Performance Bounds for MMSE Linear Macrodiversity Combining in Rayleigh Fading, Additive Interference Channels

Peter J. Smith, Hongsheng Gao, and Martin V. Clark

Abstract: **The theoretical performance of MMSE linear microdiversity combining in Rayleigh fading, additive interference channels has already been derived exactly in the literature. In the macrodiversity case the fundamental difference is that any given source may well have different average received powers at the different antennas. This makes an exact analysis more difficult and hence for the macrodiversity case we derive a bound on the mean BER and a semi-analytic upper bound on outage probabilities. Hence we provide bounds on the performance of MMSE linear microdiversity combining in Rayleigh fading with additive noise and any number of interferers with arbitrary powers.**

Index Terms: **Adaptive arrays, diversity methods, macrodiversity, interference suppression, Rayleigh channels.**

I. INTRODUCTION

Adaptive arrays and diversity combining have recently attracted great interest in wireless communication systems: By transmitting signals over a set of M channels (e.g., to an array of antennas, or over a set of frequency channels), the M received signals can be processed to combat channel impairments such as multipath fading, co-channel interference, and dispersion [1]–[17]. In turn, this kind of processing can lead to major improvements in link performance and system capacity.

One important type of array processing is MMSE linear combining, discussed in detail for the microdiversity case in [4].

In [4] the authors consider an ideal *minimum mean-square error* (MMSE) combiner which assumes exact knowledge of the instantaneous values of all channel gains[1], and of all source and noise statistics. In the idealised combiner, the MMSE can be expressed, using standard techniques, as a function of these channel gains, and can be translated into a maximum value of the combiner's output SINR (signal-to-interference-plus-noise ratio). The link BER of various modulations can generally be related to the SINR via well-known approximations and bounds [1]–[3], [16]–[17].

In [4] an exact derivation of the output SINR is given which provides a remarkably simple analytical tool for link performance. In this paper we look at the more complex case of macrodiversity where the exact SINR distribution is unknown. Hence we derive analytic bounds on the mean BER of such systems and semi-analytic bounds on the BER distribution. The bound on the mean BER is particularly appealing as it leads to valuable insights into macrodiversity combining, has a simple structure, and is numerically stable.

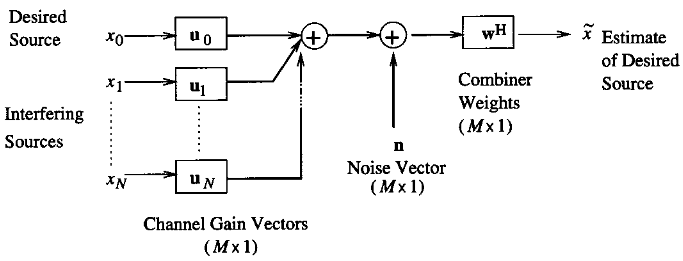

Fig. 1. Communication system.

The paper is organised as follows. In Section II, we describe the communication system, and formulate the problem. In Section III, we bound the mean BER of such systems. Section IV shows specific cases which relate our formula to standard results and reveal relevant insights. Section V shows a semi-analytic bound on the distribution of the BER, and Section VI presents a set of numerical examples and some conclusions.

II. PROBLEM FORMULATION

We consider the complex-baseband communication system shown in Fig. 1:

- x_0 and $\{x_1, x_2, \cdots, x_N\}$ are the desired source and set of interfering sources, respectively. Each source comes from some independent, identically distributed zero-mean random process with magnitude variance a^2.
- $\mathbf{u_0}, \mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_N}$ are the corresponding $M \times 1$ channel gain vectors. Each gain in the set of vectors is an independent zero-mean complex Gaussian random variable. The Rayleigh magnitudes of these gains have the variances $E[\mathbf{u_n}\mathbf{u_n^H}] = \text{diag}(P_{1n}, \cdots, P_{Mn})$, for $n = 0, 1, 2, \cdots, N$. Hence $\mathbf{u_n}$ can be written as

$$\begin{aligned}\mathbf{u_n^H} &= \left(\sqrt{P_{1n}}\,h_{1n},\ \sqrt{P_{2n}}\,h_{2n},\ \cdots,\ \sqrt{P_{Mn}}\,h_{Mn}\right)\\ &= \left(u_{1n},\ u_{2n},\ \cdots,\ u_{Mn}\right), \qquad n = 0, 1, 2, \cdots, N\end{aligned}$$

