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ON A CONNECTION ON A HYPERCONTACT MANIFOLD

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ABSTRACT. We construct the canonical connection associated with a hypercontact structure. Moreover, we discuss the canonical connection associated with a sub-Riemannian 3-structure. As an application, we study the sub-symmetry property in terms of the canonical connection.

1. Introduction

Several connections on contact structures have been studied by many geometers ([4], [5], [7]). Recently, Falbel-Gorodski ([2]) defined a connection on a contact structure in the sense of sub-Riemannian geometry, which can be considered as a generalization of the generalized Tanaka connection ([5]) on a contact Riemannian structure.

On the other hand, Geiges-Thomas ([3]) introduced a notion of a hypercontact structure as a quaternionic analogue of contact Riemannian structure.

In this paper, we shall construct a new connection on a hypercontact manifold from the view point of foliated structure. That is, if $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}$ is an almost contact 3-structure compatible with a hypercontact structure, the foliation is defined by vector fields $\{\xi_1, \xi_2, \xi_3\}$ which generate a Lie algebra locally isomorphic to so(3).

In Section 2, we give a brief review of several known connections on a contact Riemannian structure. In Section 3, we define a new connection D on a hypercontact manifold by a similar way as in [5]. In Section 4, we discuss a canonical connection associated with a sub-Riemannian 3-structure. As an application, we study the sub-symmetry property in terms of the canonical connection.

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2. The case of a contact Riemannian structure

An m-dimensional manifold M $(m = 2n+1)$ is a contact manifold if it admits a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. There is a unique vector field ξ on M such that

$$\eta(\xi) = 1, \qquad L_\xi \eta = 0, \tag{2.1}$$

where L_ξ denotes the Lie derivation by ξ. It is well known that there is a contact Riemannian structure (ϕ, η, ξ, g) such that

$$g(\xi, X) = \eta(X), \quad 2g(X, \phi Y) = d\eta(X, Y), \quad \phi^2 X = -X + \eta(X)\xi,$$

where $X, Y \in \Gamma(TM)$ on M. Here and hereafter $\Gamma(.)$ is denoted by the space of all sections of (.). The followings hold:

$$\begin{aligned} &\phi\xi = 0, \qquad \eta(\phi X) = 0, \\ &g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \\ &d\eta(X, \phi Y) = -d\eta(\phi X, Y). \end{aligned} \tag{2.2}$$

Let E be the foliation of TM generated by the Reeb vector field ξ. Then E gives the orthogonal decomposition

$$TM = E \oplus \mathcal{D}$$

with respect to g. By (2.1), E is a geodesic and transversally symplectic flow with exact transversal symplectic form $d\eta$ on a Riemannian manifold (M, g). If, moreover, ξ satisfies $L_\xi g = 0$, or equivalently, $L_\xi \phi = 0$, then E can be considered as a geodesic almost Kähler flow on (M, g).

LEMMA 2.1 ([5]). *On a contact Riemannian structure (ϕ, ξ, η, g), the Riemannian connection ∇ satisfies the following properties:*

(i) $\nabla_\xi \eta = 0$, $\nabla_\xi \xi = 0$, $\xi^r \nabla_i \eta_r = 0$,
(ii) $\nabla_r \xi^r = 0, \nabla_r \phi^r_j = -2n\eta_j$,
(iii) $\nabla_r \eta_s \phi^r_i \phi^s_j = -\nabla_j \eta_i$,
(iv) $\nabla_r \eta_i \phi^r_j$ *and* $\nabla_i \eta_r \phi^r_j$ *are symmetric in* i, j,
(v) $\nabla_\xi \phi = 0$.

Tanno ([5]) defined the generalized Tanaka connection $^*\nabla$ on a contact Riemannian manifold (M, ϕ, ξ, η, g) by

$$^*\nabla_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi \tag{2.3}$$

for $X, Y \in \Gamma(TM)$. The torsion tensor $^*T$ of $^*\nabla$ is given by

$$^*T(X,Y) = \eta(X)\phi Y - \phi X \eta(Y) - \eta(Y)\nabla_X \xi + \eta(X)\nabla_Y \xi + 2g(X, \phi Y)\xi.$$

PROPOSITION 2.2 ([5]). *With above notations, $^*\nabla$ satisfies the followings:*

(i) $^*\nabla\eta = 0$, $^*\nabla\xi = 0$,
(ii) $^*\nabla g = 0$,
(iii) $^*T(X,Y) = d\eta(X,Y)\xi$ *for* $X, Y \in \Gamma(\mathcal{D})$,
(iv) $^*T(\xi, \phi Y) = -\phi{}^*T(\xi, Y)$ *for* $Y \in \Gamma(\mathcal{D})$,
(v) $(^*\nabla_X \phi)Y = (\nabla_X \phi)Y + \eta(Y)\phi\nabla_X \xi + (\nabla_X \eta)(\phi Y)\xi$ *for* $X, Y \in \Gamma(TM)$,
(vi) $^*\nabla\phi = 0$ *if and only if* ϕ *is integrable.*

This connection is a natural generalization of the Tananka connection defined on a nondegenerate, pseudo-hermitian manifold ([6]).

