Bayesian Multiple Comparison of Normal Populations based on Bayes Factor

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요 약 이 논문에서는 정규분포를 하는 모집단에 대한 베이지안 다중비교를 개발한다. 베이지안 다중비교를 위해서는 베이즈요인의 계산이 필수적인데 베이즈 요인의 계산은 O'Hagan (1995)이 제안한 부분베이즈요인을 이용한다. 그리고 베이지안에서 필수적인 모수에 대한 사전분포로는 무정보적 사전분포를 이용한다. 또한, 비교대상이 되는 모집단의 수가 3이상인 경우에 대하여 베이즈요인의 정확한 형태를 유도했으며 정규분포를 한다고 널리 알려져 있는 자료를 제안된 방법으로 분석하는 사례를 보였으며, 모의실험을 통하여 제안된 방법의 유용성을 보였다.

Abstract In this paper, we develop the Bayesian multiple comparison procedure for the normal model. The procedure which we suggest is based on the fractional Bayes factor of O'Hagan (1995). We apply our procedure to normal populations, when noninformative prior is assumed to the model parameters. We derive explicit form of Bayes Factors when the number of populations is greater than 3. A famous data is analyzed by the proposed procedure. For this example, the suggested method is straightforward for specifying distributionally and to implement computationally, with output readily adapted for required comparison.

1. INTRODUCTION

The multiple comparisons problem (MCP) among treatment means has been studied by many authors and various multiple comparison procedures have been suggested, including Fisher's least significant difference (LDS), Duncan's multiple range test, Student-Newman -Keuls, Tukey's honestly significant difference (HSD), Scheffé's, and so on. (for descriptions of these procedure see Hochberg and Tamhane 1987).

In the Bayesian viewpoint, Duncan (1965) gave the first Bayeisan multiple comparison procedure, for the pairwise comparisons among the means in a one-way layout. Waller and Duncan (1969) modified Duncan's original procedure using a hyperprior distribution for the unknown ratio of the between-to-within variances. A Dirichlet process prior distribution approach to the MCP can be found in Berry (1988), and Gopalan and Berry (1998) developed multiple comparison procedure using Dirichlet process prior.

Generally, Bayesian model selection chooses the model with the highest posterior model probability. This model probability can be calculated by the Bayes factor. In Bayesian testing and selection problems, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constrains often the use of noninformative priors. Since noninformative priors such as Jeffrey's (1961) priors or reference priors (Berger and Bernardo 1989, 1992) are typically improper so that such priors are only up to arbitrary constants which affects the values of Bayes factors. Many people have made efforts to compensate for that arbitrariness.

Among them, fractional Bayes factor (FBF) of O'Hagan (1995) and intrinsic Bayes factor (IBF) of

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Berger and Pericchi (1996) were specially devised to overcome the arbitrariness of Bayes factor when the improper prior is used for model parameters. Berger and Pericchi (1996) introduced the intrinsic Bayes factor (IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor (FBF). For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas.

But there is a little study on Bayesian multiple comparison procedure based on Bayes factor. In developing the Bayesian procedure for MCP, we will suggest a method based on FBF rather than IBF. In MCP for a reasonable number of populations, the use of IBF encountered some of difficulties. And some of the works reveals that IBF and FBF does not make serious differences. So, we prefer to use FBF.

The primary objective of this paper is to provide a Bayesian multiple comparison procedure based on the fractional Bayes factor for normal populations when noninformative priors are used. The outline of the remaining sections is as follows. In Section 2, we review the concept of the FBF methodology and develop the Bayesian multiple comparison procedure. In section 3, we derive expressions of the Bayesian multiple comparison procedure for normal populations. And we give real example to illustrate our procedure. Finally, we give numerical example. From these results, our Bayesian multiple comparison procedure based on FBF very well select the target model.