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[1] In reality, adaptive algorithms would be used to set combiner weights.

where the h_{ij}'s are independent and identically distributed (i.i.d.) Gaussians with $E[h_{ij}] = 0$, $E[|h_{ij}|^2] = 1$, $i = 1, \cdots, M$, $j = 1, \cdots, N$.

We also define the vector $\mathbf{h_0}$ as

$$h_0 = (h_{10}, h_{20}, \cdots, h_{M0})^{\mathrm{H}}.$$

- $\mathbf{n}$ is an $M \times 1$ additive noise vector. Each noise source in this vector is an independent zero-mean complex Gaussian process with magnitude variance σ^2.
- $\mathbf{w}^{\mathrm{H}}$ is the $1 \times M$ vector of complex combiner weights. (We use the Hermitian transpose for analytical convenience.)
- $\tilde{x}_0$ is the estimate of the desired source.

As an example, the desired source could be data symbols to be detected by the receiver, the interfering sources could be co-channel interferers, the Rayleigh fading could come from multipath scattering in radio channels, and the diversity channels could come from multiple receive antennas. Note also that it is quite arbitrary which source is defined as the desired source. For instance, one could envisage an array processor comprising a set of $N+1$ combiners for estimating all $N+1$ sources.

The formulation for the MMSE combiner (i.e., optimal weights, MMSE and maximum SINR) is given in detail in [4]. We summarise the key results in the following. The combiner's MMSE is given by

$$\epsilon = E\left[|\tilde{x}_0 - x_0|^2\right] = a^2\left[1 + \mathbf{u_0^H R^{-1} u_0}\right]^{-1}, \qquad (1)$$

where

$$\mathbf{R} = \sum_{\mathbf{n=1}}^{\mathbf{N}} \mathbf{u_n u_n^H} + \frac{\sigma^2}{a^2}\mathbf{I}. \qquad (2)$$

The maximum SINR at the combiner output is

$$Z = \mathbf{u_0^H R^{-1} u_0}, \qquad (3)$$

and the mean BER can (for certain modulations) be bounded by

$$\mathrm{BER} \le a e^{-Z}, \qquad (4)$$

where a depends on the type of modulation ($a = 0.5$ for DPSK). For other modulations (i.e., BPSK or OFSK) bounds are usually written in terms of the complementary error function. At high SNR these can also be approximated by the simple exponential as in (4). Denoting $E[e^{-Z}]$ by Q_M, we can write $\langle \mathrm{BER} \rangle \le aQ_M$ where expectation is taken over all the independent zero mean complex Gaussian variables in $\{\mathbf{u_0}, \cdots, \mathbf{u_N}\}$.

To summarise, we will derive bounds on the mean of the BER given in (4) and on its distribution function.

III. A BOUND ON THE MEAN BER

Suppose that $\lambda_1, \cdots, \lambda_M$ are the eigenvalues of the matrix

$$\overline{\mathbf{R}} = \mathrm{diag}(\sqrt{P_{10}}, \sqrt{P_{20}}, \cdots, \sqrt{P_{M0}})\mathbf{R}^{-1} \times \mathrm{diag}(\sqrt{P_{10}}, \sqrt{P_{20}}, \cdots, \sqrt{P_{M0}}).$$

Then there exists a unitary matrix $\mathbf{\Phi}$ such that

$$\overline{\mathbf{R}} = \mathbf{\Phi}\mathrm{diag}(\lambda_1, \cdots, \lambda_{\mathrm{M}})\mathbf{\Phi}^{\mathrm{H}}. \qquad (5)$$

