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FIXED POINT THEOREMS, SECTION PROPERTIES AND MINIMAX INEQUALITIES ON K-G-CONVEX SPACES

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ABSTRACT. In [11] Kim obtained fixed point theorems for maps defined on some "locally G-convex" subsets of a generalized convex space. Theorem 2 in Kim's article determines us to introduce, in this paper, the notion of K-G-convex space. In this framework we obtain fixed point theorems, section properties and minimax inequalities.

1. Introduction

Motivated by the well-known works of Horvath [7, 8, 9], there have appeared many generalizations of the concept of convex subset of a topological vector space. The most general one seems to be that of *generalized convex space* or *G-convex space* introduced by Park and Kim [15], which extends many of topological spaces having generalized convexity structures.

In [11, Theorem 1] Kim extends the fixed point theorem of Kakutani-Fan-Glicksberg to maps defined on some "locally G-convex" subsets of G-convex spaces. Kim's result determines us to introduce, in this paper, the notion of K-G-convex space. In this framework we obtain a fixed point theorem for the composite of two Kakutani maps. Using this, we get a new fixed point theorem, section properties and minimax inequalities. A part of our results seem to be new even in the classical case (when the K-G-convex space is a convex subset of a locally convex topological vector space), although they are closely related to some known results.

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Let us recall the terminology from [11] needed in the sequel. For a set A let $|A|$ denote the cardinality of A and $\langle A\rangle$ the set of all nonempty finite subsets of A. Let Δ_n denote the standard n-simplex, that is,

$$\Delta_n = \left\{ u \in \mathbb{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u)e_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i(u) = 1 \right\},$$

where e_i is the i-th unit vector in $\mathbb{R}^{n+1}$.

A *generalized convex space* or a *G-convex space* $(X;\Gamma)$ consists of a topological space X and a map $\Gamma : \langle X\rangle \to X$ satisfying:

(i) $A, B \in \langle X\rangle$, $A \subset B$ implies $\Gamma_A = \Gamma(A) \subset \Gamma_B$; and
(ii) for each $A \in \langle X\rangle$ with $|A| = n+1$ there exists a continuous function $\Phi_A : \Delta_n \to \Gamma_A$ such that $J \in \langle A\rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma_J$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_1, a_2, ..., a_{n+1}\}$ and $J = \{a_{i_1}, a_{i_2}, ..., a_{i_k}\}$ then $\Delta_J = \mathrm{co}\{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$.

Note that Γ_A does not need contain A, for $A \in \langle X\rangle$.

If for each $A \in \langle X\rangle$, Γ_A is assumed to be contractible, then $(X;\Gamma)$ becomes an *H-space* [1, 2, 3] or a *c-space* [7, 8, 9]. There is a lot of other examples of G-convex spaces, see [15].

For an G-convex space $(X;\Gamma)$ a subset C of X is said to be *G-convex* if $A \in \langle C\rangle$ implies $\Gamma_A \subset C$. For a nonempty subset S of X, the *G-convex hull* of S, is denoted and defined by

$$G\text{-co}S = \cap\{Y : S \subset Y \subset X \text{ and } Y \text{ is } G\text{-convex}\}.$$

In [11] Kim defines two types of subsets of an G-convex space. An G-convex space which, in Kim's terminology, is of type II will be called in this paper an *K-G-convex space.* More exactly and *K-G-convex space* is an G-convex space $(X;\Gamma)$ satisfying the following conditions:

(i) for each $x \in X$, $\{x\}$ is G-convex; and
(ii) for any compact G-convex subset A of X and each open neighborhood V of A there exists an open neighborhood U of A such that $G\text{-co}U \subset V$.

Every convex subset X of a locally convex vector topological space is an K-G-convex space by putting $\Gamma_A = \mathrm{co}A$, where co denotes the convex hull in the usual sense.

