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ON HÖLDER-MCCARTHY-TYPE INEQUALITIES WITH POWERS

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ABSTRACT. We extend the Hölder-McCarthy inequality for a positive and an arbitrary operator, respectively. The powers of each inequality are given and the improved Reid's inequality by Halmos is generalized. We also give the bound of the Hölder-McCarthy inequality by recursion.

Let A be a positive (bounded and linear) operator (written $A \geq 0$) on a Hilbert space H. Then, for any $x \in H$ and a given positive real number γ,

$$\text{(a)} \qquad (A^\gamma x, x) \leq (Ax, x)^\gamma \|x\|^{2(1-\gamma)}, \quad \gamma \in (0, 1],$$

and

$$\text{(b)} \qquad (A^\gamma x, x) \geq (Ax, x)^\gamma \|x\|^{2(1-\gamma)}, \quad \gamma \geq 1.$$

McCarthy [7] proved the inequalities above by using the spectral resolution of A and the Hölder inequality, which justifies the terminology: the Hölder-McCarthy inequality. His proof is simple, but not elementary by no means.

In this paper, we shall generalize the inequalities (a), (b) and consider the powers of the inequalities for a positive and an arbitrary operator, respectively. Also, the improved Reid's inequality by Halmos is extended and the bound of $(A^n x, x) - (Ax, x)^n$ for $n = 1, 2, \cdots$ and $\|x\| = 1$ is

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given recursively together with the equality condition. For other recent improvements on Reid's inequality, see [3] and [5].

Before we proceed, we need to know that, if $A \geq 0$, then

(1) $A^\alpha \geq 0$ for any real number $\alpha \geq 0$,

(2) $|(Ax, y)|^2 \leq (Ax, x)(Ay, y)$ for every $x, y \in H$.

The inequality (2) is known as the Cauchy-Schwarz inequality for a positive operator A. For more information on Cauchy-Schwarz inequality for high-order and high-power, one may refer to [6]. These two properties would be frequently used throughout this paper without mentioning them. The identity operator on H is denoted by I, which is positive, and $A > 0$ means that $A \geq 0$ and A is invertible.

THEOREM 1. *For $A \geq 0$, a given positive real number $\gamma \geq 1$ and for every $x, y \in H$, we have*

$$|(Ax, y)|^\gamma \leq (A^\gamma x, x)^{\frac{1}{2}} (A^\gamma y, y)^{\frac{1}{2}} \|x\|^{\gamma-1} \|y\|^{\gamma-1}. \tag{1}$$

More generally, for $n = 1, 2, \cdots$, we have

$$\begin{aligned} &|(Ax, y)|^\gamma \\ &\leq (A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{2^n}} (A^{2^{n-1}(\gamma-1)+1} y, y)^{\frac{1}{2^n}} \\ &\quad \times (Ax, x)^{\frac{2^{n-1}-1}{2^n}} (Ay, y)^{\frac{2^{n-1}-1}{2^n}} \|x\|^{\gamma-1} \|y\|^{\gamma-1} \end{aligned} \tag{2}$$

satisfying the relation

$$\begin{aligned} &(A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{2^n}} (Ax, x)^{\frac{2^{n-1}-1}{2^n}} \\ &\leq (A^{2^n(\gamma-1)+1} x, x)^{\frac{1}{2^{n+1}}} (Ax, x)^{\frac{2^n-1}{2^{n+1}}}. \end{aligned}$$