Also, if we set $\overline{\mathbf{h}}_0^{\mathrm{H}} = \mathbf{h}_0^{\mathrm{H}}\mathbf{\Phi}$, the elements $\overline{h}_{i0}$ of $\overline{\mathbf{h}}_0$ are also i.i.d. complex Gaussian random variables with zero mean and $E[|\overline{h}_{i0}|^2] = 1$. Furthermore, the λ_n's are independent of $\overline{\mathbf{h}}_0$. Thus, we can average over the desired and interfering signal vectors separately, i.e.,

$$\begin{aligned}
Q_M &= E\left[\exp(-\mathbf{u_0^H R^{-1} u_0})\right] \\
&= E\left[\exp(-\mathbf{h_0^H \overline{R} h_0})\right] \\
&= E\left[\exp(-\mathbf{h_0^H}[\mathbf{\Phi}\mathrm{diag}(\lambda_1, \cdots, \lambda_{\mathrm{M}})\mathbf{\Phi}^{\mathrm{H}}]^{-1})\mathbf{h_0}\right] \\
&= E\left[\exp\Big(-\sum_{i=1}^{M} |\overline{h}_{i0}|^2\lambda_i\Big)\right] \\
&= E_\lambda\left\{E_{\overline{\mathbf{h}}_0}\left[\exp(-\sum_{i=1}^{M} |\overline{h}_{i0}|^2\lambda_i)\right]\right\} \\
&= E_\lambda\left[\prod_{i=1}^{M} E_{\overline{\mathbf{h}}_0}\left[\exp(-|\overline{h}_{i0}|^2\lambda_i)\right]\right] \\
&= E_\lambda\left[\prod_{i=1}^{M} \frac{1}{1+\lambda_i}\right] \\
&= E_\lambda\left[\frac{1}{\det(\mathbf{I} + \overline{\mathbf{R}})}\right] \\
&= E_\lambda\left[\frac{1}{\det\left(\mathbf{I} + \mathrm{diag}(P_{10}, P_{20}, \cdots, P_{\mathrm{M0}})\mathbf{R}^{-1}\right)}\right] \\
&= E_\lambda\left[\frac{\det(\mathbf{R})}{\det\left(\mathrm{diag}(P_{10}, P_{20}, \cdots, P_{\mathrm{M0}}) + \mathbf{R}\right)}\right].
\end{aligned}$$

Since the matrices $\mathrm{diag}(P_{10}, P_{20}, \cdots, P_{\mathrm{M0}})$, $\frac{\sigma^2}{a^2}\mathbf{I}$ and $\mathbf{R}$ are all positive definite we have

$$\det\left(\mathrm{diag}(P_{10}, P_{20}, \cdots, P_{\mathrm{M0}}) + \mathbf{R}\right) \ge \det\left(\mathrm{diag}(P_{10}, P_{20}, \cdots, P_{\mathrm{M0}}) + \frac{\sigma^2}{a^2}\mathbf{I}\right). \qquad (6)$$

Noting that

$$\det\left(\mathrm{diag}(\mathrm{P}_{10}, \mathrm{P}_{20}, \ldots, \mathrm{P}_{\mathrm{M0}}) + \frac{\sigma^2}{a^2}\mathbf{I}\right) = \prod_{i=1}^{M}\left(\frac{\sigma^2}{a^2} + P_{i0}\right), \qquad (7)$$

we get

$$Q_M \le \frac{E[\det(\mathbf{R})]}{\prod_{i=1}^{M}\left(\frac{\sigma^2}{a^2} + P_{i0}\right)}. \qquad (8)$$

Now set

$$\mathbf{X} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ \vdots & \vdots & \ldots & \vdots \\ u_{M1} & u_{M2} & \cdots & u_{MN} \end{pmatrix},$$

then we have $\mathbf{R} = \frac{\sigma^2}{a^2}\mathbf{I} + \mathbf{X}\mathbf{X}^{\mathrm{H}}$ and from a matrix result on determinants of sums we can write

$$\det(\mathbf{R}) = \sum_{\mathrm{j=0}}^{M}\left(\frac{\sigma^2}{a^2}\right)^{(M-j)} \alpha_{\mathrm{j}}, \qquad (9)$$

