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EXISTENCE OF THE THIRD POSITIVE RADIAL SOLUTION OF A SEMILINEAR ELLIPTIC PROBLEM ON AN UNBOUNDED DOMAIN

BONG SOO KO† AND YONG-HOON LEE‡

ABSTRACT. We prove the multiplicity of ordered positive radial solutions for a semilinear elliptic problem defined on an exterior domain. The key argument is to prove the existence of the third solution in presence of two known solutions. For this, we obtain some partial results related to three solutions theorem for certain singular boundary value problems. Proof are mainly based on the upper and lower solutions method and degree theory.

1. Introduction

In this paper, we prove the existence of $2N-1$ distinct ordered positive radial solutions for the following problem:

$$(P_\lambda) \qquad \begin{cases} \triangle u + \lambda g(|x|) f(u) = 0 & \text{in } \Omega = \mathbf{R}^n \setminus B(0, r_o), \\ \lim_{|x| \to \infty} u(x) = 0, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive real parameter, $B(0, r_o)$ is an open ball centered at 0 with the radius r_o, $n \geq 3$. Assume that $f \in C(I, \mathbf{R})$, $I \subset \mathbf{R}$ and $g \in C^1([r_o, \infty), \mathbf{R}^+))$ satisfy the following conditions:

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(f_0) there exists a positive constant M such that

$$f(u) - f(v) \geq -M(u-v) \quad \text{if } u, v \in I,\ u \geq v.$$

(f_1) There exist exactly N positive numbers $0 < a_1 < a_2 < \cdots < a_N$ such that $f(a_i) = 0$ for all $i = 1, \cdots, N$.

(f_2) $[0, a_N] \subset I$ and there exists a positive constant K_f such that

$$f'(a_i) \leq -K_f \quad \text{for all } i = 1, \cdots, N.$$

(f_3) $\int_s^{a_i} f(u)du > 0$ for all $s \in [0, a_i)$ and for all $i = 1, \cdots, N$.

(g) $\int_{r_o}^{\infty} rg(r)dr < \infty$.

If Ω is a bounded open domain with smooth boundary, then there have been several studies ([3], [4], [6], [7], [10], [11]) which prove generally that there exists $\lambda_o > 0$ such that if $\lambda > \lambda_o$, then problem (P_λ) has at least $2N-1$ ordered positive solutions. In those studies, the boundedness of Ω is crucial. If Ω is unbounded, we might not use the compactness on the operator or the functional which are induced from the problem (P_λ). So the question of the multiplicity of positive solutions is nontrivial.

To obtain the radial solutions, we rewrite problem (P_λ), via transformations $r = |x|,\ s = r^{2-n}$ and $t = \frac{r_o^{2-n} - s}{r_o^{2-n}}$ ([8]), as

$$(1_\lambda) \qquad \begin{cases} u'' + \lambda q(t) f(u) = 0, & 0 < t < 1 \\ u(0) = 0 = u(1), \end{cases}$$

where q is given by

$$q(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} g(r_o(1-t)^{\frac{-1}{n-2}}).$$

Transforming (P_λ) to (1_λ), we see that the coefficient function q in (1_λ) is of $C[0,1]$ if g in (P_λ) satisfies $\lim_{r\to\infty} r^{2(n-1)}g(r) < \infty$ and q is singular at $t = 1$ if $\lim_{r\to\infty} r^{2(n-1)}g(r) = \infty$. Since the result for the regular case can be proved by exactly the same way as the singular case with less restrictions, our interest will be mainly focused at the singular case and, thus, a crucial step for our goal will be to establish a theorem of existence of the third solution for problem (1_λ) in the case with two pairs of upper and lower solutions.

More precisely, suppose that $\bar{v}$, $\tilde{v}$, and $\bar{w}$, $\tilde{w}$ are pairs of lower solutions and upper solutions of (1) such that $\bar{v}(t) \leq \tilde{w}(t)$, $\bar{v}(t) \leq \tilde{v}(t)$, $\bar{w}(t) \leq \tilde{w}(t)$ for all $t \in [0,1]$ and $\bar{w}(t_0) > \tilde{v}(t_0)$ for some $t_0 \in [0,1]$. If q is Hölder continuous on the interval $[0,1]$, then there is a solution in the ordered interval $[\bar{v}, \tilde{v}] = \{u \in C([0,1]) : \bar{v}(t) \leq u(t) \leq \tilde{v}(t),\ t \in [0,1]\}$ and a solution in $[\bar{w}, \tilde{w}]$. And furthermore it is known that there exists a third solution in the set $[\bar{v}, \tilde{w}] \setminus ([\bar{v}, \tilde{w}] \cup [\bar{w}, \tilde{w}])$. ([1], [11])

In the case that q is singular at $t = 0$ and/or 1, the existence of solution given a pair of lower solution and upper solution, $\bar{v}$ and $\tilde{v}$, with $\bar{v}(t) \leq \tilde{v}(t)$ for all $t \in [0,1]$, is known under additional conditions ([8], [9]). Hence, there arises a question, does there exist the third solution if there are pairs of lower solutions and upper solutions as in the preceding paragraph? As far as the authors know, the complete answer has not been made yet and some partial results for singular problems have been studied by Parter [13] and Ben-Naoum and De Coster [2], but neither are applicable in problem (P_λ). Because. in [2] we can understand necessity of strict sense of upper and lower solutions somehow. We also note that the singular function q in [13] plays the role of the coefficient of damping term. The authors are able to get another partial answer for the multiplicity for (1_λ) using Leray-Schauder degree and Green's function.

This paper is organized as follows: In Section 2, we introduce fundamental theorems based on the method of upper and lower solutions. In Section 3, we give a theorem of existence of the third solution for singular problem (1_λ). In section 4, we study the multiplicity of positive radial solutions for problem (P_λ) on an exterior domain $\mathbf{R}^n \setminus B(0,r)$.

2. Fundamental theorems for G-upper and G-lower solutions

In this section, we prove a fundamental existence theorem in terms of general upper and lower solutions for singular boundary value problems of the form;

$$(2) \qquad \begin{cases} u'' + f(t,u) = 0, & 0 < t < 1 \\ u(0) = A, \ u(1) = B, \end{cases}$$

where $f : D \subset (0,1) \times \mathbf{R} \to \mathbf{R}$ is continuous. We denote $(0,\infty)$ by $\mathbf{R}^+$.

A solution $u(\cdot)$ means a function $u \in C[0,1] \cap C^2(0,1)$ such that

$(t,u(t)) \in D$ for all $t \in (0,1)$ and $u''(t) + f(t,u(t)) = 0$ for all $t \in (0,1)$ with $u(0) = A$ and $u(1) = B$.

