quasi-duo ring by [19, Lemma 4] and [20, Proposition 2.2].

The following proposition extends the result of [20, Proposition 4.3].

PROPOSITION 1.4. *If R is an abelian right P-exchange ring, then R is weakly right duo.*

PROOF. By [4, Corollary 2.16], R is weakly π -regular. Thus $R/J(R)$ is right weakly π -regular and $J(R)$ is nil. Since R is abelian exchange, it is right quasi-duo by [5, Theorem 3.1(ii)(b)]. Thus by [13, Theorem 22], R is weakly right duo. \square

COROLLARY 1.5. *Let R be a right P-exchange ring. Then the following statements are equivalent:*

- (1) R is weakly right duo.
- (2) R is abelian right quasi-duo.

The fact “every abelian exchange ring is a right quasi-duo ring” and Proposition 1.4 suggest that abelian exchange rings may be weakly right duo. However, the following example shows that this is not true. Thus the condition “right *P-exchange* ring” cannot be replaced by the condition “exchange ring” in Proposition 1.4 and Corollary 1.5.

EXAMPLE 1.6. [13, Example 23] Let S be the quotient field of the polynomial ring $F[t]$ over a field F with t its indeterminate and define a ring homomorphism $\sigma : S \rightarrow S$ by

$$\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t^2)}{g(t^2)}.$$

Next consider the skew power series ring $R = S[[x; \sigma]]$, every element is of the form $\sum_{n=1}^{\infty} a_n x^n$ over S and only subject to $xa = \sigma(a)x$ for $a \in S$, where x is the indeterminate. Then R is an integral domain and so 0 and 1 are the only idempotents of R ; hence R is abelian. Moreover R is local and so it is exchange with $J(R) = Rx$ not nil, therefore it is right quasi-duo. However, R is not weakly right duo.

2. Main results

Now, we study the relationship between the exchange property and the maximality of prime ideals of rings.

The following example shows that the condition “ R is an exchange ring” is not equivalent to the condition “every prime ideal of R is maximal”.

EXAMPLE 2.1. (1) Let $\text{Mat}_n(F)$ denote the ring of all $n \times n$ matrices over a field F . Consider a ring

$$R = \left\{ \left[\begin{array}{ccc} A & 0 & 0 \cdots \\ 0 & a & 0 \cdots \\ 0 & 0 & a \cdots \\ \vdots & \vdots & \ddots \end{array} \right] \mid A \in \text{Mat}_n(F), a \in F, n = 1, 2, \dots \right\}.$$

Then it can be easily checked that R is π -regular and so it is exchange. But (0) is a prime ideal of R which is not maximal.

(2) The ring R , in Example 1.2, is not exchange and so it is not π -regular. But every prime ideal of R is maximal because the prime radical $\mathbf{P}(R)$ is the unique maximal ideal of R .

Let $\mathbf{P}(R)$, $\mathbf{N}_r(R)$ and $\mathbf{N}(R)$ be the prime radical, the unique maximal nil ideal and the set of all nilpotent elements of R , respectively.

In [20], Yu proved that if R is a right quasi-duo ring, then $\mathbf{N}(R) \subseteq J(R)$. However, there exists a ring R which satisfies the condition

“ $\mathbf{N}(R) \subseteq J(R)$ ” or “ $N_2(R) \subseteq J(R)$ ” where $N_2(R)$ denotes the set of all nilpotent element of bounded index 2, but R is not right quasi-duo. For example, the ring R , in Example 1.2, is not right quasi-duo, but $\mathbf{N}(R) = N_2(R) = J(R)$.