2. THE BAYESIAN MULTIPLE COMPARISONS PROCEDURE USING FRACTIONAL BAYES FACTOR

2.1 Preliminaries

Models (Hypotheses) $M_1, M_2, \cdots, M_q$ are under consideration, with the data $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ having probability density function $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ under model M_i, $i = 1, 2, \cdots, q$. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distribution of model M_i, and let p_i be the prior probabilities of model M_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the model M_i is true is

$$P(M_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji}\right)^{-1},$$

where B_{ji} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})} = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}. \tag{1}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta}_j)$. Usually, one can use the noninformative prior, often improper, for parameters such as uniform prior, Jeffreys prior, reference prior or probability matching prior. Denote it as π_i^N. The use of improper priors $\pi_i^N(\cdot)$ in (1) cause the B_{ji} to contain arbitrary constants. To solve this problem, O'Hagan (1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow.

When the $\pi_i^N(\boldsymbol{\theta}_i)$ is noninformative prior under H_i, equation (1) becomes

$$B_{ji}^N(\boldsymbol{x}) = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j^N(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}.$$

Then the fraction Bayes factor (FBF) of model H_j versus model H_i is

$$B_{ji}^F = \frac{q_j(b, \boldsymbol{x})}{q_i(b, \boldsymbol{x})},$$

where

$$q_i(b, \boldsymbol{x}) = \frac{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}{\int f_i^b(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i},$$

and $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. One frequently suggested choice is $b=m/n$, where m is the size of the minimal training sample, assuming this is well defined (see O'Hagan, 1995 and the discussion by Berger and Mortera of O'Hagan, 1995).

2.2 Bayesian Multiple Comparison Procedure

Consider k populations with parameters $\boldsymbol{\theta}=(\theta_1,\cdots,\theta_k)'$. Let $\boldsymbol{X}_i=(x_{i1},\cdots,x_{in_i})'$ be a $n_i\times 1$ vector of independent observations on θ_i with density $f(x_{ij}\mid\theta_i)$, $i=1,\cdots,k$, $j=1,\cdots,n_i$. Then the likelihood function for $\boldsymbol{\theta}$ given $\boldsymbol{X}=(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_k)$ is

$$L(\boldsymbol{\theta}\mid\boldsymbol{x})=\prod_{i=1}^{k}\prod_{j=1}^{n_i}f(x_{ij}\mid\theta_i).$$

The multiple comparisons problem (MCP) of k populations is to make inferences concerning relationships among the θ_i's based on $\boldsymbol{X}$. Let $\Omega=\{\boldsymbol{\theta}=(\theta_1,\cdots,\theta_k):\theta_i\in R, i=1,\cdots,k\}$ be the k-dimensional parameter space. Equality and inequality relationships among the θ_i's induce statistical hypotheses that subsets of Ω: $H_0:\Omega_0=\{\theta_i:\theta_1=\cdots=\theta_k\}$, H_1: $\Omega_1=\{\theta_i:\theta_1\neq\theta_2=\cdots=\theta_k\}$, and so on up to H_Q: $\Omega_Q=\{\theta_i:\theta_1\neq\cdots\neq\theta_k\}$. The hypotheses $(H_r:\Omega_r;r=1,\cdots,Q)$, are disjoint, and $\Omega=\bigcup_{r=1}^{Q}\Omega_r$. Each hypothesis can classified $r(r=1,\cdots,Q)$ distinct groups. Let $(\theta_1^*,\cdots,\theta_r^*)$ denote the set of distinct θ_i's, where r is the number of distinct elements in the vector Ω. We define the configuration notation.

Definition 1 (Configuration). The configuration $S=\{S_1,\cdots,S_k\}$ determines a classification of $\boldsymbol{\theta}$ into r distinct groups. Write I_j for the set of indices of parameters in group j, $I_j=\{i:S_i=j\}$. Let $n_{I_j}=\{n_i:i\in I_j\}$ be the index set of observations and θ_j^* be the common parameter value for I_j.