DEFINITION 1. $\alpha \in C[0,1] \cap C^2(0,1)$ is called a *lower solution* of (2) if $(t,\alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{aligned} &\alpha''(t) + f(t,\alpha(t)) \geq 0, \\ &\alpha(0) \leq A,\ \alpha(1) \leq B. \end{aligned}$$

Similarly, $\beta \in C[0,1] \cap C^2(0,1)$ is called an *upper solution* of (2) if the above inequalities are reversed.

If $\alpha,\ \beta \in C[0,1]$ are such that $\alpha(t) \leq \beta(t)$ for all $t \in [0,1]$, we define the set

$$D_\alpha^\beta = \{(t,u) \in (0,1) \times \mathbf{R} : \alpha(t) \leq u \leq \beta(t)\}.$$

The well-known fundamental theorem on upper and lower solutions method for problem (2) is as follows;

THEOREM 1. ([9]) *Let $\alpha,\ \beta$ be, respectively, a lower solution and an upper solution of* (2) *such that*
(a_1) $\alpha(t) \leq \beta(t)$ *for all* $t \in [0,1]$.
(a_2) $D_\alpha^\beta \subset D$.
Assume also that there is a function $h \in C\big((0,1),\mathbf{R}^+\big)$ *such that*
(a_3) $|f(t,u)| \leq h(t)$ *for all* $(t,u) \in D_\alpha^\beta$ *and*
(a_4) $\int_0^1 s(1-s)h(s)ds < \infty$.
Then problem (2) *has at least one solution* $u(\cdot)$ *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for all* $t \in (0,1)$.

We give definitions of somewhat general type of upper and lower solutions.

DEFINITION 2. We say that a continuous function $\alpha(\cdot) : [0,1] \to \mathbf{R}$ is a *G-lower solution* of (2) if $\alpha \in C^2(0,1)$ except at finite points $\tau_1, \cdots, \tau_n$ with $0 < \tau_1 < \cdots < \tau_n < 1$ such that

$$(L_1) \qquad \begin{aligned} &\text{at each}\quad \tau_i,\quad \text{there exist}\,\alpha'(\tau_i{}^-),\ \alpha'(\tau_i{}^+) \text{ such that} \\ &\alpha'(\tau_i{}^-) < \alpha'(\tau_i{}^+) \quad \text{and} \end{aligned}$$

$$(L_2) \qquad \begin{cases} \alpha''(t) + f(t, \alpha(t)) \geq 0 \ \text{ for all } t \in (0,1) \setminus \{\tau_1, \cdots, \tau_n\}, \\ \alpha(0) \leq A, \ \alpha(1) \leq B. \end{cases}$$

We also say that a continuous function $\beta(\cdot) : [0,1] \to \mathbf{R}$ is a *G-upper solution* of (2) if $\beta \in C^2(0,1)$ except at finite points $\sigma_1, \cdots, \sigma_m$ with $0 < \sigma_1 < \cdots < \sigma_m < 1$ such that

(U_1) at each σ_i, there exist $\beta'(\sigma_j{}^-)$, $\beta'(\sigma_j{}^+)$ and $\beta'(\sigma_j{}^-) > \beta'(\sigma_j{}^+)$ and

$$(U_2) \qquad \begin{cases} \beta''(t) + f(t, \beta(t)) \leq 0 \ \text{ for all } t \in (0,1) \setminus \{\sigma_1, \cdots, \sigma_m\}, \\ \beta(0) \geq A, \ \beta(1) \geq B. \end{cases}$$

The fundamental theorem on G-upper and G-lower solutions is given as follows. The proof is basically similar to that of [9] and we give it for reader's convenience.

THEOREM 2. *Let α and β be, respectively, a G-lower solution and a G-upper solution of (2) satisfying (a_1) and (a_2) in Theorem 1. Also assume (a_3) and (a_4). Then (2) has at least one solution u such that*

$$\alpha(t) \leq u(t) \leq \beta(t) \ \text{ for all } t \in (0,1).$$

Proof. Define a modified function of f as follows;

$$F(t,u) = \begin{cases} f(t, \beta(t)) - \dfrac{u - \beta(t)}{1+u^2} & \text{if } u > \beta(t), \\ f(t,u) \ \ \text{if } \alpha(t) \leq u \leq \beta(t), & \\ f(t, \alpha(t)) - \dfrac{u - \alpha(t)}{1+u^2} & \text{if } u < \alpha(t). \end{cases}$$

Then $F : (0,1) \times \mathbf{R} \to \mathbf{R}$ is continuous and

$$(3) \qquad |F(t,u)| \leq m(\alpha, \beta) + h(t)$$

for all $(t,u) \in (0,1) \times \mathbf{R}$, where $m(\alpha, \beta) = \|\alpha\|_\infty + \|\beta\|_\infty + 1$.

Consider the problem

$$(4) \qquad \begin{cases} u'' + F(t,u) = 0, \quad 0 < t < 1, \\ u(0) = A, \ u(1) = B. \end{cases}$$

We claim that any solution u of (4) satisfies $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0,1]$. Suppose, by contradiction, that $\alpha \not\leq u$. So let $(\alpha - u)(t_o) = \max_{t\in[0,1]}(\alpha - u)(t) > 0$. If $t_o \in (0,1)\setminus\{\tau_1, \cdots, \tau_n\}$. Then $(\alpha - u)''(t_o) \leq 0$. Since $u(t_o) < \alpha(t_o)$,

$$\begin{aligned} 0 &\geq (\alpha - u)''(t_o) = \alpha''(t_o) + F(t_o, u(t_o)) \\ &= \alpha''(t_o) + f(t_o, \alpha(t_o)) - \frac{u(t_o) - \alpha(t_o)}{1 + u^2(t_o)} \\ &\geq \frac{\alpha(t_o) - u(t_o)}{1 + u^2(t_o)} > 0, \end{aligned}$$

a contradiction.

If $t_o = \tau_i$ for some $i = 1, \cdots, n$, then since $\alpha - u$ attains its positive maximum at τ_i,

$$(\alpha - u)'(\tau_i^-) \geq 0 \ \text{ and } \ (\alpha - u)'(\tau_i^+) \leq 0.$$

Thus

$$\begin{aligned} 0 &\leq (\alpha - u)'(\tau_i^-) - (\alpha - u)'(\tau_i^+) \\ &= \alpha'(\tau_i^-) - \alpha'(\tau_i^+). \end{aligned}$$

This leads to a contradiction to the definition of G-lower solution.