where $\alpha_0 = 1$ and $\alpha_j,\ j > 0$ denotes the sum of the leading subdeterminants of $\mathbf{XX}^{\mathrm{H}}$ with order j and can be expressed as shown at the bottom of this page, where (h) denotes a permutation of the integers 1, 2, $\cdots$, j. Using Cauchy's formula, we can derive $E[\alpha_j]$ as shown at the bottom of this page, where the second summation over i_r and i_t is over all pairs of distinct integers in $\{1, 2, \cdots, N\}$. Therefore (9) becomes

$$E[\det(\mathbf{R})] = \sum_{j=0}^{M}\left(\frac{\sigma^2}{a^2}\right)^{(M-j)} \times \sum_{1\le k_1<\cdots<k_j\le M}\left\{\sum_{1\le i_r\ne i_t\le N} P_{k_1 i_1}P_{k_2 i_2}\cdots P_{k_j i_j}\right\}, \quad (10)$$

and we have the bound

$$Q_M \le \frac{1}{\prod_{i=1}^{M}\left(\frac{\sigma^2}{a^2}+P_{i0}\right)} \sum_{j=0}^{M}\left(\frac{\sigma^2}{a^2}\right)^{(M-j)} \times \sum_{1\le k_1<\cdots<k_j\le M}\left\{\sum_{1\le i_r\ne i_t\le N} P_{k_1 i_1}P_{k_2 i_2}\cdots P_{k_j i_j}\right\}. \quad (11)$$

IV. SPECIAL CASES

A. Interference-limited System

With no noise ($\sigma^2 = 0$) the bound becomes

$$Q_M \le \frac{\sum_{1\le i_r\ne i_t\le N} P_{1i_1}P_{2i_2}\cdots P_{Mi_M}}{\prod_{i=1}^{M} P_{i0}} = \sum_{1\le i_r\ne i_t\le N} \Gamma_{1i_1}\Gamma_{2i_2}\cdots\Gamma_{Mi_M}, \quad (12)$$

where Γ_{mn} is defined by $\Gamma_{mn} = P_{mn}/P_{m0}$.

Note that the bound in (12) can be rewritten in terms of a permanent [8], namely

$$Q_M \le \mathrm{Per}(\mathbf{\Gamma}), \quad (13)$$

where $\mathbf{\Gamma}$ is the $N \times M$ matrix $(\mathbf{\Gamma})_{ij} = \Gamma_{ji}$. Unfortunately the computation of permanents is #P-complete [12] and so for large values of M, N an exact evaluation of Per($\mathbf{\Gamma}$) becomes difficult due to prohibitive run times. The established method of approaching such problems is to use randomised algorithms and several are available in the literature [6], [7], [10], [18].

$$\alpha_j = \sum_{1\le k_1\cdots<k_j\le M}\left|\begin{pmatrix} u_{k_1 1} & u_{k_1 2} & \cdots & u_{k_1 N}\\ \vdots & \vdots & \cdots & \vdots\\ u_{k_j 1} & u_{k_j 2} & \cdots & u_{k_j N}\end{pmatrix}\begin{pmatrix} u_{k_1 1} & u_{k_1 2} & \cdots & u_{k_1 N}\\ \vdots & \vdots & \cdots & \vdots\\ u_{k_j 1} & u_{k_j 2} & \cdots & u_{k_j N}\end{pmatrix}^{\mathrm{H}}\right|$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_1<\cdots<i_j\le N}\begin{vmatrix} u_{k_1 i_1} & u_{k_1 i_2} & \cdots & u_{k_1 i_N}\\ \vdots & \vdots & \cdots & \vdots\\ u_{k_j i_1} & u_{k_j i_2} & \cdots & u_{k_j i_N}\end{vmatrix}^2\right\}$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_1<\cdots<i_j\le N}\left|\sum_{(h)} u_{k_1 i_{h_1}}u_{k_2 i_{h_2}}\cdots u_{k_j i_{h_j}}\right|^2\right\},$$