We suppose that M is oriented and $\mathcal{D}$ is oriented. Let $g_{\mathcal{D}}$ be a positive definite symmetric bilinear form on $\mathcal{D}$. A triple $(M, \mathcal{D}, g_{\mathcal{D}})$ becomes a sub-Riemannian manifold. A contact manifold admits a sub-Riemannian metric $d\eta(\phi\cdot, \cdot)$. Falbel-Gorodski showed the following result.

PROPOSITION 2.3 ([2]). *There is a unique connection ∇^F on a contact sub-Riemannian manifold $(M, \mathcal{D}, \xi, \eta, g_{\mathcal{D}})$ with following properties:*

(i) $\nabla^F{}_X : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$ *for* $X \in \Gamma(TM)$,
(ii) $\nabla^F \xi = 0$,
(iii) $\nabla^F g = 0$,
(iv) $T^F(X,Y) = d\eta(X,Y)\xi$ *for* $X, Y \in \Gamma(\mathcal{D})$,
(v) *the sub-torsion τ^F of ∇^F defined by $\tau^F(X) := T^F(\xi, X)$ satisfies $\tau^F(\Gamma(\mathcal{D})) \subset \Gamma(\mathcal{D})$ and is symmetric.*

The connection ∇^F may be regarded as a natural extension of the generalized Tanaka connection defined on a contact Riemannian structure in the sense of sub-Riemannian geometry.

3. The case of a hypercontact structure

We recall the definitions of the following quaternionic analogue of an almost contact structure.

DEFINITION 3.1. A tensor field $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}$ is called an almost contact 3-structure if the following conditions are satisfied:

(i) $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$,
(ii) $\phi_\alpha \xi_\beta = \sum_\gamma \epsilon_{\alpha\beta\gamma}\xi_\gamma$,
(iii) $\eta_\alpha \circ \phi_\beta = \sum_\gamma \epsilon_{\alpha\beta\gamma}\eta_\gamma$,
(iv) $\phi_\alpha\phi_\beta = -\delta_{\alpha\beta} + \xi_\alpha \otimes \eta_\beta + \sum_\gamma \epsilon_{\alpha\beta\gamma}\phi_\gamma$,

where $\epsilon_{\alpha\beta\gamma}$ is the sign of a permutation of $(1, 2, 3)$.

The η_α define the subbundle $\mathcal{D}$ of codimension 3 in TM on which the ϕ_α satisfy the quaternionic identities. The existence of an almost contact 3-structure on a manifold M is equivalent to a reduction of the structure group of M to $Sp(n) \times Sp(1)$. In particular, M has to be of dimension $4n + 3$.

An almost contact 3-structure is said to be compatible with a Riemannian metric g if

$$g(\phi_\alpha X, \phi_\alpha Y) = g(X, Y) - \eta_\alpha(X)\eta_\alpha(Y), \quad X, Y \in \Gamma(TM). \tag{3.1}$$

DEFINITION 3.2. A triple of contact forms $(\omega_1, \omega_2, \omega_3)$ on a manifold M is called a hypercontact structure if there is a Riemannian metric g and a compatible almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}$ such that

$$g(\phi_\alpha X, Y) := d\omega_\alpha(X, Y), \quad X, Y \in \Gamma(TM). \tag{3.2}$$

The following result was proved in [3].

PROPOSITION 3.3 ([3]). *With above notations,*

(i) $d\omega_\alpha(\phi_\alpha X, \phi_\alpha Y) = d\omega_\alpha(X, Y)$,
(ii) $d\omega_\alpha(\xi_\beta, \xi_\gamma) = g(\xi_\gamma, \xi_\gamma) = 1$ *for any cyclic permutation* $\{\alpha, \beta, \gamma\}$ *of* $\{1, 2, 3\}$,
(iii) *the* ξ_α *are multiples of the Reeb vector fields of the* ω_α,
(iv) *the underlying almost contact 3-structure* $(\phi_\alpha, \eta_\alpha, \xi_\alpha)_{\alpha=1,2,3}$ *is completely determined if* $\omega_\alpha(\xi_\alpha) > 0$.

The definition of a hypercontact structure involves a triple of contact forms, a Riemannian metric and an almost contact 3-structure. Proposition 3.3 shows that the almost contact 3-structure is completely determined (up to sign) by the contact forms $(\omega_\alpha)_{\alpha=1,2,3}$ and metric g.