Let $t_o = 0$ or 1

$$\begin{aligned} 0 &< (\alpha - u)(0) = \alpha(0) - A \leq 0, \\ 0 &< (\alpha - u)(1) = \alpha(0) - B \leq 0, \end{aligned}$$

a contradiction. Therefore $\alpha(t) \leq u(t) \leq \beta(t)$, and so we can conclude u is a solution of (2). We can prove for the case that $u \not\leq \beta$ by a similar fashion. We claim that (4) has at least one solution. It is well-known that problem (4) is equivalently written as

$$u = Tu \ \text{ on } X = C[0,1],$$

where

$$Tu(t) = A + (B - A)t + \int_0^1 G(t,s)F(s,u(s))ds$$

and $G(t,s)$ is the Green's function explicitly written as

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases}$$

By (3) and (a_4), $T : X \to X$ is well-defined, continuous and TX is bounded. If T is a compact operator, then the proof of the existence of a solution is done by Schauder fixed point Theorem. To show T compact, making use of Arzela-Ascoli Theorem, it suffices to show that TX is equicontinuous. Let $t \in (0,1)$, then by (3) we get

$$\begin{aligned} &|\frac{d}{dt}Tu(t)| \\ &\le |B-A| + \int_0^t s|F(s,u(s))|ds + \int_t^1 (1-s)|F(s,u(s))|ds \\ &\le |B-A| + \frac{m(\alpha,\beta)}{2}(t^2 + (1-t)^2) + \int_0^t sh(s)ds + \int_t^1 (1-s)h(s)ds \\ &\triangleq |B-A| + \frac{m(\alpha,\beta)}{2}(t^2 + (1-t)^2) + \gamma(t). \end{aligned}$$

If $\gamma \in L^1(0,1)$, then the proof follows from that

$$\begin{aligned} \int_0^1 |\gamma(s)|ds \le& \lim_{t\to 1^-}(1-t)\int_0^t sh(s)ds + \lim_{t\to 0^+} t\int_t^1 (1-s)h(s)ds \\ &+ 2\int_0^1 s(1-s)h(s)ds \\ \le& 4\int_0^1 s(1-s)h(s)ds < \infty. \end{aligned}$$

□

REMARK 1. It is easy to see that if we replace (a_4) by the condition $\int_0^1 sh(s)ds < \infty$, then the solution u which we have found belongs to $C^1\big((0,1]\big)$. Similarly, if $\int_0^1 (1-s)h(s)ds < \infty$, then $u \in C^1\big([0,1)\big)$.

3. Multiplicity

In this section, we prove the existence of three solutions for the problem

$$(5) \qquad \begin{cases} u'' + q(t)f(u) = 0, & 0 < t < 1, \\ u(0) = 0 = u(1), \end{cases}$$

under certain assumptions for two pairs of G-lower solutions and G-upper solutions. In what follows, we assume that $q \in C\big((0,1), \mathbf{R}^+\big)$ is singular at $t = 0$ and/or 1, and $f : I \subset \mathbf{R} \to \mathbf{R}$ is continuous. We know that problem (5) is equivalent to

$$u = Tu \quad \text{on } C[0,1],$$

where T is given by

$$Tu(t) = \int_0^1 G(t,s)q(s)f(u(s))ds$$

and $G(t,s)$ is the Green's function given in the proof of Theorem 2.

It is well-known ([5], [8]) that T is completely continuous on $C[0,1]$ if q satisfies

$$(\mathrm{C}) \qquad \int_0^1 s(1-s)q(s)ds < \infty.$$

Definition 3. For any $u,\ v \in C[0,1]$, $u \prec v$ means that $u(t) \le v(t)$, for all $t \in [0,1]$ but $u \ne v$ on $[0,1]$.

Let $C_0[0,1] = \{u \in C[0,1] : u(0) = 0 = u(1)\}$. If q satisfies (C), then the problem

$$(6) \qquad \begin{cases} u'' + q(t) = 0, & 0 < t < 1, \\ u(0) = 0 = u(1) \end{cases}$$

has the unique solution $e \in C[0,1] \cap C^2(0,1)$. Let

$$C_e[0,1] = \{u \in C_0[0,1] : \gamma e(t) \le u(t) \le \delta e(t),\ t \in [0,1] \text{ and } \gamma, \delta \in \mathbf{R}\}.$$

Then $C_e[0,1]$ is a linear subspace of $C[0,1]$. For any $u \in C_e[0,1]$, we define a norm

$$\|u\|_e = \inf\{\lambda > 0 : -\lambda e(t) \leq u \leq \lambda e(t), t \in [0,1]\}.$$

Then $C_e[0,1]$ is a Banach space with respect to the norm $\|\cdot\|_e$ ([1]). We claim that

$$T(C([0,1])) \subset C_e[0,1].$$

Indeed, for any $u \in C[0,1]$,

$$\begin{aligned}(Tu - \delta e)''(t) &= -q(t)f(u(t)) + \delta q(t) \\ &= q(t)[-f(u(t)) + \delta],\end{aligned}$$

for all $t \in (0,1)$. Since $Tu(0) - \delta e(0) = 0 = Tu(1) - \delta e(1)$, if we choose $\delta > 0$ sufficiently large so that $(Tu - \delta e)''(t) \geq 0$, then

$$Tu \leq \delta e.$$

By a similar method, we can show the existence of constant γ such that

$$\gamma e \leq Tu.$$

Therefore,

$$Tu \in C_e[0,1].$$

From now on, we only consider the operator $T : C_e[0,1] \to C_e[0,1]$. Since the embedding $C_e[0,1] \hookrightarrow C[0,1]$ is continuous and $T : C[0,1] \to C_e[0,1]$ is also completely continuous, so is T on $C_e[0,1]$. ([1])

We now state and prove maximum principle for certain linear operators.

LEMMA 1. *Let $q \in C\big([a,b],(0,\infty)\big)$, and $K > 0$ given. If $u \in C^2[a,b]$ satisfies*

$$\begin{aligned}&u'' - Kq(t)u \geq 0, \quad a < t < b, \\ &u(a) \leq 0, \ u(b) \leq 0.\end{aligned}$$

Then $u(t) \leq 0$ for all $t \in [a,b]$.

Assume, moreover, that there exists $\tau \in (a,b)$ such that $u(\tau) = 0$. Then $u(t) = 0$ for all $t \in [a,b]$.