$$E[\alpha_j] = \sum_{1\le k_1\cdots<k_j\le M} E\left\{\sum_{1\le i_1<\cdots<i_j\le N}\left|\sum_{(h)} u_{k_1 i_{h_1}}u_{k_2 i_{h_2}}\cdots u_{k_j i_{h_j}}\right|^2\right\}$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_1<\cdots<i_j\le N} E\left|\sum_{(h)}(-1)^t u_{k_1 i_{h_1}}u_{k_2 i_{h_2}}\cdots u_{k_j i_{h_j}}\right|^2\right\}$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_1<\cdots<i_j\le N}\sum_{(h)} E\left|u_{k_1 i_{h_1}}u_{k_2 i_{h_2}}\cdots u_{k_j i_{h_j}}\right|^2\right\}$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_1<\cdots<i_j\le N}\sum_{(h)} P_{k_1 i_{h_1}}P_{k_2 i_{h_2}}\cdots P_{k_j i_{h_j}}\right\}$$
$$= \sum_{1\le k_1\cdots<k_j\le M}\left\{\sum_{1\le i_r\ne i_t\le N} P_{k_1 i_1}P_{k_2 i_2}\cdots P_{k_j i_j}\right\},$$

The microdiversity case.

In microdiversity combining $\Gamma_{mn} = \Gamma_n$ for $m = 1, 2, \cdots, M$ and the bound becomes

$$Q_M \leq \sum_{1\leq i_r \neq i_t \leq N} \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_M}. \quad (14)$$

In contrast to the bound in (13) involving a permanent, the bound in (14) is simple to use and a recursion is available [19], [4] which gives extremely rapid computation. Rewriting (14) as $Q_M \leq D_M$ the recursion is given by

$$D_k = \sum_{i=1}^{k}(-1)^{i-1\,k-1}P_{i-1}S_iD_{k-i},$$

where ${}^{k-1}P_{i-1} = (k-1)(k-2)\cdots(k-i+1)$, $S_i = \sum_{j=1}^{N}\Gamma_j^i$ and $D_1 = S_1$.

In fact for the microdiversity case the SINR density is known exactly [18], [19] and is given by

$$f_{SINR}(z) = \sum_{i=1}^{N}\alpha_i(1+\Gamma_i z)^{-2}, \quad (15)$$

where

$$\alpha_i = (-1)^{N-M}\Gamma_i^{N-M+1}\sum\Gamma_{i_1}\Gamma_{i_2}\Gamma_{i_{M-1}} \times\prod_{j\neq i}(\Gamma_j-\Gamma_i)^{-1}, \quad (16)$$

and the summation in (16) is over all $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{M-1} \leq N$ such that $i_j \neq i$. The value of Q_M therefore is given by

$$Q_M = \int_0^\infty \sum_{i=1}^{N}\alpha_i(1+\Gamma_i z)^{-2}exp(-z)dz,$$

and using a standard integral in [20] we have

$$Q_M = \sum_{i=1}^{N}\alpha_i\Gamma_i^{-2}(\Gamma_i - exp(1/\Gamma_i))E_1(1/\Gamma_i), \quad (17)$$

where $E_1(\cdot)$ denotes the exponential integral. Similar results can be found in [11] where the characteristic function of Z is derived. The use of (17) makes exact computation of Q_M possible for the microdiversity case (see Fig. 2). Note that although Q_M can be computed exactly, the form of (17) does not lead to insights into the relationship between BER and the interferer powers. Here the bound given in (14) may be more useful showing the relationship between performance and products of interferer powers.

B. No Interference

With no interference the bound is found from (8) as

$$Q_M \leq \frac{(\sigma^2/a^2)^M}{\prod_{i=1}^{M}(\sigma^2/a^2+P_{i0})} = \left[\prod_{i=1}^{M}(1+a^2P_{i0}/\sigma^2)\right]^{-1}, \quad (18)$$

which collapses in the microdiversity case to

$$Q_M \leq (1+a^2P_0/\sigma^2)^{-M}, \quad (19)$$

as predicted by standard results.