In the following, we consider a hypercontact structure $(\omega_\alpha, g)_{\alpha=1,2,3}$ satisfying assumptions (A) and (B).

(A) $\mathcal{D} := \bigcap_{\alpha=1}^3 \ker\ \eta_\alpha = \bigcap_{\alpha=1}^3 \ker\ \omega_\alpha$.

(B) Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}$ be the underlying almost contact 3-structure of a given hypercontact structure $(\omega_\alpha, g)_{\alpha=1,2,3}$. The vector field ξ_α is a positive multiple of Reeb vector field of ω_α for each α.

Then we may $g(\xi_\alpha, \xi_\beta) = \omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}(\alpha, \beta = 1, 2, 3)$ in the sense of Proposition 3.3 (iv). It is obvious from (3.2) and Proposition 3.3 that

$$[\xi_\alpha, \xi_\beta] = 2\epsilon_{\alpha\beta\gamma}\xi_\gamma. \tag{3.3}$$

It is well known that on contact Riemannian structure ([B]),

$$\nabla_X \xi_\beta = -\phi_\beta X - \frac{1}{2}\phi_\beta(L_{\xi_\beta}\phi_\beta)X \text{ for } X \in \ker \omega_\beta. \tag{3.4}$$

LEMMA 3.4. *Let $(\omega_\alpha, g)_{\alpha=1,2,3}$ be a hypercontact structure on M with assumptions A and B. Then we have:*

(i) $\nabla_{\xi_\alpha}\xi_\beta = \epsilon_{\alpha\beta\gamma}\xi_\gamma$,

(ii) $\nabla_X\xi_\beta \in \Gamma(\mathcal{D})$ *for* $X \in \Gamma(\mathcal{D})$.

Proof. By (3.4), we have

$$\nabla_{\xi_\alpha}\xi_\beta = -\phi_\beta\xi_\alpha - \frac{1}{2}\phi_\beta(L_{\xi_\beta}\phi_\beta)\xi_\alpha$$

for $\xi_\alpha \in \ker\ \omega_\beta$. A direct computation gives rise to

$$\begin{aligned}\phi_\beta(L_{\xi_\beta}\phi_\beta)\xi_\alpha &= \phi_\beta L_{\xi_\beta}(\phi_\beta\xi_\alpha) - \phi_\beta^2 L_{\xi_\beta}(\xi_\alpha) \\ &= \phi_\beta[\xi_\beta, \phi_\beta\xi_\alpha] + [\xi_\beta, \xi_\alpha] - \eta_\beta([\xi_\beta, \xi_\alpha])\xi_\beta \\ &= \phi_\beta[\xi_\beta, \epsilon_{\beta\alpha\gamma}\xi_\gamma] + [\xi_\beta, \xi_\alpha] \\ &= 2\epsilon_{\alpha\beta\gamma}\xi_\gamma - 2\epsilon_{\alpha\beta\gamma}\xi_\gamma = 0,\end{aligned}$$

which proves (i).

Similarly, we have

$$\begin{aligned} g(\nabla_X \xi_\beta, \xi_\gamma) &= g(X, \xi_\alpha) + g(\frac{1}{2}(L_{\xi_\beta}\phi_\beta)X, \xi_\alpha) \\ &= \frac{1}{2} g((L_{\xi_\beta}\phi_\beta)X, \xi_\alpha) \end{aligned}$$

for $X \in \Gamma(\mathcal{D})$. On the other hand, we note that $[\xi_\alpha, X] \in \Gamma(\mathcal{D})$ by means of $d\omega_\beta(\xi_\alpha, X) = 0$ for $X \in \Gamma(\mathcal{D})$. Then

$$g((L_{\xi_\beta}\phi_\beta)X, \xi_\alpha) = g([\xi_\beta, \phi_\beta X] - \phi_\beta[\xi_\beta, X], \xi_\alpha) = 0,$$

which completes the proof of (ii). □

Lemma 3.4 (i) means that E is totally geodesic with respect to g. Moreover, this, together with the metrical property of ∇, implies

$$\nabla_{\xi_\alpha} \omega_\beta = \epsilon_{\alpha\beta\gamma}\omega_\gamma. \tag{3.5}$$

Since $d\omega_\alpha(\xi_\beta, X) = 0$ for $X \in \Gamma(\mathcal{D})$, we have

$$g((\nabla_{\xi_\alpha}\phi_\beta)Y, Z) = g(\nabla_{\xi_\alpha}(\phi_\beta Y), Z) + g(\nabla_{\xi_\alpha} Y, \phi_\beta Z) \tag{3.6}$$

for $Y, Z \in \Gamma(\mathcal{D})$. Since ∇ can be expressed as

$$\begin{aligned} 2g(\nabla_{\xi_\alpha} Y, Z) = {} & \xi_\alpha d\omega_\gamma(Y, \phi_\gamma Z) + Y d\omega_\gamma(\xi_\alpha, \phi_\gamma Z) - Z d\omega_\gamma(\xi_\alpha, \phi_\gamma Y) \\ & - d\omega_\gamma([Y, Z], \phi_\gamma \xi_\alpha) + d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma Y) \\ & + d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma Z), \end{aligned} \tag{3.7}$$