Proof. (1) By the inequality (b), we have

$$\begin{aligned} |(Ax, y)|^\gamma = |(Ax, y)|^{2 \cdot \frac{\gamma}{2}} &\leq (Ax, x)^{\frac{\gamma}{2}} (Ay, y)^{\frac{\gamma}{2}} \\ &\leq (A^\gamma x, x)^{\frac{1}{2}} (A^\gamma y, y)^{\frac{1}{2}} \|x\|^{\gamma-1} \|y\|^{\gamma-1}. \end{aligned}$$

$$\begin{aligned} (2) \qquad (A^\gamma x, x)^{\frac{1}{2}} &= (AA^{\gamma-1} x, x)^{2 \cdot \frac{1}{4}} \leq (A^{2\gamma-1} x, x)^{\frac{1}{4}} (Ax, x)^{\frac{1}{4}} \\ &= (AA^{2\gamma-2} x, x)^{2 \cdot \frac{1}{8}} (Ax, x)^{\frac{1}{4}} \\ &\leq (A^{4\gamma-3} x, x)^{\frac{1}{8}} (Ax, x)^{\frac{3}{8}}. \end{aligned}$$

For $n \geq 2$, suppose that

$$\begin{aligned}&(A^{2^{n-2}(\gamma-1)+1}x,x)^{\frac{1}{2^{n-1}}}(Ax,x)^{\frac{2^{n-2}-1}{2^{n-1}}}\\ &\leq (A^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}(Ax,x)^{\frac{2^{n-1}-1}{2^n}}.\end{aligned}$$

Then we have

$$\begin{aligned}&(A^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}(Ax,x)^{\frac{2^{n-1}-1}{2^n}}\\ &= (AA^{2^{n-1}(\gamma-1)}x,x)^{2\cdot\frac{1}{2^{n+1}}}(Ax,x)^{\frac{2^{n-1}-1}{2^n}}\\ &\leq (A^{2^n(\gamma-1)+1}x,x)^{\frac{1}{2^{n+1}}}(Ax,x)^{\frac{2^n-1}{2^{n+1}}}\end{aligned}$$

as $(Ax,x)^{\frac{1}{2^{n+1}}+\frac{2^{n-1}-1}{2^n}} = (Ax,x)^{\frac{2^n-1}{2^{n+1}}}$. Similarly, we can consider the term $(A^\gamma y,y)^{\frac{1}{2}}$, and conclude that the proof is completed by induction.□

Remark 1. The Hölder-McCarthy inequality (a) and two inequalities (1) and (2) in Theorem 1 are all equivalent to one another.

Theorem 2. *Let T be an arbitrary operator. If γ is a positive real number with $\gamma \geq 1$, then, for every $x, y \in H$,*

$$(1) \qquad |(Tx,y)|^\gamma \leq ((T^*T)^\gamma x,x)^{\frac{1}{2}}\|x\|^{\gamma-1}\|y\|^\gamma.$$

More generally, for $n = 1, 2, \cdots$, we have

$$\begin{aligned}(2) \qquad &|(Tx,y)|^\gamma\\ &\leq ((T^*T)^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}(T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}}\|x\|^{\gamma-1}\|y\|^\gamma\end{aligned}$$

satisfying the relation

$$\begin{aligned}&((T^*T)^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}(T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}}\\ &\leq ((T^*T)^{2^n(\gamma-1)+1}x,x)^{\frac{1}{2^{n+1}}}(T^*Tx,x)^{\frac{2^n-1}{2^{n+1}}}.\end{aligned}$$

Proof. (1) Clearly $T^*T \geq 0$. By the inequality (b), we have

$$\begin{aligned}|(Tx,y)|^\gamma &= |(ITx,y)|^{2\cdot\frac{\gamma}{2}} \leq [(ITx,Tx)(Iy,y)]^{\frac{\gamma}{2}}\\ &= (T^*Tx,x)^{\frac{\gamma}{2}}\|y\|^\gamma\\ &\leq ((T^*T)^\gamma x,x)^{\frac{1}{2}}\|x\|^{\gamma-1}\|y\|^\gamma.\end{aligned}$$