C. No Diversity (M=1)

With no diversity we let $P_i = P_{1i}$, $\Gamma_i = P_i/P_0$, and (11) gives

$$Q_1 \leq \frac{\sum_{i=1}^{N}P_i}{(\sigma^2/a^2+P_0)} = \frac{\sum_{i=1}^{N}\Gamma_i}{1+\sigma^2/(a^2P_0)}. \quad (20)$$

V. A SEMI-ANALYTIC BOUND FOR THE BER DISTRIBUTION

Consider the output SINR given in (3).

$$\begin{aligned} Z &= \mathbf{u}_0^H\mathbf{R}^{-1}\mathbf{u}_0 \\ &= \mathbf{h}_0^H\text{diag}\left(\sqrt{P_{10}}, \cdots, \sqrt{P_{M0}}\right)\mathbf{R}^{-1} \\ &\quad\times \text{diag}\left(\sqrt{P_{10}}, \cdots, \sqrt{P_{M0}}\right)\mathbf{h}_0 \\ &= \mathbf{h}_0^H\tilde{\mathbf{R}}^{-1}\mathbf{h}_0. \end{aligned}$$

We suppose that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{N-M+1}$ are the non-zero eigenvalues of the matrix $\tilde{\mathbf{R}}$ and $\mathbf{\Phi}_1$ is a unitary matrix which satisfies

$$\mathbf{\Phi}_1\text{diag}(\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{N-M+1}, 0, \cdots, 0)\mathbf{\Phi}_1^H = \tilde{\mathbf{R}}.$$

If we set $(\hat{h}_{1,0}, \hat{h}_{2,0}, \cdots, \hat{h}_{M,0}) = (h_{1,0}, h_{2,0}, \cdots, h_{M,0})\mathbf{\Phi}_1$, then we have

$$Z = \sum_{i=1}^{M}\theta_i^{-1}|\hat{h}_{i,0}|^2.$$

From the Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{i=1}^{M}|\hat{h}_{i,0}|^2 &= \sum_{i=1}^{M}\theta_i^{-1/2}|\hat{h}_{i,0}|\theta_i^{1/2}|\hat{h}_{i,0}| \\ &\leq \sqrt{\left(\sum_{i=1}^{M}\theta_i^{-1}|\hat{h}_{i,0}|^2\right)\left(\sum_{i=1}^{M}\theta_i|\hat{h}_{i,0}|^2\right)}, \end{aligned}$$

and so

$$\sum_{i=1}^{M}\theta_i^{-1}|\hat{h}_{i,0}|^2 \geq \frac{\left(\sum_{i=1}^{M}|\hat{h}_{i,0}|^2\right)^2}{\sum_{i=1}^{M}\theta_i|\hat{h}_{i,0}|^2},$$

which gives

$$Z \geq \frac{\left(\sum_{i=1}^{M}|h_{i,0}|^2\right)^2}{\mathbf{h}_0^H\tilde{\mathbf{R}}\mathbf{h}_0}.$$

Now $\tilde{\mathbf{R}} = \text{diag}(P_{10}{}^{-1/2}\cdots P_{M0}{}^{-1/2})(\frac{\sigma^2}{a^2}\mathbf{I}+\mathbf{X}\mathbf{X}^H)\text{diag}(P_{10}^{-1/2}$ $P_{M0}{}^{-1/2})$ and so defining $\tilde{\mathbf{X}} = \text{diag}(P_{10}^{-1/2}\cdots P_{M0}^{-1/2})\mathbf{X}$ gives

$$Z \geq \frac{\left(\sum_{i=1}^{M}|h_{i0}|^2\right)^2}{\frac{\sigma^2}{a^2}\sum_{i=1}^{M}\frac{|h_{i0}|^2}{P_{i0}} + \mathbf{h}_0^H(\tilde{\mathbf{X}}\tilde{\mathbf{X}}^H)\mathbf{h}_0}. \quad (21)$$

