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Some Asymptotic Stability Theorems in the perturbed Linear Differential System

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요 약 미분시스템의 안정성에 관한 이론에서 페론 방법은 각 개념의 정의와 적분부등식을 통해서 해의 정성적 규명을 연구하는 최근에 가장 일반적 형식 중의 하나이다. 이 논문을 통해서는 특히, 두 개의 섭동 $e(t,x)$와 $f(t,x)$를 수반하는 미분 시스템의 자명해와 접근적 안정성의 여러 가지 양태를 페론 방법을 써서 조사해 보고 이들의 충분조건을 찾아 몇 가지 정리를 얻었다.

Abstract We investigate some asymptotic stabilities of the zero solution for the perturbed linear differential system $dx/dt = A(t)x + e(t,x) + f(t,x)$, by using Perron's method and integral inequalities, etc. and we also find some sufficient conditions that ensure some asymptotic stabilities of the zero solution of the system. And hence we obtain several results of it.

1. Introduction

The Qualitative research of solutions of linear differential systems and their perturbed linear differential systems may be very important for the role of handling various problems on mechanical, electronic, control engineering, or economic, and other practical problems. Many authors have been investigating these problems about them and presenting some properties. Among these properties, the following Perron's theorem is one of the most interesting theorems.

Theorem 1.1 ([12]) Let the perturbed linear differential system

(A) $dx/dt = A(t)x + f(t,x)$

hold the following condition :

(a) $A(t)$ is continuous and bounded on $0 \le t < \infty$,

(b) $f(t,x)$ is continuous on $[0,\infty) \times B_k$, $f(t,x) \equiv 0$,

(c) $\|f(t,x)\| = o(\|x\|)$ uniformly in t.

If the zero solution of the linear differential system $dx/dt = A(t)$ is uniformly asymptotic stable, then the zero solution of (A) is also uniformly asymptotically stable.

Resently, S. K. Chang, H. J. Lee, and Y. S. Oh([4, 5]) introduced several concepts of stabilities and they also extended the Taniguchi results under some conditions for the differential systems

$dy/dt = h(t)^{-1}F(t, h(t)y(t))$ and

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$dy/dt = F(t, k(t)y(t))$, respectively.

We will use the following modified lemma for the well-known integral inequalities in [9] that ensure $k(t)\varphi(t)$ stabilities of the zero solution of the perturbed linear differential systems.(cf. [4])

Lemma 1.2 ([4]) Let the following conditions (a) or (b) hold for functions $f, g \in C[[t_0, \infty), R^+]$ and $F(t,s,u)$, $t \geq s \geq t_0$, $t_0, u \geq 0$:

(a) $f(t) - \int_{t_0}^{t} F(t,s,f(s))ds$

$\leq g(t) - \int_{t_0}^{t} F(t,s,g(s))ds$ for all $t \geq t_0$

and $F(t,s,u)$ is strictly increasing in u for each fixed $t \geq s \geq 0$,

(b) $f(t) - \int_{t_0}^{t} F(t,s,f(s))ds$

$< g(t) - \int_{t_0}^{t} F(t,s,g(s))ds$ for all $t \geq t_0$ and $F(t,s,u)$ is monotone non-decreasing in u for each fixed $t \geq s \geq 0$.

If $f(t_0) < g(t_0)$ for any $t_0 \geq 0$, then $f(t) < g(t)$, for all $t \geq t_0$.

2. Preliminaries

Let R^n and R^+ be the n-dimensional Euclidean space with the convenient norm $\|\cdot\|$ and the set of all non-negative real numbers, respectively.

$C[X, Y]$ denotes the set of all continuous mappings from a topological space X to a topological space Y.

Let $A(t)$ be a continuous $n \times n$ matrix defined for all $t \in R^+$ and let $e, f \in C[[0,\infty) \times R^n, R^n]$ with $e(t,0) = 0$ and $f(t,0) = 0$ for any $t \geq 0$.

We consider a linear differential system

$$dx/dt = A(t)x \qquad (2\text{-}1)$$

and a perturbed linear differential system of (2-1)

$$dx/dt = A(t)x + e(t,x) + f(t,x). \qquad (2\text{-}2)$$

Let $U(t)$ be the fundamental matrix solution of (2-1). Then the solution $x(t)$ of (2-2) satisfies the integral equation

$$\begin{aligned} x(t) &= U(t)U^{-1}(t_0)x(t_0) \\ &+ \int_{t_0}^{t} U(t)U^{-1}(s)(e(s,x(s)) \\ &+ f(s,x(s)))ds, \\ &t \geq t_0. \end{aligned} \qquad (2\text{-}3)$$

Let us now introduce some stability concepts.