(2) We have

$$
\begin{aligned}
&((T^*T)^{\gamma}x,x)^{\frac{1}{2}}\\
&= ((T^*T)\ (T^*T)^{\gamma-1}x,x)^{2\cdot\frac{1}{4}}\\
&\leq ((T^*T)^{2\gamma-1}x,x)^{\frac{1}{4}}\ (T^*Tx,x)^{\frac{1}{4}}\\
&= ((T^*T)(T^*T)^{2\gamma-2}x,x)^{2\cdot\frac{1}{8}}\ (T^*Tx,x)^{\frac{1}{4}}\\
&\leq ((T^*T)^{4\gamma-3x,x})^{\frac{1}{8}}\ (T^*Tx,x)^{\frac{3}{8}}.
\end{aligned}
$$

For $n \geq 2$, suppose that

$$
\begin{aligned}
&((T^*T)^{2^{n-2}(\gamma-1)+1}x,x)^{\frac{1}{2^{n-1}}}\ (T^*Tx,x)^{\frac{2^{n-2}-1}{2^{n-1}}}\\
&\leq ((T^*T)^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}\ (T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&((T^*T)^{2^{n-1}(\gamma-1)+1}x,x)^{\frac{1}{2^n}}\ (T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}}\\
&= ((T^*T)(T^*T)^{2^{n-1}(\gamma-1)}x,x)^{2\cdot\frac{1}{2^{n+1}}}\ (T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}}\\
&\leq ((T^*T)^{2^n(\gamma-1)+1}x,x)^{\frac{1}{2^{n+1}}}\ (T^*Tx,x)^{\frac{2^n-1}{2^{n+1}}}
\end{aligned}
$$

as $(T^*Tx,x)^{\frac{2^{n-1}-1}{2^n}+\frac{1}{2^{n+1}}} = (T^*Tx,x)^{\frac{2^n-1}{2^{n+1}}}$ and so the proof is completed by induction. □

REMARK 2. The inequalities (1) and (2) in Theorem 2 are equivalent to one another. Also, if S is a self-adjoint operator (not necessarily positive), then Theorem 2 may be changed to the following and we shall omit the proof.

(1) $$|(Sx,y)|^{\gamma} \leq (S^{2\gamma}x,x)^{\frac{1}{2}}\|x\|^{\gamma-1}\|y\|^{\gamma}.$$

More generally, for $n = 1, 2, \cdots$, we have

(2) $$|(Sx,y)|^{\gamma} \leq (S^{2^n(\gamma-1)+2}x,x)^{\frac{1}{2^n}}(S^2x,x)^{\frac{2^{n-1}-1}{2^n}}\|x\|^{\gamma-1}\|y\|^{\gamma}$$

satisfying the relation

$$
\begin{aligned}
&(S^{2^n(\gamma-1)+2}x,x)^{\frac{1}{2^n}}(S^2x,x)^{\frac{2^{n-1}-1}{2^n}}\\
&\leq (S^{2^{n+1}(\gamma-1)+2}x,x)^{\frac{1}{2^{n+1}}}(S^2x,x)^{\frac{2^n-1}{2^{n+1}}}.
\end{aligned}
$$

Recall that the *spectral radius* of an operator T is denoted by $r(T)$, which is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T)$ is the spectrum of T. Note that clearly $0 \le r(T) \le \|T\|$ and $r(T)$ is known to be equal to $\lim_{n\to\infty}\|T^n\|^{1/n}$.

The relation $|(AEx, x)| \le \|E\|(Ax, x)$ for all $x \in H$ is known as the Reid inequality for $A \ge 0$, and an operator E such that AE is a self-adjoint operator ([8]). In [2], Halmos sharpened the inequality in that he has $r(E)$ instead of $\|E\|$. Our inequality (2) in Theorem 3 below is a further generalization with a different proof.