LEMMA 2. *Let $q \in C\big((a,b),(0,\infty)\big)$ may be singular at $t=a$ and/or b and $K>0$ given. If $u \in C[a,b]\cap C^2(a,b)$ satisfies*

$$\begin{aligned} &u'' - Kq(t)u \geq 0, \quad a<t<b,\\ &u(a)\leq 0, \ u(b)\leq 0. \end{aligned}$$

Then $u(t)\leq 0$ for all $t\in[a,b]$.

Assume, moreover, that there exists $\tau \in (a,b)$ such that $u(\tau)=0$. Then $u(t)=0$ for all $t\in[a,b]$.

Here we give the proof of Lemma 2. Lemma 1 is well known. ([14])

Proof. It is not hard to show that for $h \in C\big((a,b),[0,\infty)\big)$ $u \in C[a,b]\cap C^2(a,b)$ is a solution of

$$\begin{aligned} &u'' - Kq(t)u = h(t), \quad a<t<b,\\ &u(a)=A, \ u(b)=B, \end{aligned}$$

then u satisfies

$$u(t) \leq \max\Big\{A, B, \frac{-\inf_{t\in(a,b)} h(t)}{K\inf_{t\in(a,b)} q(t)}\Big\}.$$

This fact implies the first part of Lemma 2. We also notice that if $v\in C[t_1,t_2]\cap C^2(t_1,t_2)$ satisfies

$$v''(t) - Kq(t)v(t) > 0 \ \text{ for } t_1<t<t_2,$$

then v cannot have a nonnegative relative maximum on (t_1,t_2).

We now prove the second part by contradiction. Suppose, without loss of generality, that there exists $t_o \in (\tau,b)$ such that $u(t_o)<0$. For $\alpha>0$, let $z(t)=e^{\alpha(t-\tau)}-1$, then $z(t)>0,=0$ or <0 according to $t>\tau$, $t=\tau$ or $t<\tau$, respectively. Take ϵ with $0<\epsilon<-\frac{u(t_o)}{z(t_o)}$ and let $v(t)=u(t)+\epsilon z(t)$. Then

$$v''(t) - Kq(t)v(t) = u''(t) - Kq(t)u(t) + \epsilon[z''(t) - Kq(t)z(t)],$$

and we know that $u''(t)-Kq(t)u(t)\geq 0$.
If $t<\tau$, then $z''(t)-Kq(t)z(t)>0$, and thus $v''(t)-Kq(t)v(t)>0$ for all $t\in(a,\tau)$.

If $t = \tau$, then $z''(t) - Kq(t)z(t) = z''(t) > 0$, and thus $v''(t) - Kq(t)v(t) > 0$ for all $t \in (a, \tau)$. We notice that $z''(t) - Kq(t)z(t) = e^{\alpha(t-\tau)}[\alpha^2 - Kq(t)] + Kq(t)$, and $Kq(t) > 0$ for all $t \in (a, b)$.
If $t \in (\tau, t_o)$, consider an interval $J = [\tau - \delta, t_o]$ for some $\delta > 0$ with $\tau - \delta > a$. In this case, if $t \in J$ and $t > \tau$, then taking $\alpha^2 > Kq_m$, where $q_m = \max_J q(t)$, $\alpha^2 - Kq(t) \geq \alpha^2 - Kq_m > 0$. Thus $v''(t) - Kq(t)v(t) > 0$. Consequently,

$$v''(t) - Kq(t)v(t) > 0 \quad \text{for all } t \in J.$$

On the other hand, $v(\tau - \delta) < 0$ if δ is sufficiently small and $v(t_o) < 0$, but $v(\tau) = u(\tau) = 0$. This implies that the maximum of v on J occurs on the interior of J and is nonnegative. This leads to a contradiction by well known Maximum Principles and completes the proof. □

The following lemma seems to be well known. However, we could not find any references for that, and so we prove it.

LEMMA 3. *Assume* (C) *and* (f_0). *Suppose that* $\bar{v}$ *and* $\tilde{v}$ *are a G-lower solution and a G-upper solution satisfying* $\bar{v} \prec \tilde{v}$ *and none of them are solutions of* (5). *Let* w *be a solution of* (5) *satisfying* $\bar{v} \prec w \prec \tilde{v}$. *Then* w *satisfies*

$$\bar{v}(t) < w(t) < \tilde{v}(t) \textit{ for all } t \in (0, 1).$$

Proof. Let w be a solution of problem (5) satisfying

$$\bar{v}(t) \leq w(t) \leq \tilde{v}(t) \quad \text{for all} \quad t \in (0, 1).$$

We show that $\bar{v}(t) < w(t)$ for all $t \in (0, 1)$, and the other inequality can be shown by a similar way.
(i) At τ_i, $i = 1, \cdots, n$.
Suppose that $\bar{v}(\tau_i) = w(\tau_i)$ for some i, then $(\bar{v} - w)(\tau_i) = \max_{t \in [0,1]}(\bar{v} - w)(t)$, and $(\bar{v} - w)'(\tau_i^-) \geq 0$, $(\bar{v} - w)'(\tau_i^+) \leq 0$. Thus $\bar{v}'(\tau_i^-) - \bar{v}'(\tau_i^+) \geq 0$, and this contradiction shows that

$$\bar{v}(\tau_i) < w(\tau_i) \quad \text{for all } i = 1, \cdots, n.$$

(ii) On (τ_i, τ_{i+1}), $i = 1, \cdots, n - 1$.

We get

$$\begin{aligned}&(\bar{v}-w)''(t)-Kq(t)(\bar{v}-w)(t)\\ \geq\ & -q(t)[f(\bar{v}(t))-f(w(t))]-Kq(t)(\bar{v}(t)-w(t))\\ \geq\ & -Mq(t)(w(t)-\bar{v}(t))-Kq(t)(\bar{v}(t)-w(t))\\ =\ &(K-M)q(t)(w(t)-\bar{v}(t))\geq 0\quad \forall t\in(\tau_i,\tau_{i+1})\end{aligned}$$

for sufficiently large K. By (i) we get

$$(\bar{v}-w)(\tau_i)<0,\ (\bar{v}-w)(\tau_{i+1})<0.$$

Now suppose that there exists $\tau\in(\tau_i,\tau_{i+1})$ such that $\bar{v}(\tau)=w(\tau)$, then by Lemma 1, $\bar{v}\equiv w$ on $[\tau_i,\tau_{i+1}]$ and this contradicts to (i).

(iii) On $(0,\tau_1)$.