Lemma 3.4 together with (3.6) and (3.7) gives rise to

$$\begin{aligned} 2g((\nabla_{\xi_\alpha}\phi_\beta)Y, Z) &= 2g(\nabla_{\xi_\alpha}(\phi_\beta Y), Z) + 2g(\nabla_{\xi_\alpha} Y, \phi_\beta Z) \\ &= \xi_\alpha d\omega_\gamma(\phi_\beta Y, \phi_\gamma Z) - \xi_\alpha d\omega_\gamma(\phi_\gamma Y, \phi_\beta Z) \\ &\quad + d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma\phi_\beta Y) + d\omega_\gamma([\phi_\beta Z, \xi_\alpha], \phi_\gamma Y) \\ &\quad + d\omega_\gamma([\xi_\alpha, \phi_\beta Y], \phi_\gamma Z) + d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma\phi_\beta Z). \end{aligned}$$

By using the Jacobi identity, the right hand side of the above formula becomes

$$\begin{aligned} & \xi_\alpha d\omega_\gamma(\phi_\beta Y, \phi_\gamma Z) - \xi_\alpha d\omega_\gamma(\phi_\gamma Y, \phi_\beta Z) \\ & + d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma\phi_\beta Y) + d\omega_\gamma([\phi_\beta Z, \xi_\alpha], \phi_\gamma Y) \\ & + d\omega_\gamma([\xi_\alpha, \phi_\beta Y], \phi_\gamma Z) + d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma\phi_\beta Z) = 0. \end{aligned}$$

Thus, we have

$$(3.8) \qquad (\nabla_{\xi_\alpha}\phi_\beta)Y \in \Gamma(E) \text{ for } Y \in \Gamma(\mathcal{D}).$$

It follows from Lemma 3.4 and (3.8) that

$$(\nabla_{\xi_\alpha}\phi_\beta)Y = 0 \text{ for } Y \in \Gamma(\mathcal{D}).$$

On the other hand,

$$\begin{aligned}(\nabla_{\xi_\alpha}\phi_\beta)\xi_\gamma &= \nabla_{\xi_\alpha}(\phi_\beta\xi_\gamma) - \phi_\beta(\nabla_{\xi_\alpha}\xi_\gamma)\\ &= \nabla_{\xi_\alpha}\epsilon_{\beta\gamma\alpha}\xi_\alpha - \phi_\beta\epsilon_{\alpha\gamma\beta}\xi_\beta = 0\end{aligned}$$

for distinct $\{\alpha,\beta,\gamma\}$. It is easy to see that

$$\begin{aligned}(\nabla_{\xi_\alpha}\phi_\beta)\xi_\alpha &= \nabla_{\xi_\alpha}(\phi_\beta\xi_\alpha) - \phi_\beta(\nabla_{\xi_\alpha}\xi_\alpha)\\ &= \nabla_{\xi_\alpha}\epsilon_{\beta\alpha\gamma}\xi_\gamma = \xi_\beta.\end{aligned}$$

By a similar way as in [4], we can show that $(\nabla_X\phi_\alpha)Y = 0$ if and only if ϕ_α is integrable. Summing up, we have

LEMMA 3.5. *Under the same station as in Lemma 3.4, the Riemannian connection ∇ satisfies*

(i) $\nabla_{\xi_\alpha}\omega_\beta = \epsilon_{\alpha\beta\gamma}\omega_\gamma$,
(ii) $(\nabla_{\xi_\alpha}\phi_\beta)X = 0$,
(iii) $(\nabla_{\xi_\alpha}\phi_\beta)\xi_\gamma = 0$, $\quad(\nabla_{\xi_\alpha}\phi_\beta)\xi_\alpha = \xi_\beta$ *for distinct* α,β,γ,
(iv) $(\nabla_X\phi_\alpha)Y = 0$ *if and only if* ϕ_α *is integrable,*

where $X, Y \in \Gamma(\mathcal{D})$.

Now, we can construct a new connection on a hypercontact structure by a similar way as in [5].