THEOREM 3. *Let $A \ge 0$ and let E, F be any operators such that AE and AF are self-adjoint. Then, for every $x, y \in H$, a positive real number $\gamma \ge 1$ and $n = 1, 2, \cdots$, we have*

$$(1)\quad |(AEx, Fy)|^{\gamma} \le (AE^{2^n}x, x)^{\frac{\gamma}{2^n}}(Ax, x)^{\frac{(2^{n-1}-1)\gamma}{2^n}}(AF^{2^n}y, y)^{\frac{\gamma}{2^n}}(Ay, y)^{\frac{(2^{n-1}-1)\gamma}{2^n}}$$

satisfying the relation

$$(AE^{2^n}x, x)^{\frac{\gamma}{2^n}}(Ax, x)^{\frac{(2^{n-1}-1)\gamma}{2^n}} \le (AE^{2^{n+1}}x, x)^{\frac{\gamma}{2^{n+1}}}(Ax, x)^{\frac{(2^n-1)\gamma}{2^{n+1}}}$$

and

$$(AE^{2^n}y, y)^{\frac{\gamma}{2^n}}(Ay, y)^{\frac{(2^{n-1}-1)\gamma}{2^n}} \le (AE^{2^{n+1}}y, y)^{\frac{\gamma}{2^{n+1}}}(Ay, y)^{\frac{(2^n-1)\gamma}{2^{n+1}}}.$$

In particular,

$$(2)\qquad |(AEx, Fy)| \le r(E)r(F)(Ax, x)^{\frac{1}{2}}(Ay, y)^{\frac{1}{2}}.$$

Proof. (1) Notice first that $(E^*)^iAE^i = AE^{2i}$ and $(F^*)^iAF^i = AF^{2i}$ for $i = 1, 2, \cdots$ due to the self-adjointness of AE and AF. Next, we see that

$$|(AEx, Fy)|^{\gamma} = |(AEx, Fy)|^{2\cdot\frac{\gamma}{2}} \le (AEx, Ex)^{\frac{\gamma}{2}}(AFy, Fy)^{\frac{\gamma}{2}} = (AE^2x, x)^{\frac{\gamma}{2}}(AF^2y, y)^{\frac{\gamma}{2}}.$$

Now, consider the term $(AE^2x,x)^{\frac{\gamma}{2}}$ as follows:

$$\begin{aligned}(AE^2x,x)^{\frac{\gamma}{2}} &= (AE^2x,x)^{2\cdot\frac{\gamma}{4}} \leq (AE^2x,E^2x)^{\frac{\gamma}{4}}(Ax,x)^{\frac{\gamma}{4}} \\ &= (AE^4x,x)^{\frac{\gamma}{4}}(Ax,x)^{\frac{\gamma}{4}} = (AE^4x,x)^{2\cdot\frac{\gamma}{8}}(Ax,x)^{\frac{\gamma}{4}} \\ &\leq (AE^8x,x)^{\frac{\gamma}{8}}(Ax,x)^{\frac{3\gamma}{8}}.\end{aligned}$$

For $n \geq 2$, suppose that

$$(AE^{2^{n-1}}x,x)^{\frac{\gamma}{2^{n-1}}}(Ax,x)^{\frac{(2^{n-2}-1)\gamma}{2^{n-1}}} \leq (AE^{2^n}x,x)^{\frac{\gamma}{2^n}}(Ax,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}}.$$

Then we have

$$\begin{aligned}&(AE^{2^n}x,x)^{\frac{\gamma}{2^n}}(Ax,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}} \\ &= (AE^{2^n}x,x)^{2\cdot\frac{\gamma}{2^{n+1}}}(Ax,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}} \\ &\leq (AE^{2^n}x,E^{2^n}x)^{\frac{\gamma}{2^{n+1}}}(Ax,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}+\frac{\gamma}{2^{n+1}}} \\ &= (AE^{2^{n+1}}x,x)^{\frac{\gamma}{2^{n+1}}}(Ax,x)^{\frac{(2^n-1)\gamma}{2^{n+1}}}.\end{aligned}$$

This together with a similar consideration for the term $(AF^2y,y)^{\frac{\gamma}{2}}$ implies the inequality (1) by induction.