By setting $\mathbf{x}_0^{\mathrm{H}} = \mathbf{h}_0^{\mathrm{H}}\tilde{X} = (x_1, x_2, \cdots, x_N)^{\mathrm{H}}$ it is clear that conditional on $\mathbf{h}_0$, the variables $x_1, x_2, \cdots, x_N$ are independent zero mean complex Gaussian variables with $E[|x_i|^2] = \sum_{j=1}^{M} |h_{j0}|^2 P_{ji}$. Hence if $z_1, z_2, \cdots, z_N$ are i.i.d. zero-mean complex Gaussian random variables we can rewrite $\mathbf{x}_0$ by

$$\mathbf{x}_0^{\mathrm{H}} = (z_1^* \cdots z_N^*) \times \mathrm{diag}\left(\left(\sum_{i=1}^{M} |\mathrm{h_{j0}}|^2 \mathrm{P_{j1}}\right)^{\frac{1}{2}}, \cdots, \left(\sum_{i=1}^{M} |\mathrm{h_{j0}}|^2 \mathrm{P_{jN}}\right)^{\frac{1}{2}}\right).$$

Using this result, (21) becomes

$$Z \geq \left(\sum_{i=1}^{M} |h_{i0}|^2\right)^2 \times \left\{\frac{\sigma^2}{a^2}\sum_{i=1}^{M}\frac{|h_{i0}|^2}{P_{i0}} + \sum_{i=1}^{N}|z_i|^2 \sum_{j=1}^{M} |h_{j0}|^2 P_{ji}\right\}^{-1}, \quad (22)$$

where $h_{10}, \cdots, h_{M0}$ and $z_1, \cdots, z_N$ are all i.i.d.. From (4) we can bound the distribution of the BER in the following way. Consider the outage probability

$$P(\mathrm{BER} \geq \mathrm{BER_0}) = P(Z \leq Z_0),$$

where $Z_0 = \log(a/\mathrm{BER_0})$ then from (22) we have $P(BER \geq BER_0)$ as shown at the bottom of this page.

This probability is intractable, but if we condition on $\{h_{i0}\}$, then it can be computed since it collapses to a weighted sum of exponentials. Let us define the following constants:

$$K = \left(\sum_{i=1}^{M} |h_{i0}|^2\right)^2 - Z_0 \frac{\sigma^2}{a^2}\sum_{i=1}^{M}\frac{|h_{i0}|^2}{P_{i0}}, \quad (23)$$

$$c_i = Z_0 \sum_{j=1}^{M} P_{ji}|h_{j0}|^2, \quad (24)$$

then we have

$$P(\mathrm{BER} \geq \mathrm{BER_0}) \leq P\left(\sum_{i=1}^{N} c_i |z_i|^2 \geq K\right). \quad (25)$$

Assuming the c_i's are all distinct (which occurs with probability 1 as they are continuous random variables) we can express (25) as

$$P(\mathrm{BER} \geq \mathrm{BER_0}) \leq (-1)^{N-1}\sum_{i=1}^{N}\frac{c_i^{N-1}\exp(-2K/c_i)}{\prod_{j\neq i}(c_j - c_i)}, \quad (26)$$

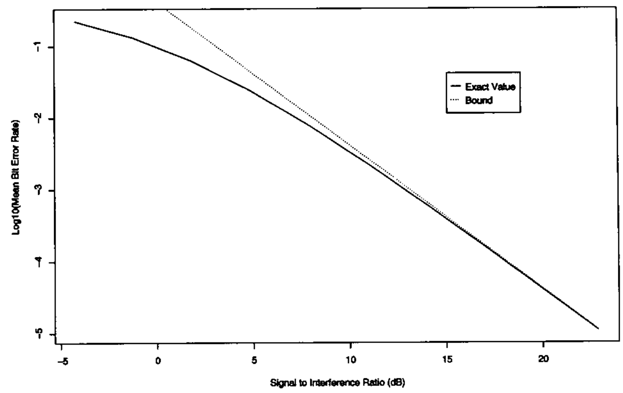

Fig. 2. Microdiversity example.