Define a connection D on a hypercontact structure $(\omega_\alpha, g)_{\alpha=1,2,3}$ by

$$(3.9) \quad D_YZ = \nabla_YZ + \sum_{\alpha=1}^{3}\{\omega_\alpha(Y)\phi_\alpha Z - \omega_\alpha(Z)\nabla_Y\xi_\alpha + (\nabla_Y\omega_\alpha)(Z)\xi_\alpha\},$$

where $Y, Z \in \Gamma(TM)$. Then torsion tensor T^D of D is given by

$$(3.10) \qquad \begin{aligned}T^D(Y,Z) = \sum_{\alpha=1}^{3}\{&\omega_\alpha(Y)\phi_\alpha Z - \omega_\alpha(Z)\phi_\alpha Y - \omega_\alpha(Z)\nabla_Y\xi_\alpha\\ &+ \omega_\alpha(Y)\nabla_Z\xi_\alpha - 2g(\phi_\alpha Y, Z)\xi_\alpha\}\end{aligned}$$

for $Y, Z \in \Gamma(TM)$.

THEOREM 3.6. *Let $(\omega_\alpha, g)_{\alpha=1,2,3}$ be as in Lemma 3.4. The connection D defined by (3.9) is a unique linear connection satisfying the followings:*

(i) $D\omega_\alpha = 0,\ D\xi_\alpha = 0,$
(ii) $Dg = 0,$
(iii) $T^D(X,Y) = \sum_{\alpha=1}^{3} d\omega_\alpha(X,Y)\xi_\alpha,$
(iv) $T^D(\xi_\alpha, \phi_\alpha Y) = -\phi_\alpha T^D(\xi_\alpha, Y),$
(v) $T^D(\xi_\alpha, \phi_\beta Y) - \phi_\beta\, T^D(\xi_\alpha, Y)$
$= 2\epsilon_{\alpha\beta\gamma}\phi_\gamma\, Y - (L_{\xi_\alpha}\phi_\beta)Y,\ T^D(\xi_\alpha, \phi_\beta Y) + \phi_\beta T^D(\xi_\alpha, Y)$
$= \phi_\beta[\xi_\alpha, Y] + [\xi_\alpha, \phi_\beta Y]$ *for* $\alpha \neq \beta,$
(vi) $T^D(\xi_\alpha, \xi_\beta) = -2\epsilon_{\alpha\beta\gamma}\xi_\gamma,$
(vii) $(D\phi_\alpha)\xi_\beta = 0,$
(viii) $(D_{\xi_\alpha}\phi_\beta)X = 2\epsilon_{\alpha\beta\gamma}\phi_\gamma X,$
(ix) $(D_X\phi_\alpha)Y = 0$ *if and only if* ϕ_α *is integrable,*

where $X, Y \in \Gamma(\mathcal{D})$.

Proof. $D\xi_\alpha = 0$ is proved by (3.9) and Lemma 3.4. By (3.9) and Lemma 3.5, for $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}
&(D_Z g)(X,Y)\\
&= Zg(X,Y) - g(D_Z X, Y) - g(X, D_Z Y)\\
&= Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)\\
&\quad - g(\{\omega_\alpha(Z)\phi_\alpha X - \omega_\alpha(X)\nabla_Z\xi_\alpha + (\nabla_Z\omega_\alpha)(X)\xi_\alpha\}, Y)\\
&\quad - g(X, \sum_{\alpha=1}^{3}\{\omega_\alpha(Z)\phi_\alpha Y - \omega_\alpha(Y)\nabla_Z\xi_\alpha + (\nabla_Z\omega_\alpha)(Y)\xi_\alpha\})\\
&= (\nabla_Z g)(X,Y) = 0.
\end{aligned}$$

Thus (ii) is proved.

(iii) can be easily verified by (3.10). From Proposition 2.2 and (3.10), we see (iv).

For the case that $\alpha \neq \beta$, we have from (3.10) that

$$T^D(\xi_\alpha, \phi_\beta Y) = \epsilon_{\alpha\beta\gamma}\phi_\gamma Y + \nabla_{\phi_\alpha Y}\xi_\alpha$$

and

$$\begin{aligned}
\phi_\beta T^D(\xi_\alpha, Y) &= \phi_\beta\phi_\alpha Y + \phi_\beta\nabla_Y\xi_\alpha\\
&= -\epsilon_{\alpha\beta\gamma}\phi_\gamma Y + \phi_\beta\nabla_Y\xi_\alpha.
\end{aligned}$$

Thus, we have

$$T^D(\xi_\alpha, \phi_\beta Y) - \phi_\beta T^D(\xi_\alpha, Y) = 2\epsilon_{\alpha\beta\gamma}\phi_\gamma Y + \phi_\beta[\xi_\alpha, Y] - [\xi_\alpha, \phi_\beta Y]$$
$$= 2\epsilon_{\alpha\beta\gamma}\phi_\gamma Y - (L_{\xi_\alpha}\phi_\beta)Y$$

and

$$T^D(\xi_\alpha, \phi_\beta Y) + \phi_\beta T^D(\xi_\alpha, Y) = \phi_\beta[\xi_\alpha, Y] + [\xi_\alpha, \phi_\beta Y].$$