Let I be a nonempty finite index set. For each $i \in I$, let $(X_i;\Gamma^i)$ be an K-G-convex space and $X = \prod_{i\in I} X_i$. Define $\Gamma : \langle X\rangle \to X$ by

$$\Gamma_A = \prod_{i\in I} \Gamma^i_{A_i},$$

where $A_i = p_i(A)$ and $p_i : X \to X_i$ is the canonical projection. Then $(X;\Gamma)$ becomes an K-G-convex space with the product topology (see [16]).

A *map* (or a *multifunction*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is, a function with the *values* $Tx \subset Y$ for $x \in X$ and the *fibers* $T^{-}y = \{x \in X : y \in Tx\}$ for $y \in Y$. Given two maps $S : X \multimap Y$, $T : Y \multimap Z$ the *composite* $T \circ S : X \multimap Z$ is defined by $(T \circ S)x = T(Sx) = \cup\{Ty : y \in Sx\}$.

Let X and Y be topological spaces. A *continuous selection* $p : X \to Y$ of a map $T : X \multimap Y$ is a continuous function such that $p(x) \in Tx$ for all $x \in X$. A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $F \subset Y$ the *lower inverse* of F under T, that is $T^{-1}(F) = \{x \in X : Tx \cap F \neq \emptyset\}$, is a closed subset of X or, equivalently, if for each open set $G \subset Y$ the *upper inverse* of G under T, that is $T^{+1}(G) = \{x \in X : Tx \subset G\}$, is an open subset of X. Note that if Y is compact Hausdorff and Tx is closed for each $x \in X$, then T is upper semicontinuous if and only if the graph of T, that is $\{(x,y) \in X \times Y : y \in Tx\}$, is closed in $X \times Y$. Recall also that the composite and the product of two u.s.c. maps are u.s.c., too.

Throughout this paper, we assume that any topological space is Hausdorff.

2. Fixed points for composite maps in K-G-convex spaces

If X is a topological space and $(Y;\Gamma)$ an G-convex space we define the classes of maps $\widehat{K}(X,Y)$ and $K(X,Y)$ as follows:

$T \in \widehat{K}(X,Y) \Leftrightarrow T$ is u.s.c. with compact G-convex values.

$T \in K(X,Y) \Leftrightarrow T \in \widehat{K}(X,Y)$ and $Tx \neq \emptyset$ for each $x \in X$.

We remark that in a special case the class $K(X,Y)$ was considered for the first time by Kakutani [10]. For this reason a map $T \in K(X,Y)$ is called a *Kakutani map*.

The following result established in [11, Theorem 2] is the starting point of our investigations. It extends to K-G-convex spaces the classical Kakutani-Fan-Glicksberg fixed point theorem.

THEOREM 2.1. *Let $(X;\Gamma)$ be a compact K-G-convex space. Then any $T \in K(X,X)$ has a fixed point.*

THEOREM 2.2. *Let $(X;\Gamma_1)$, $(Y;\Gamma_2)$ be two compact K-G-convex spaces. Then for every two maps $S \in K(X,Y)$, $T \in K(Y,X)$ the composite $T \circ S$ has a fixed point.*

Proof. Consider the diagram

$$X \times Y \xrightarrow{p} Y \times X \xrightarrow{T\times S} X \times Y,$$

where $p(x,y) = (y,x)$ and $[T \times S](y,x) = Ty \times Sx$. It is easy to see that $[T \times S] \circ p \in K(X \times Y, X \times Y)$. By Theorem 2.1, the map $[T \times S] \circ p$ has a fixed point, i.e., for some $(x_0, y_0) \in X \times Y$ we have $(x_0, y_0) \in (T \times S)(y_0, x_0)$. Hence $x_0 \in Ty_0$, $y_0 \in Sx_0$ and consequently, $x_0 \in (T \circ S)x_0$. □

Since any fixed point for the composite $T \circ S$ is a coincidence point for the maps T and S^{-1}, Theorem 2.2 generalizes Granas and Liu [4, Theorem 5.1].