There is a one-to-one correspondence between hypotheses and configurations. Therefore the Bayes factor for MCP can easily compute by this configuration. As an illustration, let $k=5$ and $S=\{1,2,1,2,3\}$. Then $r=3$, $I_1=\{1,3\}$, θ_1^*, $n_{I_1}=\{n_1,n_3\}$, $I_2=\{2,4\}$, θ_2^*, $n_{I_2}=\{n_2,n_4\}$, $I_3=\{5\}$, θ_3^* and $n_{I_3}=\{n_5\}$. Then the noninformative prior for a model with r distinct groups denoted by $\pi_r^N(\theta_1^*,\cdots,\theta_r^*)$.

Now we will develop Bayesian multiple comparisons procedure using fractional Bayes factor. Suppose that a model classified r distinct groups. Then the likelihood function is given by

$$L(\theta_1^*,\cdots,\theta_r^*\mid\boldsymbol{x})=\prod_{l=1}^{r}\prod_{\{i:i\in I_l\}}\prod_{j\in n_{I_l}}f(x_{ij}\mid\theta_l).$$

And the noninformative prior for the model is $\pi_r^N(\theta_1^*,\cdots,\theta_r^*)$. Thus the element of the FBF is given by

$$q(b,\boldsymbol{x})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L(\theta_1^*,\cdots,\theta_r^*\mid X)\pi_r^N(\theta_1^*,\cdots,\theta_r^*)d\theta_1^*\cdots d\theta_r^*\Big/\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L^b(\theta_1^*,\cdots,\theta_r^*\mid X)\pi_r^N(\theta_1^*,\cdots,\theta_r^*)d\theta_1^*\cdots d\theta_r^*.$$

Thus if a model H_i classified r_i distinct groups and a model H_j classified r_j distinct groups then the FBF of H_j versus H_i is given by

$$B_{ji}^F(\boldsymbol{x})=\frac{q_j(b,\boldsymbol{x})}{q_i(b,\boldsymbol{x})},\tag{2}$$

where

$$q_i(b,\boldsymbol{x})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L(\theta_1^*,\cdots,\theta_{r_i}^*\mid X)\pi_{r_i}^N(\theta_1^*,\cdots,\theta_{r_i}^*)d\theta_1^*\cdots d\theta_{r_i}^*\Big/\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L^b(\theta_1^*,\cdots,\theta_{r_i}^*\mid X)\pi_{r_i}^N(\theta_1^*,\cdots,\theta_{r_i}^*)d\theta_1^*\cdots d\theta_{r_i}^*$$

Hence the FBF for all comparisons can computed by equation (2). Using these FBF, we calculate the posterior probability for model H_i, $i=1,\cdots,Q$. Thus for MCP we select the hypothesis with highest posterior probability. Note that as the number k of populations increase, the number of hypotheses increases exponentially. The number of hypotheses as a function of k is given by the Bell exponential number B_k (see Berge 1971). The sequence $\{B_k\}$ can be generated by the recursion $B_{k+1}=\sum_{i=0}^{k} {}_kC_i B_i$, $k=0,1,\cdots$, where $B_0=1$ and $Q=B_k-1$ for $k\geq 2$. For a reasonably moderate number of treatments, such as 8 and 9, the number of hypotheses to be considered (4,140 and 21,147) is very large, even so the developed procedure in this section runs quickly and given a correct results. However for a large number of treatments $(k\geq 10)$, the proposed procedure has a some time on our computer of Pentium II process. Next sections, we deal with MCP for the normal populations.