Recall that a ring R is called to be *2-primal* if $\mathbf{P}(R) = \mathbf{N}(R)$ [2]. We refer to [3], [4], [8], [10], [11] and [12] for more details on 2-primal rings. Observe that R is a 2-primal ring if and only if $\mathbf{P}(R) = \mathbf{N}_r(R) = \mathbf{N}(R)$ if and only if $\mathbf{P}(R)$ is a completely semiprime ideal of R (i.e., $a^2 \in \mathbf{P}(R)$ implies $a \in \mathbf{P}(R)$ for $a \in R$). Also, $\mathbf{N}_r(R) = \mathbf{N}(R)$ if and only if $\mathbf{N}_r(R)$ is completely semiprime. Observe that the class of rings which satisfies $\mathbf{N}_r(R) = \mathbf{N}(R)$ properly contains the class of 2-primal rings (see Example 2.9). Clearly, if R is 2-primal, then $N_2(R) \subseteq \mathbf{P}(R)$. The converse does not hold by [10, Example 1]. Moreover, the following example, using the example [10, Example 1], shows that there exists a ring R such that $N_2(R) \subseteq \mathbf{N}_r(R)$, but R does not satisfy $\mathbf{N}_r(R) = \mathbf{N}(R)$.

EXAMPLE 2.2. Let $F\langle X, Y \rangle$ be the free algebra on X, Y over a field F and I denote the ideal $(X^2)^2$ of $F\langle X, Y \rangle$, where $(X^2)^2$ is the ideal of $F\langle X, Y \rangle$ generated by X^2 . Consider the ring $R = F\langle X, Y \rangle / I$. Then $N_2(R) = Rx^2R = \mathbf{P}(R)$, where $x = X + I$ and $\mathbf{N}(R) = xRx + Rx^2R + Fx$. Now, we claim $N_2(R) = \mathbf{N}_r(R)$. If not, that is, $\mathbf{P}(R) = N_2(R) \subsetneq \mathbf{N}_r(R)$, then $\mathbf{N}_r(R)/\mathbf{P}(R)$ is a nonzero nil ideal of $S = R/\mathbf{P}(R) (\cong F\langle X, Y \rangle / \langle X^2 \rangle)$. But S is a prime ring and $\mathbf{N}(S) = N_2(S)$. Since $\mathbf{N}_r(R)/\mathbf{P}(R) \subseteq \mathbf{N}(S)$, we have $\bar{a}^2 = \bar{0}$ for all $\bar{a} \in \mathbf{N}_r(R)/\mathbf{P}(R)$. By [11, Lemma 11], $\bar{a}S\bar{a} = \bar{0}$. Since S is prime, $\bar{a} = \bar{0}$ which is a contradiction. Therefore $\mathbf{P}(R) = N_2(R) = \mathbf{N}_r(R)$, but $\mathbf{N}_r(R) \subsetneq \mathbf{N}(R)$.

However we have the following result.

PROPOSITION 2.3. *Let R be an exchange ring. Then the following statements are equivalent:*

- (1) R is right quasi-duo.
- (2) $\mathbf{N}(R) \subseteq J(R)$.
- (3) $N_2(R) \subseteq J(R)$.

PROOF. (1) \Rightarrow (2): It follows from [20, Lemma 2.3].

(2) \Rightarrow (3): It is trivial.

(3) \Rightarrow (1): Since R is exchange, $R/J(R)$ is also a semiprimitive exchange ring. So by [10, Theorem 2], $R/J(R)$ is reduced and so $R/J(R)$ is abelian exchange. Thus $R/J(R)$ is right quasi-duo and hence R is right quasi-duo. \square

The condition “ R is an exchange ring” in Proposition 2.3 is not superfluous by Example 1.2.

COROLLARY 2.4. *Let R be an exchange ring whose prime ideals are maximal. Then the following statements are equivalent:*

- (1) R is 2-primal.
- (2) $\mathbf{N}_r(R) = \mathbf{N}(R)$.
- (3) $\mathbf{N}(R)$ is a two-sided ideal of R .
- (4) $N_2(R) \subseteq \mathbf{P}(R)$.
- (5) $N_2(R) \subseteq \mathbf{N}_r(R)$.
- (6) R is right quasi-duo.

PROOF. (1) \Rightarrow (2) \Leftrightarrow (3) and (1) \Rightarrow (4) \Rightarrow (5): These are obvious.
 (2) \Rightarrow (6) and (5) \Rightarrow (6): These follow from Proposition 2.3 since $\mathbf{N}_r(R) \subseteq J(R)$.
 (6) \Rightarrow (1): Since R is a right quasi-duo ring whose prime ideals are maximal, $\mathbf{N}(R) \subseteq J(R) = \mathbf{P}(R)$ and so $\mathbf{P}(R) = \mathbf{N}(R)$. Therefore R is 2-primal. \square

Observe that the condition “all prime ideals are maximal” in Corollary 2.4 is not superfluous by the following example and Example 2.9.