We consider a differential system of a general form:

$$dy/dt = g(t,y) \qquad (2\text{-}4)$$

where $g \in C[[0,\infty) \times R^n, R^n]$, $g(t,0) = 0$ for any $t \geq 0$, and we assume that there exists the unique solution for the system (2-4) with sufficiently small positive initial values.

Let $y(t) \equiv y(t; t_0, y_0)$ denote a solution of (2-4) with an initial value (t_0, y_0).

Definition 2.1. ([4]) Let $\varphi(t)$ be a continuous positive real function for all $t \in R^+$. The zero solution of (2-4) is said to be

$\varphi(t)$-stable if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that if $\|y(t_0)\| < \delta(t_0, \varepsilon)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_0$;

$\varphi(t)$-quasi-asymptotically stable if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exist $T(t_0, \varepsilon) > 0$ and $\delta(t_0) > 0$ such that if $\|y(t_0)\| < \delta(t_0)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_0 + T(t_0, \varepsilon)$;

$\varphi(t)$-quasi-uniformly asymptotically stable if the $\delta(t_0)$ and the $T(t_0, \varepsilon)$ in $\varphi(t)$-quasi-asymptotically stable are independent of time

t_0;

$\varphi(t)$*-asymptotically stable* if it is $\varphi(t)$-stable and is $\varphi(t)$-quasi-asymptotically stable.

From now on, some other definitions and terminologies throughout this paper follow those of [1, 2, 3, 6, 7, 8, 10, 11, etc.].

3. Some Asymptotic Stability Theorems

In this section, we discuss $\varphi(t)$-asymptotic stability of the zero solution for the perturbed linear differential system (2-2).

Assume that $f(t,0)\equiv 0$ for all $t\geq 0$ throughout this section.

Let us consider the following comparison differential systems

$$dy/dt = h(t)^{-1}F(t, k(t)y(t)) \qquad (3\text{-}1)$$

$$dy/dt = F(t, k(t)y(t)) \qquad (3\text{-}2)$$

where $k(t)$ and $h(t)$ is a continuous positive real functions on R^+.

Theorem 3.1. Let the following conditions hold for the differential system (2-2)

(i) The zero solution of the differential equation (2-1) is $k(t)$-stable for a continuous positive real function $k(t)$ on R^+, that is, there exists a continuous function $h(t)>0$ on R^+ such that

$$\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1}, \quad t\geq s\geq 0.$$

(ii) there exists a sufficiently small $L>0$ such that $\|f(t,x)\| \leq L\|x\|$ for any $x \in R^n$;

(iii) $\|e(t,x)\| \leq h(t)$ *for each* $(t,x) \in [T,\infty)\times R^n$.

If the zero solution of (3-1) is $\varphi(t)$-quasi-asymptotically stable for a continuous positive real function $\varphi(t)$ on R^+, then the zero solution of (2-2) is $k(t)\varphi(t)$-quasi-asymptotically stable.

If $k(t)=K\exp(\mu t)$ *(in (i) and (3-1)) for a real* number μ and a positive constant K and if the zero solution of (3-1) is $T(\nu)$-quasi-asymptotically stable for a real number ν, then the zero solution of (2-2) is $T(\mu+\nu)$-quasi-asymptotically stable.

Proof. Let $F(t,\|x\|)=L\|x\|$, for any $x\in R^n$, $L>0$. By the hypothesis, since the conditions (i) and (ii) hold for (2-2), we conclude that for all $t\geq T\geq 0$,

$$\begin{aligned}\|x(t)\| &\leq k(t)h(T)^{-1}\|x(T)\| + \int_T^t \|U(t)U^{-1}(s)\|\|e(s,x(s))+f(s,x(s))\|ds \\ &\leq k(t)h(T)^{-1}\|x(T)\| + \int_T^t k(t)h(s)^{-1}h(s)ds \\ &\quad + \int_T^t k(t)h(s)^{-1}L\|x(s)\|ds\end{aligned}$$

for $L>0$.

That is,

$$k(t)^{-1}\|x(t)\| \leq h(T)^{-1}\|x(T)\| + \int_T^t ds + \int_T^t h(s)^{-1}L\|x(s)\|ds.$$

Since the condition (iii) holds, for $T>0$, there exists $M>0$ such that $\int_T^t ds < M$.