3. THE NORMAL SAMPLING

Let $X=(X_1,\cdots,X_k)$ be a set of conditionally independent sample, where $X_i=(x_{i1},\cdots,x_{in_i})$ is a sample from a normal distribution with mean θ_i and variance σ^2. Suppose that a model H_i classified r distinct groups. Then the priors for $\theta_i^*,\cdots,\theta_r^*$ and σ^2 are

$$\pi_r(\theta_1^*,\cdots,\theta_r^*,\sigma^2)=\frac{1}{\sigma^2},\ \sigma^2>0.$$

The likelihood function is

$$L(\theta_1^*,\cdots,\theta_r^*,\sigma^2 \mid \boldsymbol{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \times \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}(x_{ij}-\theta_t)^2\right\},$$

where $n=\sum_{i=1}^{k} n_i$. Then the elements of FBF are given by

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L(\theta_1^*,\cdots,\theta_r^*,\sigma^2\mid\boldsymbol{x}) \times\pi_r(\theta_1^*,\cdots,\theta_r^*,\sigma^2)d\theta_1^*\cdots d\theta_r^* d\sigma^2$$
$$=(\sqrt{2\pi})^{-n+r}\Gamma\left(\frac{n-r}{2}\right)\left[\prod_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}\right]^{-1/2} \left[\frac{1}{2}\sum_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}(x_{ij}-\bar{x}_t)^2\right]^{-\frac{n-r}{2}},$$

and

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L^b(\theta_1^*,\cdots,\theta_r^*,\sigma^2\mid\boldsymbol{x}) \times\pi_r(\theta_1^*,\cdots,\theta_r^*,\sigma^2)d\theta_1^*\cdots d\theta_r^* d\sigma^2$$
$$=(\sqrt{2\pi})^{-bn+r}\Gamma\left(\frac{bn-r}{2}\right)\left[\prod_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}b\right]^{-1/2} \times\left[\frac{1}{2}\sum_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}b(x_{ij}-\bar{x}_t)^2\right]^{-\frac{bn-r}{2}},$$

where $\bar{x}_t=\dfrac{\sum_{i\in I_t}\sum_{j\in n_i}x_{ij}}{\sum_{i\in I_t}\sum_{j\in n_i}}$, $t=1,\cdots,r$. Thus

$$q(b,\boldsymbol{x})=\frac{(\sqrt{2\pi})^{-n+r}\Gamma\left(\frac{n-r}{2}\right)\left[\prod_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}\right]^{-1/2}}{(\sqrt{2\pi})^{-bn+r}\Gamma\left(\frac{bn-r}{2}\right)\left[\prod_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}b\right]^{-1/2}} \times\frac{\left[\frac{1}{2}\sum_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}(x_{ij}-\bar{x}_t)^2\right]^{-\frac{n-r}{2}}}{\left[\frac{1}{2}\sum_{t=1}^{r}\sum_{i\in I_t}\sum_{j\in n_i}b(x_{ij}-\bar{x}_t)^2\right]^{-\frac{bn-r}{2}}}.$$

Therefore if a model H_i classified r_i distinct groups and a model H_j classified r_j distinct groups then the FBF of H_j versus H_i is given by

$$B_{ji}^F(\boldsymbol{x}) = \frac{\Gamma\left(\frac{n-r_j}{2}\right)\Gamma\left(\frac{bn-r_i}{2}\right)}{\Gamma\left(\frac{n-r_i}{2}\right)\Gamma\left(\frac{bn-r_j}{2}\right)} \times\frac{\left[\prod_{t=1}^{r_j}\sum_{i\in I_t}\sum_{j\in n_i}\right]^{-1/2}\left[\prod_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_i}b\right]^{-1/2}}{\left[\prod_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_i}\right]^{-1/2}\left[\prod_{t=1}^{r_j}\sum_{i\in I_t}\sum_{j\in n_i}b\right]^{-1/2}}$$