Recall that a ring R is said to be *of bounded index (of nilpotency)* if there exists a positive integer n such that $a^n = 0$ for all nilpotent elements a of R .

EXAMPLE 2.5. Let $S = \text{Mat}_2(F)$ denote the ring of 2×2 matrices over a field F and $T = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in S \mid a \in F \right\}$. Consider the ring $R = T + xS[[x]]$, where $S[[x]]$ denotes the ring of all formal power series over S . Then $xS[[x]]$ is the unique maximal left and right ideal of R . Thus R is a right quasi-duo ring and $J(R) = xS[[x]]$ is not nil. Also, R is a prime exchange ring of bounded index 2. Note that the ideal (0) is a prime ideal of R which is not maximal. Moreover, R does not satisfy $\mathbf{N}_r(R) = \mathbf{N}(R)$ and so it is not 2-primal. Note that R is not (right weakly) π -regular because $J(R)$ is not nil.

THEOREM 2.6. *Let R be an exchange ring of bounded index. Then the following statements are equivalent:*

- (1) R is strongly π -regular.
- (2) R is π -regular.
- (3) $R/P(R)$ is π -regular.
- (4) Every prime ideal of R is maximal.

PROOF. (1) \Rightarrow (2) \Rightarrow (3): These implications are clear.

(3) \Rightarrow (4): Since R is of bounded index, $R/\mathbf{P}(R)$ is also of bounded index. Thus by [9, Theorem 2], every prime ideal of $R/\mathbf{P}(R)$ is maximal. Therefore every prime ideal of R is maximal.

(4) \Rightarrow (1): Let P be a prime ideal of R . By [21, Theorem 3.11], R/P is Artinian. So it is strongly π -regular. Thus by [7, Theorem 2.1], R is strongly π -regular. \square

The rings R , in Example 2.1(1) and (2), show that the condition “ R is a ring of bounded index” and the condition “ R is an exchange ring” in Theorem 2.6 are not superfluous, respectively.

COROLLARY 2.7. *Let R be a ring of bounded index. Then the following statements are equivalent:*

- (1) R is strongly π -regular.
- (2) R is π -regular.
- (3) R is exchange whose prime ideals are maximal.

PROOF. It follows directly from Theorem 2.6. \square

The condition “every prime ideal of R is maximal” in Corollary 2.7(3) is not superfluous by Example 2.5. Also, the condition “ R is π -regular” in Corollary 2.7(2) cannot be replaced by the condition “ R is right weakly π -regular” by Example 1.2. In fact, the ring R , in Example 1.2, is right weakly π -regular of bounded index 2 which is not exchange, thus R is not π -regular. Observe that every prime ideal of R is maximal because $\mathbf{P}(R)$ is the unique maximal ideal of R .

Moreover, the condition “ R is a ring of bounded index” cannot be dropped from Corollary 2.7. In fact, the ring $R = \text{End}_F(V)$, where V is an infinite dimensional vector space over a field F . Then R is a von Neumann regular ring of no bounded index. Thus R is π -regular, but it is not strongly π -regular.

Note that if R is an exchange ring of bounded index, then every primitive factor ring of R is Artinian. However, the condition “ R is a ring of bounded index” in Theorem 2.6 and Corollary 2.7 cannot be replaced by the condition “every primitive factor ring of R is Artinian” by Example 2.9.

THEOREM 2.8. *Let R be a 2-primal exchange ring. The following statements are equivalent:*

- (1) R is strongly π -regular.

- (2) R is π -regular.
- (3) R is weakly π -regular.
- (4) R is right weakly π -regular.
- (5) $R/J(R)$ is right weakly π -regular and $J(R)$ is nil.
- (6) $R/\mathbf{P}(R)$ is right weakly π -regular.
- (7) Every prime ideal of R is maximal.