Accordingly, $k(t)^{-1}\|x(t)\| - \int_T^t h(s)^{-1}L\|x(s)\|ds \leq h(T)^{-1}\|x(T)\|+M < y_0$ and $y_0 = y(t) - \int_T^t h(s)^{-1}F(s,k(s)y(s))ds.$ And hence we obtain that

$$k(t)^{-1}\|x(t)\| - \int_T^t h(s)^{-1}F(s,\|x(s)\|)ds < y(t) - \int_T^t h(s)^{-1}F(s,k(s)y(s))ds.$$

Let $v(t)=k(t)y(t)$ for all $t\geq T$. Then $y(t)=k(t)^{-1}v(t)$ for all $t\geq T$.

S o ,

$$\|x(t)\| - k(t)\int_T^t h(s)^{-1}F(s,\|x(s)\|)ds < v(t) - k(t)\int_T^t h(s)^{-1}F(s,v(s))ds.$$

Thus, applying Lemma 1.2, we get $\|x(t)\| < v(t) = k(t)y(t)$ for all $t\geq T$.

Also, by the hypothesis, for any $t_0\geq 0$ and for any $\varepsilon>0$, there exists $\delta_0(t_0,\varepsilon)>0$ and

$T \equiv T(t_0, \varepsilon) > 0$n such that if $\|y(T)\| < \delta_0(t_0, \varepsilon)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_0 + T$.

Accordingly, we take

$\delta(t_0, \varepsilon) = h(T)(\delta_0(t_0, \varepsilon) - M)$, if $\|x(T)\| < \delta(t_0, \varepsilon)$, then

$h(T)^{-1}\|x(T)\| + M < \delta_0(t_0, \varepsilon)$ and hence there exists $y_0 = y(T) > 0$ such that $h(T)^{-1}\|x(T)\| + M < y_0 < \delta_0(t_0, \varepsilon)$.

Accordingly, $\|x(t)\| < k(t)y(t)$ for all $t \geq T$ and $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_0 + T(t_0, \varepsilon)$.

Thus

$$\|x(t)(k(t)\varphi(t))^{-1}\| < k(t)y(t)k(t)^{-1}\varphi(t)^{-1}$$
$$= \|y(t)\varphi(t)^{(-1)}\| < \varepsilon \text{ for all } t \geq t_0 + T,$$

which is complete.

Corollary 3.2. Suppose the conditions (ii) and (iii) hold for the differential system (2-2).

If $k(t) = K\exp(\mu t)$ for a real number μ and a positive constant K in (i) and (3-1) and if the zero solution of (3-1) is $T(\nu)$-quasi-asymptotically stable for a real number ν, then the zero solution of (2-2) is $T(\mu+\nu)$-quasi-asymptotically stable.

Theorem 3.3. Suppose that the condition (ii) holds for the differential system (2-2).

Furthermore, suppose that the following conditions are satisfied:

(i)′ the zero solution of the differential system (2-1) is $k(t)$-uniformly stable for a continuous positive real function $k(t)$ on R^+, that is, there exists function $k(t) > 0$ such that $\|U(t)U^{-1}(s)\| \leq k(t),\ t \geq s \geq 0$.

(iii)′ $\|e(t, x)\| \leq h(t)$ for $(t, x) \in [T, \infty) \times R^n$ and for any $\varepsilon > 0$, there exists $T \equiv T(\varepsilon) > 0$ such that $\int_{T(\varepsilon)}^{t} ds < \varepsilon$.

If the zero solution of (3-2) is $\varphi(t)$-quasi-uniformly asymptotically stable for a continuous positive function $\varphi(t)$ on R^+, then the zero solution of (2-2) is $k(t)\varphi(t)$-quasi-uniformly asymptotically stable.

Proof. In the similar method of the theorem 3.1.

Corollary 3.4. Suppose the conditions (ii) and (iii)′ hold for the differential system (2-2).

If $k(t) = K\exp(\mu t)$ for a real μ and a positive constant K in (i)′ and (3-2) and if the zero solution of (3-2) is $T(\nu)$-quasi-uniformly asymptotically stable for real ν, then the zero solution of (2-2) is $T(\mu+\nu)$-quasi-uniformly asymptotically stable.

Theorem 3.5. Suppose that the condition (ii) holds for the differential system (2-2).

Furthermore, suppose that the following conditions are satisfied:

(i)″ There exist positive real number μ and bounded continuous positive real functions $k(t)$ as positive constant $K > 0$ such that

$$\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1}\exp(-\mu(t-s))$$
$$\text{for all } t \geq s \geq 0.$$

(iii)″ $\|e(h, x)\| \leq h(t)$ for $(t, x) \in [0, \infty) \times R^n$ and for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $T \equiv T(t_0, \varepsilon) > 0$ such that $\int_{T(T_0, \varepsilon)}^{t} \exp(\mu s)ds < \varepsilon$ for all $t \geq T$.