<Table 1> Rhizobium Data

	3DOK13	3DOK4	composite	3DOK7	3DOK5	3DOK1
Treatments	1	2	3	4	5	6
	14.3	17.0	17.3	20.7	17.4	19.4
	14.4	19.4	19.4	21.0	24.8	32.6
	11.8	9.1	19.1	20.5	27.9	27.0
	11.6	11.9	16.9	18.8	25.2	32.1
	14.2	15.8	20.8	18.6	24.3	33.0
Mean	13.26	14.64	18.70	19.92	23.98	28.82
SD	1.43	4.12	1.60	1.13	3.78	5.80

$$\times \frac{[\frac{1}{2}\sum_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_{I_t}}(x_{ij}-\bar{x}_t)^2]^{-\frac{n-r_i}{2}}}{[\frac{1}{2}\sum_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_{I_t}}(x_{ij}-\bar{x}_t)^2]^{-\frac{n-r_i}{2}}} \times \frac{[\frac{1}{2}\sum_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_{I_t}}b(x_{ij}-\bar{x}_t)^2]^{-\frac{bn-r_i}{2}}}{[\frac{1}{2}\sum_{t=1}^{r_i}\sum_{i\in I_t}\sum_{j\in n_{I_t}}b(x_{ij}-\bar{x}_t)^2]^{-\frac{bn-r_i}{2}}}$$

The next example analyzed by Gopalan and Berry (1998) using Dirichlet process priors.

Example 1. Steel and Torrie (1981) reported an experiment measuring nitrogen content in milligrams of red clover plants innoculated with cultures of Rhizobium trifolli plus a composite of five Rhizobium meliloti strains. The treatments were each of five red cover cultures R. trifolli tested individually with a composite of five alfalfa strains (3DOK1, 3DOK4, 3DOK5, 3DOK7, 3DOK13), R. meliloti, and a composite of red clover strains also tested with a composite of the alfalfa strains, making six in all. The experiment was conducted in a greenhouse using completely randomized design with five pots per treatment. The objective was to compare the nitrogen levels for the different treatments. Table 1 gives the data.

We assume that the prior probabilities are equal. Then Table 2 gives the posterior probabilities for hypotheses under the equal variances condition. The number of possible hypotheses for the MCP is 203. To save the space, we write down the posterior probabilities greater than 0.01 in Table 2. In Table 2, a hypothesis "1 1 1 2 2 3" stands for the hypothesis $\theta_1=\theta_2=\theta_3\neq\theta_4=\theta_5\neq\theta_6$. The Hypothesis "1 1 2 2 3 4" $(\theta_1=\theta_2\neq\theta_3=\theta_4\neq\theta_5\neq\theta_6)$ has the largest posterior probability.

The frequentist multiple comparison procedures give the following groupings, where treatments in parentheses are not significantly different at $\alpha=0.05$.

(1) Fisher's LSD and Duncan's multiple range test: (θ_1,θ_2), (θ_2,θ_3), (θ_3,θ_4), (θ_4,θ_5), θ_6

<Table 2> Posterior Probabilities for Hypotheses

Hypothesis	Posterior Probability	Hypothesis	Posterior Probability
1 1 1 2 2 3	0.0181	1 2 2 2 3 4	0.0221
1 1 1 2 3 3	0.0112	1 2 2 3 3 4	0.0183
1 1 1 2 3 4	0.0197	1 2 2 3 4 4	0.0107
1 1 2 2 2 3	0.0601	1 2 2 3 4 5	0.0183
1 1 2 2 3 3	0.0948	1 2 3 3 3 4	0.0221
1 1 2 2 3 4	0.2460	1 2 3 3 4 4	0.0351
1 1 2 3 2 4	0.0235	1 2 3 3 4 5	0.0753
1 1 2 3 3 4	0.0612	1 2 3 4 4 5	0.0183
1 1 2 3 4 4	0.0339	1 2 3 4 5 5	0.0101
1 1 2 3 4 5	0.0722	1 2 3 4 5 6	0.0144
1 2 2 2 3 3	0.0123		

(2) Student-Newman-Keuls:

$(\theta_1, \theta_2, \theta_3)$, $(\theta_2, \theta_3, \theta_4)$, $(\theta_3, \theta_4, \theta_5)$, θ_6