PROOF. (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) and (4) \Rightarrow (6): These implications are trivial.

(5) \Leftrightarrow (7): Assume (5). Since $J(R)$ is nil and R is 2-primal, $J(R) = \mathbf{P}(R)$. By [3, Corollary 9], every prime ideal of R is maximal.

Conversely, Assume (7). By [3, Corollary 9], $R/\mathbf{P}(R)$ is right weakly π -regular and so $R/J(R)$ is right weakly π -regular. Now let $a \in J(R)$. Consider $\bar{a} \in \bar{R} = R/\mathbf{P}(R)$. Since \bar{R} is right weakly π -regular, there exists a positive integer n such that $\bar{a}^n = \bar{a}^n \bar{r}$ for some $\bar{r} \in \bar{R} \bar{a}^n \bar{R} \subseteq \bar{J}(R)$, where $\bar{J}(R) = J(R)/\mathbf{P}(R)$. Then $\bar{a}^n (\bar{1} - \bar{r}) = \bar{0}$ and so $a^n \in \mathbf{P}(R)$. Since R is 2-primal, $a \in \mathbf{P}(R)$ and hence $J(R)$ is nil.

(6) \Rightarrow (1): Since $R/\mathbf{P}(R)$ is reduced exchange, it is right quasi-duo. By [11, Proposition 7], $R/\mathbf{P}(R)$ is strongly π -regular. Note that every prime factor ring of R is strongly π -regular. Therefore R is strongly π -regular by [7, Theorem 2.1].

(7) \Rightarrow (1): Note that R is a right quasi-duo ring by Corollary 2.4. Thus by [20, Theorem 2.5], R is strongly π -regular. \square

Related to (5) and (6) in Theorem 2.8, we noted that there exists a 2-primal ring whose prime ideals are maximal, but it is neither right nor left weakly π -regular [3, Example 12].

The following example shows that there exists an abelian exchange (and so right quasi-duo) ring R which is not 2-primal such that $R/J(R)$ is right weakly π -regular and $J(R)$ is nil, but R contains a prime ideal which is not maximal. Hence the condition “ R is 2-primal” in Theorem 2.8 is not superfluous and cannot be replaced by the condition “ $\mathbf{N}_r(R) = \mathbf{N}(R)$ ”.

EXAMPLE 2.9. [4, Example 3.3] Let G be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups and denote by $\{b(0), b(1), b(-1), \dots, b(i), b(-i), \dots\}$ of basis of G . Then there exists one and only one homomorphism $u(i)$ of G , for $i = 1, 2, \dots$ such that $u(i)(b(j)) = 0$ if $j \equiv 0 \pmod{2^i}$ and $u(i)(b(j)) = b(j-1)$ if $j \not\equiv 0 \pmod{2^i}$. Denote U the ring of endomorphisms of G generated

by the endomorphisms $u(1), u(2), \dots$. Now let A be the ring obtained from U by adjoining the identity map of G . Let $R = A \otimes_{\mathbb{Z}} \mathbb{Q}$, where \mathbb{Z} is the integers and \mathbb{Q} is the rational numbers. Then R is a strongly π -regular right quasi-duo ring which satisfies $\mathbf{N}_r(R) = \mathbf{N}(R)$, but it is not 2-primal. Thus R is an exchange ring such that $R/J(R)$ is right weakly π -regular and $J(R)$ is nil. However, there exists a prime ideal of R which is not maximal. Note that R is of no bounded index and it is a local ring. Moreover, every primitive factor ring of R is Artinian because R is right quasi-duo.

The class of 2-primal rings and the class of abelian rings are different from each other by the ring of 2×2 upper triangular matrices over a field and Example 2.9. The proof of the following theorem is an adaptation from [13, Theorem 22].