We get

$$\begin{aligned}&(\bar{v}-w)''(t)-Kq(t)(\bar{v}-w)(t)\geq 0\ \text{ on } (0,\tau_1),\\ &(\bar{v}-w)(0)\leq 0,\quad (\bar{v}-w)(\tau_1)<0.\end{aligned}$$

Suppose that there exists $\tau\in(0,\tau_1)$ such that $\bar{v}(\tau)=w(\tau)$, then by Lemma 2, $\bar{v}\equiv w$ on $[0,\tau_1]$ and this also contradicts to (i). We can show similarly that $\bar{v}(t)-w(t)<0$ for all $t\in[\tau_n,1)$. Consequently,

$$\bar{v}(t)<w(t)\ \text{ for all } t\in(0,1). \qquad \square$$

We first give an existence theorem of the third solution when q is integrable. In this case, the solutions of problem (5) and the solution e of problem (6) are of $C^1([0,1])$ and Theorem 1 and Theorem 2 are obviously make use of. We furthermore, give a restriction on the coefficient function q so that $\lim_{t\to 0^+} tq(t)$ and $\lim_{t\to 1^-}(1-t)q(t)$ exist. For example, if q is decreasing near 0 and increasing near 1, then the limits exist by the Monotone Convergence Theorem. Generally, if the limits exist, then they are 0, since q is integrable.

THEOREM 3. *Assume* (f_0) *and*

(H_0) $q\in L^1(0,1)\cap C(0,1)$ *and both* $\lim_{t\to 0^+} tq(t)$ *and* $\lim_{t\to 1^-}(1-t)q(t)$ *exist.*

Suppose also that there exist two pairs of G-lower and G-upper solutions $\{\bar{v},\tilde{v}\}$ *and* $\{\bar{w},\tilde{w}\}$ *such that* $\bar{v}\prec\tilde{v}$, $\bar{w}\prec\tilde{w}$, $\bar{v}\prec\tilde{w}$, $\bar{w}\not\preceq\tilde{v}$, *none of them are solutions of* (5), *and* $[\bar{v}(t),\tilde{w}(t)]\subset I$ *for all* $t\in[0,1]$. *Then* (5) *has at least 3 distinct solutions* $u_1\prec u_2\prec u_3$ *such that* $u_1\in[\bar{v},\tilde{v}]$, $u_2\in[\bar{w},\tilde{w}]$ *and* $u_3\in[\bar{v},\tilde{w}]\setminus([\bar{v},\tilde{v}]\cup[\bar{w},\tilde{w}])$, *where* $[v,w]=\{u\in C_e[0,1]: v\prec u\prec w\}$.

Proof. Let $\bar{v}$ and $\tilde{v}$ be a G-lower solution and a G-upper solution of (5) and let

$$\Omega = \{u \in C_e[0,1] : \bar{v}(t) \le u(t) \le \tilde{v}(t),\ t \in [0,1]\},$$
$$\Omega_1 = \{u \in C_e[0,1] : \bar{v}(t) < u(t) < \tilde{v}(t),\ t \in (0,1)\}.$$

We denote Ω_1°, the interior of Ω_1 in $C_e([0,1])$. We know by Theorem 2 and Lemma 3 that there exists a solution w of problem (5) satisfying $w \in \Omega_1$. Since $w, e \in C^1(0,1)$, it is obvious that $w \in C_e([0,1])$. We claim that $w \in \Omega_1^\circ$. For this, we first show that there exists $\epsilon_1 > 0$ such that

$$w - \bar{v}(t) > \epsilon_1 e(t) \text{ for all } t \in (0,1). \tag{7}$$

The function $w - \bar{v}$ is of $C[0,1]$ and $(w - \bar{v})(t) > 0$ for all $t \in (0,1)$. On the other hand, the function e is of $C^1[0,1]$, $e > 0$ and concave on $(0,1)$ so that

$$0 < e'(0) < \infty, \quad -\infty < e'(1) < 0.$$

If $\bar{v}(t) < 0$ for $t = 0$ and 1, then $w - \bar{v}(t) > 0$ for all $t \in [0,1]$ and (7) is easily verified.

Let $\bar{v}(t) = 0$ for $t = 0$ or 1. Assume $\bar{v}(0) = 0$ and $\bar{v}(d) - w(d) < 0$ for some $d \in (0, \tau_1)$. Then for given $\alpha > 0$, we may choose a positive numbers ϵ so that

$$0 < \epsilon < \frac{w(d) - \bar{v}(d)}{\exp(\alpha d) - 1}.$$

Let us define the function z by

$$z(t) = \exp(\alpha t) - 1.$$

Then

$$\begin{aligned}&(\bar{v} - w + \epsilon z)''(t) - Kq(t)(\bar{v} - w + \epsilon z)(t)\\ &= (\bar{v} - w)''(t) - Kq(t)(\bar{v} - w)(t) + \epsilon(z''(t) - Kq(t)z(t)).\end{aligned}$$

As we see in the proof of Lemma 3,

$$(\bar{v} - w)''(t) - Kq(t)(\bar{v} - w)(t) > 0 \text{ for all } t \in (0, \tau_1)$$

if $K > M$.

On the other hand, (H_0) implies that $tq(t)$ is bounded on $(0,c]$, for arbitrarily fixed $c \in (0,1)$, thus for $\alpha > K\sup\{tq(t) : t \in (0,\tau_1]\}$ and $t \in (0,d)$, we get the following inequalities by Mean Value Theorem;

$$\begin{aligned} z''(t) - Kq(t)z(t) &= \exp(\alpha t)\big(\alpha^2 - Kq(t)(1-\exp(-\alpha t))\big) \\ &= \exp(\alpha t)\big(\alpha^2 - \alpha K\exp(-\alpha\xi)tq(t)\big) \\ &\geq \alpha\exp(\alpha t)(\alpha - Ktq(t)) \geq 0 \end{aligned}$$

for some $\xi \in (0,t)$. Therefore,

$$(\bar{v} - w + \epsilon z)''(t) - Kq(t)(\bar{v} - w + \epsilon z)(t) \geq 0$$

for all $t \in (0,d)$ and $(\bar{v} - w + \epsilon z)(0) = 0$ and $(\bar{v} - w + \epsilon z)(d) < 0$. Lemma 2 implies that the function $\bar{v} - w + \epsilon z$ has the zero maximum value at $t = 0$ in the interval $[0,d]$. Thus

$$\limsup_{t\to 0^+} \frac{(\bar{v} - w + \epsilon z)(t)}{t} = \limsup_{t\to 0^+} \frac{\bar{v}(t) - w(t)}{t} + \epsilon\alpha \leq 0$$

and consequently

$$\liminf_{t\to 0^+} \frac{w(t) - \bar{v}(t)}{t} \geq \epsilon\alpha > 0.$$

For the case $\bar{v}(1) = 0$, defining $z(t) = \exp(\alpha(1-t)) - 1$ and taking $\alpha > K\sup\{(1-t)q(t) : t \in [\tau_n, 1)\}$, we obtain

$$\liminf_{t\to 1^-} \frac{w(t) - \bar{v}(t)}{1-t} > 0.$$

Combining this fact and properties of $w - \bar{v}$ and e described above, we get (7). Similarly, for the functions $\tilde{v} - w$ and e, there exists $\epsilon_2 > 0$ such that

$$\tilde{v} - w(t) > \epsilon_2 e(t) \quad \text{for all } t \in (0,1). \tag{8}$$