By (3.3) and (i), we have (vi) and (vii). (viii) and (ix) are verified from Lemma 3.5. The uniqueness of the connection is obvious. □

PROPOSITION 3.7. *On a hypercontact manifold $(M, \omega_\alpha, \xi_\alpha, g)_{\alpha=1,2,3}$ of dim $4n+3$, the following holds:*

$$2\|T^D\|^2 = \|L_{\xi_\alpha} g\|^2 + 16(4n) + 12. \tag{3.11}$$

In particular, $\|T^D\|^2$ attains its minimum $16(4n)+12$ if and only if ξ_α is a Killing vector field for each α.

Proof.

$$\|T^D\|^2 = 2\|\nabla\xi_\alpha\|^2 + 6(4n) + 6.$$

By the formula $(L_{\xi_\alpha} g)(X,Y) = (\nabla_X\omega_\alpha)(Y) + (\nabla_Y\omega_\alpha)(X)$, we obtain

$$\|L_{\xi_\alpha} g\|^2 = 2\|\nabla\xi_\alpha\|^2 + 2\nabla_X\xi_\alpha(\nabla_Y\omega_\alpha)X$$

for $X, Y \in \Gamma(\mathcal{D})$.

Since $\nabla_X\xi_\alpha(\nabla_Y\omega_\alpha)X = \|\nabla\xi_\alpha\|^2 - 2(4n)$, we get

$$\|L_{\xi_\alpha} g\|^2 = 4\|\nabla\xi_\alpha\|^2 - 4(4n),$$

which yields (3.11). □

4. The case of a sub-Riemannian 3-structure

Let $(\phi_\alpha, \eta_\alpha, \xi_\alpha)_{\alpha=1,2,3}$ be an almost contact 3-structure on an oriented manifold M. We suppose that $d\eta_\beta(\xi_\alpha, X) = 0$ for $X \in \Gamma(\mathcal{D})$. Then the ϕ_α satisfy the quaternionic identities on $\mathcal{D}$. We consider a smoothly varying positive definite symmetric bilinear form $g_{\mathcal{D}}$ on $\mathcal{D}$. Then $(\mathcal{D}, g_{\mathcal{D}})$ is called a sub-Riemannian 3-structure on M. From (3.3) and Lemma 3.4, a hypercontact structure $(\omega_\alpha, g)_{\alpha=1,2,3}$ satisfying the assumptions A and B, is an example of a sub-Riemannian 3-structure whose sub-Riemannian metric is the restriction of g to $\mathcal{D}$.

Note that the sub-Riemannian metric g has a natural extension to a Riemannian metric $\langle\, ,\, \rangle$ on M by setting ξ_α $(\alpha = 1,2,3)$ to be orthonormal to $\mathcal{D}$.

THEOREM 4.1. *Let $(M, \mathcal{D}, g_{\mathcal{D}})$ be a sub-Riemannian 3-structure with the underlying almost contact 3-structure $(\phi_\alpha, \eta_\alpha, \xi_\alpha)_{\alpha=1,2,3}$. Then there is a unique connection D with following properties:*

(i) $D_X : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$ *for* $X \in \Gamma(TM)$,
(ii) $D\xi_\alpha = 0$,
(iii) $Dg_{\mathcal{D}} = 0$,
(iv) $T^D(X,Y) = \sum_{\alpha=1}^{3} d\eta_\alpha(X,Y)\xi_\alpha, \ X, Y \in \Gamma(\mathcal{D})$,
(v) *the sub-torsion* τ_α^D *of* D *defined by* $\tau_\alpha^D(X) := T^D(\xi_\alpha, X)$ *satisfies* $\tau_\alpha^D(\Gamma(\mathcal{D})) \subset \Gamma(\mathcal{D})$ *and symmetric,*
(vi) $T^D(\xi_\alpha, \xi_\beta) = -2\epsilon_{\alpha\beta\gamma}\xi_\gamma$.