THEOREM 2.10. *Let R be an abelian exchange ring. If $R/J(R)$ is right weakly π -regular and $J(R)$ is nil, then R is strongly π -regular.*

PROOF. Since R is an abelian exchange ring, it is right quasi-duo. By [11, Theorem 7], $R/J(R)$ is strongly π -regular and so $R/J(R)$ is strongly regular. Thus for each $x \in R$, there exists $y \in R$ such that $x - xyx \in J(R)$. Denote $x + J(R)$ by \bar{x} . Since R is exchange, every idempotent can be lifted to an idempotent of R . Then there exists $e^2 = e \in R$ such that $\bar{e} = \bar{x}\bar{y}$ and we get $\bar{x} = \bar{e}\bar{x}$. But $x - ex$ is nilpotent so there exists a positive integer n such that $(x - ex)^n = 0$. Thus $x^n \in eR$ because e is central. Now $\bar{e} = \bar{x}\bar{y}$ is central in $R/J(R)$ whence we obtain $\bar{e} = \bar{x}\bar{y} = \bar{x}^2\bar{y}^2 = \dots = \bar{x}^n\bar{y}^n$. Then it follows that $e - x^n y^n \in J(R)$, and $(e - x^n y^n)^m = 0$ for some positive integer m . Now we have $e \in x^n R$ and consequently $x^n R = eR$. Thus $x^n = er, e = x^n s$ for some $r, s \in R$. Then $x^n = x^n e = x^{2n} s \in x^{2n} R \subseteq x^{n+1} R$. Therefore R is strongly π -regular. \square

Recall that a ring R is called to be *biregular* if for each $x \in R$, the ideal RxR is generated by a central idempotent. Biregular rings are right weakly π -regular rings. Note that the biregularity of a ring R implies that all prime ideals of R are maximal, while the converse is not true in general.

COROLLARY 2.11. *Let R be an abelian exchange ring. The following statements are equivalent:*

- (1) R is strongly π -regular.
- (2) R is π -regular.
- (3) $R/\mathbf{P}(R)$ is π -regular.

- (4) R is weakly π -regular.
- (5) R is right weakly π -regular.
- (6) $R/\mathbf{P}(R)$ is right weakly π -regular.
- (7) $R/J(R)$ is right weakly π -regular and $J(R)$ is nil.
- (8) $R/J(R)$ is biregular and $J(R)$ is nil.

PROOF. (2) \Leftrightarrow (3): It follows directly from [1, Corollary 2].

(5) \Leftrightarrow (6): It follows from [11, Theorem 7] and [1, Corollary 2] because R is right quasi-duo.

(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (7) and (8) \Rightarrow (7): These implications are clear.

(7) \Rightarrow (1): It follows from Theorem 2.10.

(7) \Rightarrow (8): Note that $R/J(R)$ is strongly π -regular by [11, Theorem 7] because $R/J(R)$ is right quasi-duo. Moreover, $R/J(R)$ is reduced by [20, Lemma 2.3]. Therefore $R/J(R)$ is strongly regular and so it is biregular. \square

PROPOSITION 2.12. *Let R be an abelian exchange ring of bounded index. Then the following statements are equivalent:*

- (1) $R/J(R)$ is right weakly π -regular and $J(R)$ is nil.
- (2) Every prime ideal of R is maximal.

PROOF. (1) \Rightarrow (2): Since R is a right quasi-duo ring of bounded index with $J(R)$ nil, R is 2-primal by [11, Proposition 12]. By Theorem 2.8, every prime ideal of R is maximal.

(2) \Rightarrow (1): It follows from [20, Theorem 2.5] because R is right quasi-duo. \square

In Proposition 2.12, the condition “ R is of bounded index” is not superfluous by Example 2.9. In fact, the ring R , in Example 2.9, is an abelian exchange (and so right quasi-duo) ring with $J(R)$ nil and $R/J(R)$ is right weakly π -regular. But there exists a prime ideal of R which is not maximal.

We also note that the condition “ R is a ring of bounded index” in Proposition 2.12 cannot be replaced by the condition “every primitive factor ring of R is Artinian” by Example 2.9.

Related to Proposition 2.12, observe that if R is an abelian P -exchange ring of bounded index, then every prime ideal of R is maximal; but the ring R , in Example 1.1, is an abelian exchange ring of bounded index and there exists some prime ideal of R which is not maximal.

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