(3) Tukey's HSD and Scheffé's test:

$(\theta_1, \theta_2, \theta_3, \theta_4)$, $(\theta_3, \theta_4, \theta_5)$, (θ_5, θ_6)

The Dirichlet process priors procedure (Gopalan and Berry 1998) give the following groupings under different values of M (concentration parameter):

(1) $(\theta_1 = \theta_2 = \theta_3 = \theta_4 \neq \theta_5 \neq \theta_6)$, $(\theta_1 = \theta_2 \neq \theta_3 = \theta_4 = \theta_5 \neq \theta_6)$, $(\theta_1 = \theta_2 \neq \theta_3 = \theta_4 \neq \theta_5 = \theta_6)$, and $(\theta_1 = \theta_2 \neq \theta_3 = \theta_4 \neq \theta_5 \neq \theta_6)$ have large posterior probabilities compared to their prior probabilities

4. MONTE CARLO SIMULATION STUDY

We examine whether the our procedure for MCP work well under Normal sampling distribution considered in order. Although all configurations for $k(\geq 3)$ populations are considered, we examine our procedure for the MCP under some population and configuration to save the space. We consider $k=5$ populations. For the populations, consider some configuration, $\theta_1 = \theta_2 = \theta_3 \neq \theta_4 = \theta_5$.

We consider that $X = (X_1, \cdots, X_5)$ be a set of independent sample, where $X_i = (x_{i1}, \cdots, x_{in_i})$ is a sample from a normal distribution with mean θ_i and variance σ^2. Put $\sigma_i^2 = 1, i = 1, \cdots, 5$ and $1 = \theta_1 = \theta_2 = \theta_3 \neq \theta_4 = \theta_5 = 2$ with the sample sizes $n_1 = \cdots = n_5 = n = 10, 30$.

We assume that the prior probabilities are equal. Under 1,000 replications, Table 3 and Table 4 gives the posterior probabilities for hypotheses. From the results, our procedure work well for small and moderate sample sizes.

5. CONCLUDING REMARKS

We have considered the problem of developing a Bayesian multiple comparison for normal populations. We proposed the Bayesian multiple comparison procedure based on fraction Bayes factor when the noninformative prior is used. In multiple comparison problem for a reasonable number of populations, the use of the intrinsic Bayes factor encountered some of difficulties at least two reasons as follows. Firstly, to obtain a stable IBF, it needs to decide more complex model between models. In some cases, such as nested model, it is obvious. But, in multiple comparison procedure, it is a difficult to recognize which is the more complex model, and some models have the same level of complexity. And so, the IBF may do not have multiple model coherence. Secondly, it takes too much time to compute IBF because of averaging out, geometrically or arithmetically, all possible cases of the minimal training sample. The proposed method does not encountered such problem in IBF. The suggested Bayesian method allow for probability calculations of hypotheses of equality and inequality under the moderate number of populations $(k \leq 9)$ in our Pentium II computer and give a correct results. However for the large number of populations $(k \geq 10)$, our method run quite long times in Pentium II computer.