THEOREM 2.3. *Let $(X;\Gamma_1)$, $(Y;\Gamma_2)$ be two compact K-G-convex spaces, and $S \in K(X,Y)$. Let $T : Y \multimap X$ be a map having one of the following properties:*

(i) *T has a continuous selection.*
(ii) *There exists a map $R : Y \multimap X$ such that*
  (ii_1) *$G - \mathrm{co}(Ry) \subset Ty$ for each $y \in Y$;*
  (ii_2) *$Y = \cup\{\mathrm{Int}R^{-1}x : x \in X\}$.*
(iii) *T has nonempty G-convex values and open fibers.*

Then $T \circ S$ has a fixed point.

Proof. Clearly (iii) implies (ii) and by assertion (i) of Theorem 1 in [12] it follows that (ii) implies (i). Therefore it suffices to prove that $T \circ S$ has a fixed point if T has a continuous selection p. Since $p \in K(Y,X)$, by Theorem 2.2 there exists $x_0 \in X$ such that $x_0 \in (p \circ S)x_0$, whence $x_0 \in (T \circ S)x_0$. □

3. Selection properties, minimax inequalities

As a direct consequence of Theorem 2.2 we have:

THEOREM 3.1. *Let $(X;\Gamma_1)$, $(Y;\Gamma_2)$ be two compact K-G-convex spaces and M, N be two open subsets of $X \times Y$ such that $M \cup N = X \times Y$. Suppose that the following conditions are satisfied:*

(i) *For each $x \in X$, $\{y \in Y : (x,y) \notin M\}$ is G-convex.*
(ii) *For each $y \in Y$, $\{x \in X : (x,y) \notin N\}$ is G-convex.*

Then at least one of the following assertions holds:

(a) *There exists a point $x_0 \in X$ such that $\{x_0\} \times Y \subset M$.*
(b) *There exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.*

Proof. Let $M' = (X \times Y) \setminus M$ and $N' = (X \times Y) \setminus N$. Define $S : X \to Y$, $T : Y \to X$ by putting

$$Sx = \{y \in Y : (x, y) \in M'\},$$
$$Ty = \{x \in X : (x, y) \in N'\}.$$

Since M' is closed in $X \times Y$, each Sx is closed in Y and the graph of S is closed in $X \times Y$. Hence S is u.s.c. and by (i) it follows that $S \in \widehat{K}(X, Y)$. Similarly we can prove that $T \in \widehat{K}(Y, X)$.

Suppose that both assertions (a) and (b) are not true. Then for each $x \in X$ there exists $y \in Y$ such that $(x, y) \in M'$, that is $S \in K(X, Y)$ and similarly $T \in K(Y, X)$. By Theorem 2.2, $T \circ S$ has a fixed point, or equivalently there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in Sx_0$ and $x_0 \in Ty_0$. Then $(x_0, y_0) \in M' \cap N'$ which contradicts $M \cup N = X \times Y$. □

COROLLARY 3.2. *Let $(X; \Gamma_1)$, $(Y; \Gamma_2)$ be two compact K-G-convex spaces and N be an open subset of $X \times Y$ satisfying:*

(i) *There exists a map $T \in K(X, Y)$ such that graph$T \subset M$.*
(ii) *For each $y \in Y$, $\{x \in X : (x, y) \notin N\}$ is G-convex.*

Then there exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.

Proof. Consider the set

$$M = X \times Y \setminus \operatorname{graph} T$$

Since $T \in K(X, Y)$ it readily follows that:

$$\begin{cases} M \text{ is an open subset of } X \times Y; \\ \text{for each } x \in X,\ \{y \in Y : (x, y) \notin M\} \text{ is } G\text{-convex;} \\ \text{for each } x \in X,\ \{x\} \times Y \not\subset M. \end{cases}$$

Moreover $M \cup N = X \times Y$. The conclusion follows from Theorem 3.1. □

COROLLARY 3.3. *Let $(X; \Gamma)$ be a compact K-G-convex space and M be an open subset of $X \times X$ satisfying:*

(i) $\Delta = \{(x, x) : x \in X\} \subset M$.
(ii) *For each $x \in X$, $\{y \in X : (x, y) \notin M\}$ is G-convex.*

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset M$.