Proof. Let $X, Y, Z \in \Gamma(\mathcal{D})$. As is Riemannian geometry, (i), (iii) and (iv) uniquely determine $D_X Y$ by virtue of the formula

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &= 2\langle D_X Y, Z\rangle + \langle Y, [X, Z] + T(X, Z)\rangle \\ &\quad + \langle X, [Y, Z] + T(Y, Z)\rangle + \langle Z, [Y, X] + T(Y, X)\rangle. \end{aligned}$$

Because of (ii), it remains only to define $D_{\xi_\alpha} X$. Since $D_{\xi_\alpha} X = [\xi_\alpha, X] + \tau_\alpha(X)$, the formula

$$\begin{aligned} \xi_\alpha\langle X, Y\rangle &= \langle D_{\xi_\alpha} X, Y\rangle + \langle X, D_{\xi_\alpha} Y\rangle \\ &= \langle[\xi_\alpha, X], Y\rangle + \langle[\xi_\alpha, Y], X\rangle + 2\langle\tau_\alpha^D(X), Y\rangle \end{aligned}$$

determines $\tau_\alpha^D(X)$ By (ii), $T^D(\xi_\alpha, \xi_\beta) = -[\xi_\alpha, \xi_\beta]$, which determines $T^D(\xi_\alpha, \xi_\beta)$. □

COROLLARY 4.2. *The connection D has following properties:*

(i) $d\eta_\alpha(X,Y) = \eta_\alpha(T^D(X,Y))$,
(ii) $2\langle T^D(\xi_\alpha, X), Y\rangle = (L_{\xi_\alpha} g_{\mathcal{D}})(X, Y)$

for $X, Y \in \Gamma(\mathcal{D})$.

The curvature of this connection is given by

$$R^D(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

Now we study the sub-symmetry property in a situation of a sub-Riemannian 3-structure. Recalled notion of sub-Riemannian symmetric space.

A *local isometry* between sub-Riemannian manifolds $(M, \mathcal{D}, g_{\mathcal{D}})$ and $(M', \mathcal{D}', g'_{\mathcal{D}})$ is a diffeomorphism $\psi : U \subset M \to U' \subset M'$ between open sets such that $\psi_*(\mathcal{D}) = \mathcal{D}'$ and $\psi^* g_{\mathcal{D}} = g'_{\mathcal{D}}$. In the sub-Riemannian 3-structure case, it can be seen that $\psi^*(\omega_\alpha) = \pm\omega_\alpha{}'$ and $\psi_*(\xi_\alpha) = \pm\xi_\alpha{}'$ for each α. Indeed, if $\psi'(\omega_\alpha) = \pm\omega_\beta{}'$ with $\alpha \neq \beta$ then such a ψ^* contradicts to the sub-symmetric property. If ψ is globally defined of M to M', we say that ψ is isometry.

Note that an isometry $\psi : M \to M'$ is an affine map with respect to the adapted connection, that is, $D'_{\psi_* X}\psi_* Y = \psi_*(D_X Y)$ for $X, Y \in \Gamma(TM)$.

A *sub-Riemannian symmetric space* (or *sub-symmetric space*) is an homogeneous sub-Riemannian manifold $(M, \mathcal{D}, g_{\mathcal{D}})$ such that for every point $x_0 \in M$ there is an isometry ψ such that $\psi(x_0) = x_0$ and $\psi_* |_{\mathcal{D}_{x_0}} = -1$, which is called a *sub-symmetry* at x_0. Then we have:

THEOREM 4.3. *A sub-Riemannian manifold with sub-Riemannian 3-structure is a locally sub-symmetric space if and only if the following conditions are verified:*

(i) $D_X T^D = 0$,
(ii) $D_X R^D = 0$

for all $X \in \Gamma(\mathcal{D})$.

Proof. Suppose that M is a sub-symmetry space. The sub-symmetry ψ is an affine map with respect to the canonical connection D given in Theorem 4.1. We compute for $X, Y, Z \in \Gamma(\mathcal{D})$

$$\psi_*(\nabla_Z T^D)(X, Y) = (D_{\psi_* Z} T^D)(\psi_* X, \psi_* Y) = -(D_Z T^D)(X, Y).$$

By Theorem 4.1 (vi), we have that $\psi_*(D_Z T^D(X, Y)) = D_Z T^D(X, Y)$. Therefore we have

$$(D_Z T^D)(X, Y) = 0.$$

Now, it follows from Theorem 4.1 (v) that

$$\psi_*(D_Z T^D(X, \xi_\alpha)) = -D_Z T^D(X, \xi_\alpha).$$

On the other hand,

$$\begin{aligned} &\psi_*(D_Z T^D(X, \xi_\alpha)) \\ &= D_{\psi_* Z} T^D(\psi_* X, \psi_* \xi_\alpha) \\ &= D_{(-Z)} T^D(-X, \xi_\alpha) = D_Z T^D(X, \xi_\alpha), \end{aligned}$$

so that $(D_Z T^D)(X, \xi_\alpha) = 0$. Finally by Theorem 4.1 (ii), we have

$$D_Z T^D(\xi_\alpha, \xi_\beta) = D_Z(T^D(\xi_\alpha, \xi_\beta)) - T^D(D_Z\xi_\alpha, \xi_\beta) - T^D(\xi_\alpha, D_Z\xi_\beta) = 0.$$

Hence, $(D_Z T^D)(\xi_\alpha, \xi_\beta) = 0$. We have (ii) by a similar way.