(2) We may replace γ in (1) above by 2^n to get

$$\begin{aligned}&|(AEx,Fy)|^{2^n} \\ &\leq (AE^{2^n}x,x)(Ax,x)^{2^{n-1}-1}(AF^{2^n}y,y)(Ay,y)^{2^{n-1}-1} \\ &\leq \|A\|^2\|E^{2^n}\|\|x\|^2(Ax,x)^{2^{n-1}-1}\|F^{2^n}\|\|y\|^2(Ay,y)^{2^{n-1}-1}.\end{aligned}$$

The desired inequality follows by taking the 2^n-th root of both sides above and passing to the limit as $n \to \infty$. This completes the proof. □

REMARK 3. The positive operator A in Theorem 3 may be relaxed to a self-adjoint operator S. In other words, if E and F are any operators such that S^2E is self-adjoint, then, for any $x, y \in H$, a positive real number $\gamma \geq 1$ and $n = 1, 2, \cdots$, we have

(1) $|(SEx,Fy)|^\gamma$

$$\leq (S^2E^{2^n}x,x)^{\frac{\gamma}{2^n}}(S^2x,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}}((F^*F)^{2^{n-1}}y,y)^{\frac{\gamma}{2^n}}\|y\|^{\frac{(2^{n-1}-1)\gamma}{2^{n-1}}}$$

satisfying the relation

$$
\begin{aligned}
&(S^2E^{2^n}x,x)^{\frac{\gamma}{2^n}}(S^2x,x)^{\frac{(2^{n-1}-1)\gamma}{2^n}} \\
&\le (S^2E^{2^{n+1}}x,x)^{\frac{\gamma}{2^{n+1}}}(S^2x,x)^{\frac{(2^n-1)\gamma}{2^{n+1}}}.
\end{aligned}
$$

Note that

$$
(2) \qquad |(SEx,Fy)| \le r(E)r(F^*F)^{\frac{1}{2}}(S^2x,x)^{\frac{1}{2}}\|y\|.
$$

We shall omit the proof.

The next result depends on the Hölder-McCarthy inequality (b) wherever is appropriate.

THEOREM 4. *For every $x, y \in H$, we have the following:*

(1) *If $A \ge 0$, then, for a positive real number $\gamma \in (0,1]$,*

$$
|(A^\gamma x,y)| \le (Ax,x)^{\frac{\gamma}{2}}(Ay,y)^{\frac{\gamma}{2}}\|x\|^{1-\gamma}\|y\|^{1-\gamma}.
$$

(2) *If $A > 0$, then, for $n = 1, 2, \cdots$ and any real number μ,*

$$
\begin{aligned}
|(A^\mu x,y)|^{2^n} &\le (A^{2^{n-1}\mu-2^{n-1}+1}x,x) \\
&\quad \times (Ax,x)^{2^{n-1}-1}(A^{2^{n-1}\mu-2^{n-1}+1}y,y)(Ay,y)^{2^{n-1}-1}.
\end{aligned}
$$

(3) *If $A > 0$, then, for $n = 1, 2\cdots$ and a positive real number $\gamma \in (\frac{2^{n-1}-1}{2^{n-1}}, 1]$,*

$$
|(A^\gamma x,y)|^{2^n} \le (Ax,x)^{2^{n-1}\gamma}(Ay,y)^{2^{n-1}\gamma}\|x\|^{2^n(1-\gamma)}\|y\|^{2^n(1-\gamma)}.
$$

Proof. (1) By the inequality (a), we have

$$
\begin{aligned}
|(A^\gamma x,y)|^2 &\le (A^\gamma x,x)(A^\gamma y,y) \\
&\le (Ax,x)^\gamma(Ay,y)^\gamma\|x\|^{2(1-\gamma)}\|y\|^{2(1-\gamma)}.
\end{aligned}
$$