Proof. Apply Theorem 3.1 with $Y = X$, $N = X \times X \setminus \Delta$ and observe that the assertion (b) in the conclusion of this theorem cannot take place. □

THEOREM 3.4. *Let $(X; \Gamma_1)$, $(Y; \Gamma_2)$, M, N be as in Theorem 3.1. Suppose that for each $x \in X$ there exists an open subset (possibly empty) O_x of Y such that:*

(iii) *For each* $x \in X$, $O_x \subset \{y \in Y : (x,y) \notin N\}$.
(iv) $\cup_{x \in X} O_x = Y$.

Then there exists $x_0 \in X$ *such that* $\{x_0\} \times Y \subset M$.

Proof. It suffices to prove that under conditions (iii) and (iv) the assertion (b) of the conclusion of Theorem 3.1 does not hold.

Since Y is compact there exists a finite set $A = \{x_1, x_2, ..., x_{n+1}\} \subset X$ such that $Y = \cup_{i=1}^{n+1} O_{x_i}$. Let $\{\alpha_i : 1 \leq i \leq n+1\}$ be a continuous partition of unity subordinated to the open covering $\{O_{x_i} : 1 \leq i \leq n+1\}$ of the compact Y, that is,

$$\begin{cases} \text{for each } i,\ \alpha_i : Y \to [0,1] \text{ is continuous;} \\ \alpha_i(y) > 0 \Rightarrow y \in O_{x_i}; \\ \sum_{i=1}^{n+1} \alpha_i(y) = 1 \text{ for each } y \in Y. \end{cases}$$

Define a continuous map $p : Y \to \Delta_n$ by

$$p(y) = \sum_{i=1}^{n+1} \alpha_i(y) e_i$$

(recall that the e_i are vertices of Δ_n). Let $J(y) = \{x_i \in A : \alpha_i(y) > 0\}$. Then $p(y) \in \Delta_{J(y)}$. By the definition of G-convex space, there exists a continuous function $\Phi_A : \Delta_n \to \Gamma_A$ such that $\Phi_A(\Delta_J) \subset \Gamma_J$ for each $J \in \langle A \rangle$. Therefore

$$(*) \qquad (\Phi_A \circ p)(y) \in \Phi_A(\Delta_{J(y)}) \subset \Gamma_{J(y)}.$$

For each $x_i \in J(y)$ we have $y \in O_{x_i}$, hence by (iii), $(x_i, y) \notin N$. Since the sets $\{x \in X : (x,y) \notin N\}$ are G-convex (see condition (ii) in Theorem 3.1), from (*) we get

$$((\Phi_A \circ p)(y), y) \notin N \text{ for each } y \in Y.$$

Hence $X \times \{y\} \not\subset N$ for each $y \in Y$. □

Let $(X;\Gamma)$ be an G-convex space. A function $f : X \to \mathbb{R}$ will be called *G-quasiconcave* if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \geq \lambda\}$ is G-convex and *G-quasiconvex* if $-f$ is G-quasiconcave.

THEOREM 3.5. *Let* $(X;\Gamma_1)$, $(Y;\Gamma_2)$ *be two compact K-G-convex spaces, and* $f, g : X \times Y \to \mathbb{R}$ *two functions satisfying:*

(i) $f \leq g$.
(ii) f *is upper semicontinuous and* g *is lower semicontinuous on* $X \times Y$.
(iii) *For each* $x \in X$, $f(x, \cdot)$ *is G-quasiconcave on* Y.
(iv) *For each* $y \in Y$, $g(\cdot, y)$ *is G-quasiconvex on* X.

Then, given any $\alpha, \beta \in \mathbb{R}$, $\beta < \alpha$, at least one of the following assertions holds:

(a) *There exists $x_0 \in X$ such that $f(x_0, y) < \alpha$ for each $y \in Y$.*
(b) *There exists $y_0 \in Y$ such that $g(x, y_0) > \beta$ for each $x \in X$.*

Proof. Apply Theorem 3.1 with the sets:

$$\begin{aligned} M &= \{(x, y) \in X \times Y : f(x, y) < \alpha\}, \\ N &= \{(x, y) \in X \times Y : g(x, y) > \beta\}. \end{aligned}$$

From the hypothesis (i)-(iv) it follows readily that M, N are open in $X \times Y$, $M \cup N = X \times Y$ and assumptions (i) and (ii) of Theorem 3.1 are verified. The desired result follows now from Theorem 3.1. □

It would be of some interest to compare the next minimax inequality with Theorem 18 of Park [14].