Conversely, suppose the conditions (i) and (ii). We will find differential equation which must be satisfied by the curvature and torsion tensors of the connection D along the geodesic rays. Suppose $\{X_i\} = \{X_1, \cdots X_{4n}, X_{4n+1} = \xi_1, X_{4n+2} = \xi_2, X_{4n+3} = \xi_3\}$ is an adapted frame at the point $p \in M$ where $d\eta_\alpha(X_1, X_2) \neq 0$ for each $\alpha = 1, 2, 3$ and denote by the same symbols $\{X_i\}$ the frame obtained by parallel translation along geodesic rays. Our basic arguments follow [2].

Let $Z = \sum_j a^j X_j$ be a direction at p. Then $Z = \sum_j a^j X_j$ is the tangent along the geodesic ray in this direction. Write also $Z = Z' + a\xi_1 + b\xi_2 + c\xi_3$ where $Z' \in \Gamma(\mathcal{D})$. Using condition (i), we get

$$\begin{aligned}
&D_Z(R(X_i, X_j)X_l) \\
&= D_{Z'+a\xi_1+b\xi_2+c\xi_3}(R(X_i, X_j)X_l) \\
&= aD_{\xi_1}(R(X_i, X_j)X_l) + bD_{\xi_2}(R(X_i, X_j)X_l) + cD_{\xi_3}(R(X_i, X_j)X_l) \\
&= ah_1^{-1}D_{[X_1,X_2]}(R(X_i, X_j)X_l) + bh_2^{-1}D_{[X_1,X_2]}(R(X_i, X_j)X_l) \\
&\quad + ch_3^{-1}D_{[X_1,X_2]}(R(X_i, X_j)X_l) \\
&\quad - (ah_1^{-1} + ch_3^{-1})h_2 D_{\xi_1}(R(X_i, X_j)X_l) \\
&\quad - (ah_1^{-1} + bh_2^{-1})h_3 D_{\xi_2}(R(X_i, X_j)X_l) \\
&\quad - (ah_2^{-1} + ch_3^{-1})h_1 D_{\xi_3}(R(X_i, X_j)X_l)
\end{aligned}$$

and analogously for the torsion tensor, where $h_\alpha := \eta_\alpha([X_1, X_2])$ for each α is a function. Moreover it may be simplified as followings:

$$\begin{aligned}
&- (ah_1^{-1} + ch_3^{-1})h_2 D_{\xi_2}(R(X_i, X_j)X_l) \\
&- (ah_1^{-1} + bh_2^{-1})h_3 D_{\xi_3}(R(X_i, X_j)X_l) \\
&- (ah_2^{-1} + ch_3^{-1})h_1 D_{\xi_1}(R(X_i, X_j)X_l) \\
= &- \frac{ah_2h_3 + bh_3h_1 + ch_1h_2}{h_1h_2h_3} D_{[X_1,X_2]}(R(X_i, X_j)X_l) \\
&- D_Z(R(X_i, X_j)X_l).
\end{aligned}$$

Therefore

$$(4.1)\quad \begin{aligned} &2D_Z(R(X_i,X_j)X_l)\\ &= ah_1^{-1}D_{[X_1,X_2]}(R(X_i,X_j)X_l)\\ &\quad + bh_2^{-1}D_{[X_1,X_2]}(R(X_i,X_j)X_l)\\ &\quad + ch_3^{-1}D_{[X_1,X_2]}(R(X_i,X_j)X_l)\\ &\quad - (\frac{ah_2h_3+bh_3h_1+ch_1h_2}{h_1h_2h_3})D_{[X_1,X_2]}(R(X_i,X_j)X_l). \end{aligned}$$

Next, to find the function h_α along the geodesic ray determined by $Z = Z' + a\xi_1 + b\xi_2 + c\xi_3$, we compute

$$(4.2)\quad \begin{aligned} 2\dot{h_\alpha} &= -2\eta_\alpha(D_ZT^D)(X_1,X_2)\\ &= -\eta_\alpha(ah_1^{-1}(D_{[X_1,X_2]}T^D)(X_1,X_2)\\ &\quad + (bh_2^{-1}(D_{[X_1,X_2]}T^D)(X_1,X_2)\\ &\quad + ch_3^{-1}(D_{[X_1,X_2]}T^D)(X_1,X_2))\\ &\quad + \eta_\alpha(\frac{ah_2h_3+bh_3h_1+ch_1h_2}{h_1h_2h_3})(D_{[X_1,X_2]}T^D)(X_1,X_2) \end{aligned}$$

for each $\alpha = 1,2,3$. Notice (4.1) and (4.2), the rest of the proof follows [2, Theorem 2.1]. □

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