If the zero solution of (2-2) is $\varphi(t)$-stable for a continuous positive real funtion $\varphi(t)$, then the zero soiution of (2-2) is asymptotically stable.

Proof. By the hypothesis, we obtain that

$$\|x(t)\|\exp(\mu t) \leq Kh(T)^{-1}\exp(\mu T)\|x(T)\|$$
$$+ \int_T^t Kh(s)^{-1}h(s)\exp(\mu s)ds$$
$$+ \int_T^t Kh(s)^{-1}L\exp(\mu s)\|x(s)\|ds.$$

And also we have for some positive M, there exists $T > 0$, such that $\int_T^t K\exp(\mu s)ds < M$ from the condition (iii)′ .

Let $Kh(T)^{-1}\exp(\mu T)\|x(T)\|+M < y_1$. Then we conclude that in the similar method as in the proof of theorem 3.1,

$$\|x(t)\|\exp(\mu t) - \int_T^t KL\exp(\mu s)\,h(s)^{-1}\|x(s)\|ds$$
$$< \; y(t) - \int_T^t KLh(s)^{-1}y(s)ds.$$

Accordingly, applying Lemma 1.2, we get $\|x(t)\|\exp(\mu t) < y(t)$ for all $t \geq T$.

While, since $L > 0$, we may be that $KL < \mu$. And also, since $dy/dt = KLy$,

$$\int_T^t y'(s)/y(s)\,ds = \int_T^t KLds = KL(t-T).$$

S o

$$y(t) = \exp(\log(\exp y(T)))\exp(KL(t-T))$$
$$= y(T)\exp(KL(t-T)) = y_0\exp(KL(t-T))$$

And hence

$$y(t)\exp(-\mu t) = y_0\exp(KL(t-T))\exp(-\mu t)$$
$$< y_0\exp(KL(t-T))\exp(-\mu(t-T))$$
$$= y_0\exp((KL-\mu)(t-T)).$$

Thus,

for a continuous positive real function $\varphi(t) \geq c$,

$$\|x(t)\varphi(t)^{-1}\| < y(t)\exp(-\mu t)\varphi(t)^{-1}$$
$$< y_0/c \cdot \exp(KL-\mu)(t-T), \quad \text{where}$$

$c > 0$

Therefore, the zero solution of (2-2) is $\varphi(t)$ -asymptotically stable.

Corollary 3.6. Suppose that the conditions (i)′ , (ii), and (iii)′ hold for the differential system (2-2).

Then $\|x(t)\| \to 0$ as $t \to \infty$. Furthermore, if the zero solution of (2-2) is stable, then the zero solution of (2-2) is asymptotically stable.

Reference

[1]. R. Bellman, Stability theory of differential equations, McGraw-Hill, Book Company Inc., New York, 1953.

[2]. N. P. Bhatia and G. D. Szego, Stability theory of dynamical systems, Springer, Berlin, 1970.

[3]. F. Brauer and J. A. Nohel, Qulitative theory of ordinary differential equations, Benjamin, New York, 1969.

[4]. S. K. Chang, H. J. Lee, and Y. S. Oh, A generalization of Perron's stability theorems for perturbed linear differential equations, Kyungpook Math. J. 33, No. 2 (1993), 163-171.

[5]. S. K. Chang, Y. S. Oh, and H. J. Lee, Exponential behavior of solutions for perturbed linear ordinary differential equations, Kyungpook Math. J. 32 (1992), 3, 311-320

[6]. S. K. Choi, K. S. Koo, and K. H. Lee, Lipschitz stability and exponential asymptotic stability in perturbed systems, J. Koren Math. Soc., 29 (1992), 1, 175-190.

[7]. W. Hahn, Stability of motion, Springer, Beylin, 1967.

[8]. A. Halanay, Differential equations: stability, Oscillation, Time Lag. Academic Press, New York, 1966.

[9]. V. Lakshmikantham and S. Leela, Differential and integral inequalities, Vol. I, Academic Press, New York, 1969.

[10]. V. Lakshmikantham, S. Leela, and A. A. Martynyuk, Stability analysis of nonlinear system, Marcel Dekker, Inc., 1989.

[11]. Y. S. Oh and J. H. An, On $T(\mu)$-stability and bounded theorems for perturbed linear differential equations, J. of Nat. Sci. 14, No. 1 (1997), 43-50

[12]. T. Taniguchi, Stability theorems of perturbed linear ordinary differential equations, J. Math. Anal. and Appl. 14´9 (1990), 583-598.

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