(2) Note that

$$
\begin{aligned}
|(A^\mu x,y)|^4 &\le (A^\mu x,x)^2(A^\mu y,y)^2 = (AA^{\mu-1}x,x)^2(AA^{\mu-1}y,y)^2 \\
&\le (A^{2\mu-1}x,x)(Ax,x)(A^{2\mu-1}y,y)(Ay,y)
\end{aligned}
$$

and

$$\begin{aligned}|(A^{\mu}x,y)|^8 &\le (A^{2\mu-1}x,x)^2(Ax,x)^2(A^{2\mu-1}y,y)^2(Ay,y)^2\\ &= (AA^{2\mu-2}x,x)(Ax,x)^2(A^{2\mu-2}y,y)(Ay,y)^2\\ &\le (A^{4\mu-3}x,x)(Ax,x)^3(A^{4\mu-3}y,y)(Ay,y)^3.\end{aligned}$$

For $n \ge 2$, suppose that

$$\begin{aligned}|(A^{\mu}x,y)|^{2^{n-1}} \le\ & (A^{2^{n-2}\mu-2^{n-2}+1}x,x)(Ax,x)^{2^{n-2}-1}\\ &\times (A^{2^{n-2}\mu-2^{n-2}+1}y,y)(Ay,y)^{2^{n-2}-1}.\end{aligned}$$

Then it follows that

$$\begin{aligned}|(A^{\mu}x,y)|^{2^n} &\le (A^{2^{n-2}\mu-2^{n-2}+1}x,x)^2(Ax,x)^{2^{n-1}-2}\\ &\quad\times (A^{2^{n-2}\mu-2^{n-2}+1}y,y)^2(Ay,y)^{2^{n-1}-2}\\ &\le (A^{2^{n-1}\mu-2^{n-1}+1}x,x)(Ax,x)^{2^{n-1}-1}\\ &\quad\times (A^{2^{n-1}\mu-2^{n-1}+1}y,y)(Ay,y)^{2^{n-1}-1}\end{aligned}$$

since we have

$$\begin{aligned}(A^{2^{n-2}\mu-2^{n-2}+1}x,x)^2 &= (AA^{2^{n-2}\mu-2^{n-2}}x,x)^2\\ &\le (A^{2^{n-1}\mu-2^{n-1}+1}x,x)(Ax,x).\end{aligned}$$

The claim is thus proved.

(3) If $2^{n-1}\gamma - 2^{n-1} + 1 \in (0,1]$, i.e., $\gamma \in (\frac{2^{n-1}-1}{2^{n-1}}, 1]$, then we have

$$(A^{2^{n-1}\gamma-2^{n-1}+1}x,x) \le (Ax,x)^{2^{n-1}\gamma-2^{n-1}+1}\|x\|^{2^n(1-\gamma)}$$

and

$$(A^{2^{n-1}\gamma-2^{n-1}+1}y,y) \le (Ay,y)^{2^{n-1}\gamma-2^{n-1}+1}\|y\|^{2^n(1-\gamma)}$$

as $2[1-(2^{n-1}\gamma-2^{n-1}+1)] = 2^n(1-\gamma)$ for the power of $\|x\|$ and $\|y\|$. The required inequality is clear now due to (2) above. This completes the proof. □

REMARK 4. The Hölder-McCarthy inequality (a) and two inequalities (1) and (3) in Theorem 4 are all equivalent to one another.

Finally, we are going to find the bound of the Hölder-McCarthy inequality (b) by recursion. Now we assume that $\|x\| = 1$ in order to simplify the expression. First, we require the next lemma, for which the tool of the proof is the Cauchy-Schwarz inequality.

LEMMA. *Let $A \geq 0$ and let x be a unit vector. Then, for $n = 1, 2, \cdots$,*

$$\begin{aligned}&[(Ax, A^n x) - (Ax, x)(A^n x, x)]^2 \\ &\leq [\|Ax\|^2 - (Ax, x)^2][\|A^n x\|^2 - (A^n x, x)^2].\end{aligned}$$

The equality holds if and only if $A^n x = cx + dAx$ for some real numbers c and d.