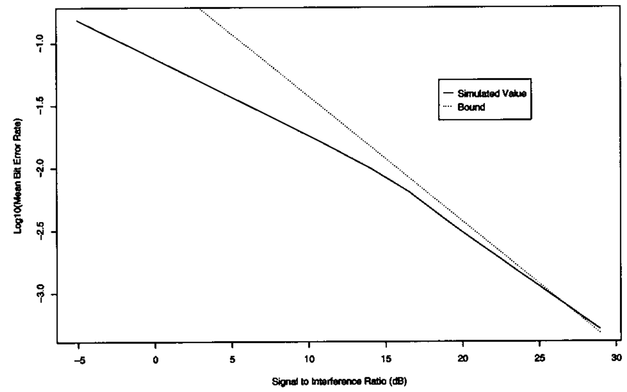

Fig. 3. Macrodiversity example.

or use a published algorithm [21] to compute it. Hence we have a semi-analytic bound in the sense that the bound in (26) must be evaluated by simulating over M variables to give the coefficients. This compares very favourably with direct simulation where $M(N+1)$ variables are required. Hence the saving is MN random number generations for each simulation of the SINR. In simulating P points, with n replicates at each point you save $nPMN$ generations and this embodies a huge runtime reduction.

VI. NUMERICAL EXAMPLES AND CONCLUSIONS

The bound on the mean BER derived in Section III depends critically on the inequality in (6), where $\det(\mathrm{diag}(P_{10}, P_{20}, \cdots P_{M0}) + \sigma^2 \mathrm{I}/a^2 + \mathrm{XX^H})$ is bounded below by $\det(\mathrm{diag}(P_{10}, P_{20} \cdots, P_{M0}) + \sigma^2\mathrm{I}/a^2)$. This bound can be expected to be tight when the terms in $\mathrm{diag}(P_{10}, P_{20}, \cdots, P_{M0})$ and $\sigma^2 I/a^2$ are large compared to those in XX^H. Hence we might predict useful bounds for large signal-plus-noise to interference ratios. In Fig. 2 and Fig. 3, we give examples of this result for DPSK modulation.

$$P(\mathrm{BER} \geq \mathrm{BER_0}) \leq P\left\{\left(\sum_{i=1}^{M}|h_{i0}|^2\right)^2 - Z_0\left[\frac{\sigma^2}{a^2}\sum_{i=1}^{M}\frac{|h_{i0}|^2}{P_{i0}} + \sum_{i=1}^{N}|z_i|^2\sum_{j=1}^{M}|h_{j0}|^2 P_{ji}\right] \leq 0\right\}.$$

Fig. 2 shows the exact mean BER and the bound in Section III for a microdiversity example with $M = 2$ antennas and $N = 6$ interferers ($\sigma^2 = 0, \Gamma_0 = 1, \Gamma_i = \beta^{i-1}\Gamma_0$ for $i = 1, 2, 3, 4, 5, 6$, $\text{SIR} = \left(\sum_{i=1}^{6}\Gamma_i\right)^{-1}$). Varying β allows the SIR to be varied and produces the curves given in Fig. 2. The exact result in Fig. 2 is computed using (17) but in Fig. 3, we have the macrodiversity case and no known analytic solution exists. Hence simulation is used and Fig. 3 shows the simulated mean BER and the bound in Section III for a macrodiversity example with $M = 2$, $N = 6$ and $\sigma^2 = 0$. The powers are given by $P_{10} = P_{20} = 1$ and

$$\mathbf{\Gamma} = \frac{1}{\gamma}\begin{bmatrix} 1 & 2 \\ 1 & 0.5 \\ 0.7 & 0.8 \\ 0.5 & 0.2 \\ 0.2 & 0.1 \\ 0.07 & 0.07 \end{bmatrix}.$$

with $\text{SIR} = 2\gamma\left(\sum_{i=1}^{6}\sum_{j=1}^{2}\Gamma_{ij}\right)^{-1}$. The constant γ is altered to let the SIR vary between 0 and 30dB. The purpose of this example is to investigate the bound when the columns in $\mathbf{\Gamma}$ are not identical (as in the microdiversity case) but can vary markedly. In the given example deviations up to 150% can occur.

As expected the bounds perform well in the high signal-plus-noise to interference region. In these examples there is no additive noise so a high SIR is necessary for the bound to be tight. In practice the bound is useful up to mean BERs of 10^{-2} and performs well up to 10^{-3}.

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