Therefore, by (7) and (8), we obtain $w \in \Omega_1^\circ$ and, thus, $\Omega_1^\circ \neq \emptyset$. If $u \in \partial_{C_e}\Omega_1^\circ$, the boundary of Ω_1° with respect to C_e-topology and satisfies $u = Tu$. Then $u \in \Omega$, since $\partial_{C_e}\Omega_1^\circ \subset \Omega$ by the continuous embedding of $C_e[0,1]$ into $C[0,1]$. This implies that u is a solution of problem (5) with

$u \in \Omega$ and thus, by the above argument, $u \in \Omega_1^\circ$ and this contradiction shows that the Leray-Schauder degree $d_{LS}(I - T, \Omega_1^\circ, 0)$ is well defined.

Now we compute $d_{LS}(I-T, \Omega_1^\circ, 0)$. Let us consider the modified problem

$$u'' + F(t,u) = 0, \quad 0 < t < 1$$
$$u(0) = 0 = u(1),$$

where

$$F(t,u) = \begin{cases} q(t)f(\tilde{v}(t)) - \dfrac{u - \tilde{v}(t)}{1+u^2} & \text{if } u > \tilde{v}(t) \\ q(t)f(u) \quad \text{if } \bar{v}(t) \le u \le \tilde{v}(t) \\ q(t)f(\bar{v}(t)) - \dfrac{u - \bar{v}(t)}{1+u^2} & \text{if } u < \bar{v}(t). \end{cases}$$

Then $F : (0,1) \times \mathbf{R} \to \mathbf{R}$ is continuous. Define the operator

$$\tilde{T}u(t) = \int_0^1 G(t,s)F(s,u(s))ds.$$

Then $\tilde{T}$ is completely continuous on $C_e[0,1]$. Choose a positive real constant δ as

$$\delta > \max\left\{ \sup_{\min \bar{v} \le u \le \max \tilde{v}} f(u), \frac{\|\bar{v}\|_\infty + 1}{q_o} + \|f \circ \bar{v}\|_\infty, \frac{\|\tilde{v}\|_\infty + 1}{q_o} + \|f \circ \tilde{v}\|_\infty \right\}$$

where $q_o = \inf_{t \in (0,1)} q(t) (> 0)$. Then

$$-\delta e \prec \tilde{T}(u) \prec \delta e \text{ on } [0,1] \text{ for all } u \in C[0,1].$$

Thus $\text{Im}\tilde{T}$ is contained in $C_e[0,1]$ and bounded by δ with respect to $\|\cdot\|_e$. Choose a ball $B_e(0,R)$ in $C_e[0,1]$ such that

$$\text{Im}\tilde{T} \cap \Omega_1 \subset B_e(0,R)$$

and consider the homotopy

$$\tilde{h}_\mu u = u - \mu \tilde{T} u.$$

Obviously $\tilde{h}_\mu$ is completely continuous on $C_e[0,1]$ for all $\mu \in [0,1]$. If $u \in \partial B_e(0,R)$ and $\tilde{h}_\mu u = 0$, then

$$R = \|u\|_e = \mu \|\tilde{T}u\|_e \le \mu\delta.$$

Taking R big enough so that $\mu\delta < R$, $d_{LS}(\tilde{h}_\mu, B_e(0,R), 0)$ is well-defined and by the homotopy invariance,

$$d_{LS}(I - \tilde{T}, B_e(0,R), 0) = d_{LS}(I, B_e(0,R), 0) = 1.$$

Since $I - T$ is equivalent to $I - \tilde{T}$ on Ω, so is on $\overline{\Omega_1^\circ}$ and by the excision property for $I - \tilde{T}$, we get

$$\begin{aligned} d_{LS}(I - T, \Omega_1^\circ, 0) &= d_{LS}(I - \tilde{T}, \Omega_1^\circ, 0) \\ &= d_{LS}(I - \tilde{T}, B_e(0,R), 0) \\ &= d_{LS}(I, B_e(0,R), 0) = 1. \end{aligned}$$

Let Ω_2° be the interior of

$$(\bar{w}, \tilde{w}) = \{u \in C_e([0,1]) \,:\, \bar{w} \prec u \prec \tilde{w}\}$$

and Ω_3° be the interior of

$$(\bar{v}, \tilde{w}) = \{u \in C_e([0,1]) \,:\, \bar{v} \prec u \prec \tilde{w}\}$$

in $C_e([0,1])$. Then we also have

$$\begin{aligned} d_{LS}(I - T, \Omega_2^\circ, 0) = 1, \\ d_{LS}(I - T, \Omega_3^\circ, 0) = 1. \end{aligned}$$

Thus by the excision and the additivity properties,

$$\begin{aligned} 1 &= d_{LS}(I - T, \Omega_3^\circ, 0) \\ &= d_{LS}(I - T, \Omega_1^\circ, 0) + d_{LS}(I - T, \Omega_2^\circ, 0) \\ &\quad + d_{LS}(I - T, \Omega_3^\circ \setminus \overline{(\Omega_1^\circ \cup \Omega_2^\circ)}, 0). \end{aligned}$$

Therefore,

$$d_{LS}(I - T, \Omega_3^\circ \setminus \overline{(\Omega_1^\circ \cup \Omega_2^\circ)}, 0) = -1.$$

This is for the proof of existence of three distinct solutions of (5).

If we choose u_1 be the minimal solution in $[\bar{v}, \tilde{v}]$ and u_3 be the maximal solution in $[\hat{w}, \tilde{w}]$, then the order of three solutions can be proved. □

The following theorem is another multiplicity of positive solutions which can be applied to find $2N - 1$ distinct ordered positive solutions of the semilinear problem (P_λ) on the exterior domain.