<Table 3> Posterior Probabilities for Hypotheses

	Hypothesis	Posterior Probability	Hypothesis	Posterior Probability	Hypothesis	Posterior Probability
$n_1=10$	1 1 1 1 1	0.0121	1 2 1 2 1	0.0055	1 2 3 1 2	0.0036
$n_2=10$	1 1 1 1 2	0.0183	1 2 1 2 2	0.0333	1 2 3 1 3	0.0029
$n_3=10$	1 1 1 2 1	0.0222	1 2 1 2 3	0.0175	1 2 3 1 4	0.0041
$n_4=10$	1 1 1 2 2	0.1930	1 2 1 3 1	0.0096	1 2 3 2 1	0.0035
$n_5=10$	1 1 1 2 3	0.0686	1 2 1 3 2	0.0192	1 2 3 2 2	0.0117
	1 1 2 1 1	0.0104	1 2 1 3 3	0.0703	1 2 3 2 3	0.0035
	1 1 2 1 2	0.0051	1 2 1 3 4	0.0200	1 2 3 2 4	0.0050
	1 1 2 1 3	0.0094	1 2 2 1 1	0.0285	1 2 3 3 1	0.0030
	1 1 2 2 1	0.0054	1 2 2 1 2	0.0054	1 2 3 3 2	0.0039
	1 1 2 2 2	0.0276	1 2 2 1 3	0.0145	1 2 3 3 3	0.0096
	1 1 2 2 3	0.0158	1 2 2 2 1	0.0051	1 2 3 3 4	0.0045
	1 1 2 3 1	0.0110	1 2 2 2 2	0.0107	1 2 3 4 1	0.0048
	1 1 2 3 2	0.0166	1 2 2 2 3	0.0095	1 2 3 4 2	0.0055
	1 1 2 3 3	0.0712	1 2 2 3 1	0.0170	1 2 3 4 3	0.0046
	1 1 2 3 4	0.0205	1 2 2 3 2	0.0102	1 2 3 4 4	0.0204
	1 2 1 1 1	0.0085	1 2 2 3 3	0.0702	1 2 3 4 5	0.0038
	1 2 1 1 2	0.0057	1 2 2 3 4	0.0197		
	1 2 1 1 3	0.0081	1 2 3 1 1	0.0098		

<Table 4> Posterior Probabilities for Hypotheses

	Hypothesis	Posterior Probability	Hypothesis	Posterior Probability	Hypothesis	Posterior Probability
$n_1=30$	1 1 1 1 1	0.0003	1 2 1 2 1	0.0001	1 2 3 1 2	0.0000
$n_2=30$	1 1 1 1 2	0.0029	1 2 1 2 2	0.0052	1 2 3 1 3	0.0001
$n_3=30$	1 1 1 2 1	0.0019	1 2 1 2 3	0.0042	1 2 3 1 4	0.0005
	1 1 1 2 2	0.4565	1 2 1 3 1	0.0008	1 2 3 2 1	0.0000
$n_4=30$	1 1 1 2 3	0.1028	1 2 1 3 2	0.0038	1 2 3 2 2	0.0012
$n_5=30$	1 1 2 1 1	0.0002	1 2 1 3 3	0.1030	1 2 3 2 3	0.0001
	1 1 2 1 2	0.0001	1 2 1 3 4	0.0185	1 2 3 2 4	0.0007
	1 1 2 1 3	0.0010	1 2 2 1 1	0.0042	1 2 3 3 1	0.0001
	1 1 2 2 1	0.0001	1 2 2 1 2	0.0001	1 2 3 3 2	0.0001
	1 1 2 2 2	0.0041	1 2 2 1 3	0.0031	1 2 3 3 3	0.0009
	1 1 2 2 3	0.0034	1 2 2 2 1	0.0001	1 2 3 3 4	0.0006
	1 1 2 3 1	0.0009	1 2 2 2 2	0.0003	1 2 3 4 1	0.0008
	1 1 2 3 2	0.0034	1 2 2 2 3	0.0011	1 2 3 4 2	0.0006
	1 1 2 3 3	0.1032	1 2 2 3 1	0.0041	1 2 3 4 3	0.0006
	1 1 2 3 4	0.0187	1 2 2 3 2	0.0010	1 2 3 4 4	0.0182
	1 2 1 1 1	0.0005	1 2 2 3 3	0.1037	1 2 3 4 5	0.0021
	1 2 1 1 2	0.0001	1 2 2 3 4	0.0181		
	1 2 1 1 3	0.0014	1 2 3 1 1	0.0008		

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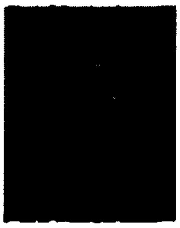

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