Proof. Let $u = \|A^n x\|^2 - (A^n x, x)^2$, which is nonnegative by the Cauchy-Schwarz inequality. The required inequality is trivial if $u = 0$ (equivalently, x and $A^n x$ are proportional). So, let $u > 0$ and put $v = (Ax, A^n x) - (Ax, x)(A^n x, x)$. Then we have

$$\begin{aligned}0 &\leq \|uAx - vA^n x\|^2 - (uAx - vA^n x, x)^2 \\ &= u^2\|Ax\|^2 - 2uv(Ax, A^n x) + v^2\|A^n x\|^2 \\ &\quad - [u^2(Ax, x)^2 - 2uv(Ax, x)(A^n x, x) + v^2(A^n x, x)^2] \\ &= u\{u[\|Ax\|^2 - (Ax, x)^2] - v^2\},\end{aligned}$$

which yields $u[\|Ax\|^2 - (Ax, x)^2] \geq v^2$ and so we have the desired inequality.

The equality holds if and only if $\|uAx - vA^n x\| = |(uAx - vA^n x, x)|$. Equivalently, $uAx - vA^n x$ and x are proportional and so the equality condition follows. This completes the proof. □

REMARK 5. The equality condition in Lemma can be checked as follows:

Necessity is trivial by the proof. Since $A^n x = cx + dAx$, a straightforward computation shows that both sides of the inequality are equal to $d^2[\|Ax\|^2 - (Ax, x)^2]^2$ and so sufficiency is proved.

THEOREM 5. *Let $A \geq 0$ and let x be a unit vector. Then, for $n = 1, 2, \cdots$,*

$$\begin{aligned}&(A^n x, x) - (Ax, x)^n \\ &\leq [\|Ax\|^2 - (Ax, x)^2]^{\frac{1}{2}} [\|A^{n-1}x\|^2 - (A^{n-1}x, x)^2]^{\frac{1}{2}} \\ &\quad + (Ax, x)[(A^{n-1}x, x) - (Ax, x)^{n-1}].\end{aligned}$$

The equality holds if and only if $A^{n-1}x = cx + dAx$ for some real numbers c and d.

Proof. The proof is a straightforward application of Lemma as follows:

$$\begin{aligned}&(A^n x, x) - (Ax, x)^n \\ &= (Ax, A^{n-1}x) - (Ax, x)(A^{n-1}x, x) + (Ax, x)(A^{n-1}x, x) - (Ax, x)^n \\ &\leq [\|Ax\|^2 - (Ax, x)^2]^{\frac{1}{2}} [\|A^{n-1}x\|^2 - (A^{n-1}x, x)^2]^{\frac{1}{2}} \\ &\quad + (Ax, x)[(A^{n-1}x, x) - (Ax, x)^{n-1}].\end{aligned}$$

The equality holds if and only if

$$\begin{aligned}&[(Ax, A^{n-1}x) - (Ax, x)(A^{n-1}x, x)]^2 \\ &= [\|Ax\|^2 - (Ax, x)^2][\|A^{n-1}x\|^2 - (A^{n-1}x, x)^2],\end{aligned}$$

which, in turn, implies that if and only if $A^{n-1}x = cx + dAx$ for some real numbers c and d by Lemma again. □

REMARK 6. If $A \geq 0$ and $0 < m \leq A \leq M$ in particular for some real numbers m and M, then it can be shown from Theorem 5 that $(A^n x, x) - (Ax, x)^n$ is bounded by a function of m and M. This result was precisely obtained in [1, Theorem 2] by the fact that the covariance of A and A^n is bounded by a function of m and M. For further developments of the variance-covariance inequality, refer to [4].

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