THEOREM 4. *Assume that* (C), (f_0) *and* $f(0) > 0$. *Suppose also that there exist two pairs of G-lower and G-upper solutions* $\{\bar{v}, \tilde{v}\}$ *and* $\{\bar{w}, \tilde{w}\}$ *satisfying the following conditions:*

(i) $\bar{v} \prec \tilde{v}$, $\bar{w} \prec \tilde{w}$, $\bar{v} \prec \tilde{w}$, $\bar{w} \not\preceq \tilde{v}$, *and none of them are solutions of* (5);
(ii) $[\bar{v}(t), \tilde{w}(t)] \subset I$ *for all* $t \in [0,1]$,
(iii) $\tilde{v}(i) > 0$, $\tilde{w}(i) > 0$ *for* $i = 0, 1$, *and*
(iv) *there is a small positive number* δ *such that* $\bar{v}(t)$, $\bar{w}(t) \leq 0$ *if* $0 \leq t \leq \delta$ *and* $0 \leq 1 - t \leq \delta$.

Then (5) *has at least 3 distinct solutions* $u_1 \prec u_2 \prec u_3$ *such that* $u_1 \in [\bar{v}, \tilde{v}]$, $u_2 \in [\bar{w}, \tilde{w}]$ *and* $u_3 \in [\bar{v}, \tilde{w}] \setminus \big([\bar{v}, \tilde{v}] \cup [\bar{w}, \tilde{w}]\big)$.

Proof. We choose the positive function e which is the unique solution of the problem: $u'' + q(t) = 0$, $0 < t < 1$ and $u(0) = u(1) = 0$. Then we prove this theorem in the space $C_e([0,1])$. We note that the proof of this theorem can be shown using the exact same technique in Theorem 3 if we get the following: If $w \in [\bar{v}, \tilde{v}]$ is a solution of (5), then w belongs to the interior of Ω_1 in the space $C_e[0,1]$.

First, we prove that $w \in C_e([0,1])$. To show that, we will find a positive number c so that $w(t) \leq ce(t)$ for all $t \in (0,1)$. It can be shown from the following calculations: By L'Hospital rule,

$$\lim_{t \to 0} \frac{e(t)}{w(t)} = \lim_{t \to 0} \frac{e'(t)}{w'(t)} > 0$$

if $w'(t) < \infty$ as $t \to 0$. Let $w'(t) \to \infty$ as $t \to 0$. Then

$$\lim_{t \to 0} \frac{e'(t)}{w'(t)} = \lim_{t \to 0} \frac{e''(t)}{w''(t)} = \lim_{t \to 0} \frac{1}{f(w(t))} > 0.$$

Similarly, we have the following limit:

$$\lim_{t \to 1} \frac{e(t)}{w(t)} > 0.$$

Thus, we get a positive constant c so that $w(t) \leq ce(t)$ for all $t \in (0,1)$.

Since $w(t) \geq 0$ on some neighborhood of $t = 0$ or $t = 1$ in $[0,1]$, we can easily find a negative constant d so that $de(t) \leq w(t)$ for all $t \in (0,1)$. Consequently, $w \in C_e([0,1])$.

Secondly, we show that w lies in the interior of Ω_1. Hence, we prove the existence of a positive number ϵ_1 so that

$$\epsilon_1 e(t) + \bar{v}(t) < w(t)$$

for all $t \in (0,1)$. Let

$$\inf_{t\in(0,1)} \frac{w(t)-\bar{v}(t)}{e(t)} = \gamma.$$

We will show $\gamma > 0$.

Suppose that $\gamma = 0$. Then for any $\epsilon > 0$, there is a point $t_\epsilon \in (0,1)$ such that

$$\frac{w(t_\epsilon)-\bar{v}(t_\epsilon)}{e(t_\epsilon)} < \epsilon.$$

Then

$$(*) \qquad \epsilon e(t_\epsilon) + \bar{v}(t_\epsilon) - w(t_\epsilon) > 0.$$

We note that if either $t = 0$ or $t = 1$, then

$$\epsilon e(t) + \bar{v}(t) - w(t) \le 0.$$

Now, if we calculate the followings:

$$\begin{aligned}
&[\epsilon e(t) + \bar{v}(t) - w(t)]'' - \delta[\epsilon e(t) + \bar{v}(t) - w(t)] \\
&= \epsilon(e''(t) - \delta e(t)) + \bar{v}''(t) - w''(t) - \delta(\bar{v}(t) - w(t)) \\
&\ge \epsilon(-q(t) - \delta e(t)) - q(t)f(\bar{v}(t)) + q(t)f(w(t)) - \delta(\bar{v}(t) - w(t)) \\
&= \epsilon(-q(t) - \delta e(t)) + q(t)[f(w(t)) - f(\bar{v}(t))] - \delta(\bar{v}(t) - w(t)) \\
&\ge \epsilon(-q(t) - \delta e(t)) + (\delta - Mq(t))(w(t) - \bar{v}(t)).
\end{aligned}$$

Without loss of generality, we assume that t_ϵ is a maximizer of $\epsilon e(t) + \bar{v}(t) - w(t)$ in $[0,1]$. Let $t_0 = \liminf_{\epsilon\to 0} t_\epsilon$. Suppose that $t_0 \ne 0$, $t_0 \ne 1$, and t_0 is not a singular interior point of $\bar{v}$, i.e. $t_0 \ne \tau_i$, $i = 1, \cdots, n$ in Definition 2. Then if we choose large $\delta > 0$ so that $\delta - Mq(t) > 0$ on some neighborhood of t_0 in $(0,1)$ and choose very small $\epsilon > 0$ so that

$$\epsilon(-q(t) - \delta e(t)) + (\delta - Mq(t))(w(t) - \bar{v}(t)) \ge 0$$

on the neighborhood. It is always possible from the result of Lemma 3. If $\epsilon > 0$ is sufficiently small, then t_ϵ belongs to the neighborhood of t_0, and then by Maximum Principle, it leads to a contradiction.

Suppose $t_0 = 0$. By L'Hospital rule,

$$\lim_{t_\epsilon\to 0} \frac{w(t_\epsilon)-\bar{v}(t_\epsilon)}{e(t_\epsilon)} = \lim_{t_\epsilon\to 0} \frac{w'(t_\epsilon)}{e'(t_\epsilon)} > 0$$

if $\lim_{t_\epsilon \to 0} e'(t_\epsilon) < \infty$, and

$$\begin{aligned}\lim_{t_\epsilon \to 0} \frac{w'(t_\epsilon)}{e'(t_\epsilon)} &= \lim_{t_\epsilon \to 0} \frac{w''(t_\epsilon)}{e''(t_\epsilon)} \\ &= \lim_{t_\epsilon \to 0} \frac{-q(t_\epsilon)f(w(t_\epsilon))}{-q(t_\epsilon)} \\ &= f(w(0)) > 0\end{aligned}$$

if $\lim_{t_\epsilon \to 0} e'(t_\epsilon) = \infty$. This leads to a contradiction to the fact $\gamma = 0$. Similarly, we can get a contradiction in the case $t_0 = 1$.