COROLLARY 3.6. *Under the hypothesis of Theorem 3.5 the following inequality holds:*

$$\inf_{x \in X} \max_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} g(x, y).$$

Proof. First let us observe that if f is upper semicontinuous on $X \times Y$, then for each $x \in X$, $f(x, \cdot)$ is also an upper semicontinuous function of y on Y and therefore its maximum $\max\limits_{y \in Y} f(x, y)$ on the compact set Y exists. Similarly $\inf\limits_{x \in X} g(x, y)$ can be replaced by $\min\limits_{x \in X} g(x, y)$.

Suppose the conclusion were false and choose two real numbers α, β such that

$$\sup_{y \in Y} \min_{x \in X} g(x, y) < \beta < \alpha < \inf_{x \in X} \max_{y \in Y} f(x, y).$$

We prove that neither the assertion (a) nor the assertion (b) of the conclusion of Theorem 3.5 cannot take place.

If (a) happens, then

$$\inf_{x \in X} \max_{y \in Y} f(x, y) \leq \max_{y \in Y} f(x_0, y) \leq \alpha, \text{ a contradiction.}$$

If (b) happens, then

$$\sup_{y \in Y} \min_{x \in X} g(x, y) \geq \min_{x \in X} g(x, y_0) \geq \beta, \text{ a contradiction again.}$$ □

THEOREM 3.7. *Let X, Y, f, g be as in Theorem 3.5. If $T : X \to Y$ is a map with nonempty values then the following inequality holds:*

$$\inf_{y \in Tx} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} g(x, y).$$

Proof. We may assume that $\inf_{y\in Tx} f(x,y) > -\infty$. Apply Theorem 3.5 with the case $\alpha = \inf_{y\in Tx} f(x,y)$, $\beta = \inf_{y\in Tx} f(x,y) - \varepsilon$, where $\varepsilon > 0$ is arbitrarily fixed. Since the values of T are nonempty, the assertion (a) of the conclusion of Theorem 3.5 cannot take place. It follows that there exists $y_0 \in Y$ such that

$$\min_{x\in X} g(x,y_0) > \inf_{y\in Tx} f(x,y) - \varepsilon.$$

Clearly this implies the desired minimax inequality. □

Close results in topological vector spaces have been obtained by Granas and Liu [5, Theorem 7.1] and Ha [6, Theorem 1]. In both mentioned results T is a Kakutani map while in our theorems T is only a map with nonempty values.

In [14, Theorem 13] Park extends to G-convex spaces Fan's minimax inequality. The next result can be considered as a variant of Park's result.

COROLLARY 3.8. *Let $(X;\Gamma)$ be a compact K-G-convex space and $f,g : X\times X \to \mathbb{R}$ two functions satisfying:*

(i) $f \le g$.
(ii) *f is upper semicontinuous and g is lower semicontinuous on $X\times X$.*
(iii) *For each $x\in X$, $f(x,\cdot)$ is G-quasiconcave.*
(iv) *For each $y\in X$, $g(\cdot,y)$ is G-quasiconvex.*

Then we have

$$\inf_{x\in X} f(x,y) \le \sup_{y\in X}\min_{x\in X} g(x,x).$$

Proof. Apply Theorem 3.7 with $X=Y$, $Tx=\{x\}$. □

REMARK. All the results obtained in this paper are applications of Theorem 2.1. For this reason they remain true in other particular compact G-convex spaces for which Theorem 2.1 holds. For instance they remain true in compact Φ-spaces (see [13, Theorem 4]), and in any LC-space for which every singleton is G-convex (see [13, Theorem 5]).

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