Suppose that t_0 is a singular interior point of $\bar{v}$. $(*)$ implies that $\bar{v}(t_0) = w(t_0)$. This is impossible from Lemma 3.

From condition (iii), we also can easily show the existence of a positive number ϵ_2 so that

$$w(t) < \tilde{v}(t) - \epsilon_2 e(t)$$

for all $t \in (0, 1)$. Therefore,

$$\epsilon_1 e(t) + \bar{v}(t) < w(t) < \tilde{v}(t) - \epsilon_2 e(t)$$

for all $t \in (0, 1)$, and this implies that $w \in \Omega_1^\circ$. □

4. Applications

Throughout this section, we assume that $f(0) > 0$. As an application of Theorem 4, we prove the existence of $2N - 1$ distinct ordered positive solutions of the following problems ;

$$(1_\lambda) \qquad \begin{cases} u'' + \lambda q(t) f(u) = 0, & 0 < t < 1, \\ u(0) = 0 = u(1), \end{cases}$$

where $\lambda > 0$ is a real parameter, $q \in C((0,1), \mathbf{R}^+)$ is singular at $t = 0$ and/or 1.

THEOREM 5. ([11]) *For $a, b \in \mathbf{R}$ with $a < b$, consider*

$$(9_\lambda) \qquad \begin{cases} u'' + \lambda \tilde{q}(t) f(u) = 0, & a < t < b, \\ u(a) = 0 = u(b). \end{cases}$$

Assume that $\tilde{q} \in C^1([a,b],[0,\infty))$ does not vanish identically on some subinterval of $[a,b]$ and f satisfies $(f_0) \sim (f_3)$. Then there exists $\lambda_o > 0$ such that for all $\lambda > \lambda_o$, problem (9_λ) has $2N-1$ distinct positive ordered solutions u_i, $u_{i+\frac{1}{2}}$ and u_{i+1} $(i = 1, \cdots, N-1)$ such that

$$u_i(t) \le u_{i+\frac{1}{2}}(t) \le u_{i+1}(t)$$

for all $t \in [a,b]$. Moreover, for $j = 1, \cdots, N$, $u_j(t) \le a_j$ for all $t \in [a,b]$, and u_j converges to a_j as $\lambda \to \infty$ uniformly on every compact subset of (a,b).

Now we have a similar result for singular problem (1_λ).

THEOREM 6. *Assume that $q \in C^1((0,1),(0,\infty))$ satisfies (C) and f satisfies $(f_0) \sim (f_3)$. Then there exists $\lambda_o > 0$ such that for all $\lambda > \lambda_o$, the singular boundary value problem (1_λ) has $2N-1$ distinct positive ordered solutions u_i, $u_{i+\frac{1}{2}}$ and u_{i+1} $(i = 1, \cdots, N-1)$ such that*

$$u_i(t) \le u_{i+\frac{1}{2}}(t) \le u_{i+1}(t)$$

for all $t \in [0,1]$.

Proof. We prove this theorem by constructing N many G-lower solutions of (1_λ). Choose a closed bounded interval $[a,b] \subset (0,1)$ with $a < b$. Then by Theorem 5, there is $\lambda_0 > 0$ such that for all $\lambda > \lambda_o$, problem (9_λ) has N distinct positive ordered solutions $v_1, \cdots, v_N$ such that

$$v_1(t) \le v_2(t) \le \cdots \le v_N(t) \quad \text{for all } t \in [a,b]$$

and $v_j(t) \le a_j$ for all $t \in [0,1]$ and v_j converges to a_j as $\lambda \to \infty$ uniformly on every compact subset of (a,b). Let

$$\bar{u}_j(t) = \begin{cases} v_j(t), & a \le t \le b, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \cdots, N$. Then $\bar{u}_j$ is a G-lower solution and a_j is a G-upper solution of (1_λ), respectively. Thus for each $j = 1, \cdots, N-1$, we have two pairs of G-lower and G-upper solutions $\{\bar{u}_j, a_j\}$ and $\{\bar{u}_{j+1}, a_{j+1}\}$ with $\bar{u}_{j+1} \not\le a_j$, none of them are solutions of (1_λ). Since those pairs satisfies all the conditions of Theorem 4, we obtain three solutions u_j,

$u_{j+\frac{1}{2}}$, u_{j+1} for each j such that $u_j \in [\bar{u}_j, a_j]$, $u_{j+1} \in [\bar{u}_{j+1}, a_{j+1}]$, $u_{i+\frac{1}{2}} \in [\bar{u}_j, a_{j+1}] \setminus ([\bar{u}_j, a_j] \cup [\bar{u}_{j+1}, a_{j+1}])$, and

$$u_i(t) \leq u_{i+\frac{1}{2}}(t) \leq u_{i+1}(t)$$

for all $t \in [0,1]$. This completes the proof. □

As an application of Theorem 6, we show the existence of $2N-1$ distinct ordered positive radial solutions of problem (P_λ).

THEOREM 7. *Assume that g satisfies (g) and f satisfies $(f_0) \sim (f_3)$. Then there exists $\lambda_o > 0$ such that for all $\lambda > \lambda_o$, (10_λ) has $2N-1$ distinct ordered positive radial solutions.*

REMARK 2. The above problem is either regular or singular depending on the coefficient function g. Necessary and sufficient condition in order that g satisfies the integral condition in (g) is that q fulfills the condition (C). We also note that g satisfies the integral condition in (g) if $\int_{\{|x| \geq r_0\}} g(|x|)|x|^{2-n} dx < \infty$. Furthermore, if we let $h(x) = g(|x|)$ in the exterior of the ball $B(0, r_o)$, we get the following statement: if for some p with $1 < p < \frac{n}{2}$, $h \in L^p$ on the exterior domain, then g satisfies the condition (g).

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Bong Soo Ko
Department of Mathematics Education
Cheju National University
Cheju 690-756, Korea
E-mail: bsko@cheju.cheju.ac.kr

Yong-Hoon Lee
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: yhlee@pusan.ac.kr