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Smooth Ideals

A.A. Ramadan*, Y.C. Kim** and M.K. El-Gayyar***

*Department of Mathematics, Faculty of Science, Cairo University, Beni-Suef , Egypt
**Department of Mathematics, Kangnung National University, Korea
***Department of Mathematics, Faculty of Engineering, Suez-Canal University, Port Said , Egypt

Abstract

In this paper, we show that there exists a one-to-one correspondence between the collection of generalized ideals on a set and the collection of smooth ideals on a set. We study some properties of the images and preimages of smooth ideals.

Key Words : Smooth ideals, Generalized ideals, Images and preimages of smooth ideals

1. Introduction and Preliminaries

Since the notion of a fuzzy set was introduced by Zadeh [7], there has been an attempt to extend useful mathematical notions to this wider setting, replacing sets by fuzzy sets. Hoehle and Sostak [4] introduced the concept of an L-fuzzy filter and established a theory of a convergence for L-fuzzy topological spaces. Gutierrez et al. [3] introduced the relationship between various filter notions.

In this paper, our purpose is to outline how some aspects of the theory of ideal can be extended to fuzzy sets. We investigate the relationship between generalized ideals and smooth ideals. We study some properties of the images and preimages of smooth ideals.

Throughout this paper, let X be a non-empty set and L will be denoted a fuzzy lattice,i.e., a completely distributive lattice with a smallest element 0 and a greatest element 1 and $I=[0,1]$ be the closed unit interval. The family L^X denotes the set of all fuzzy subsets of a given set X and 1_A be the characteristic function of $A\subseteq X$. For each $\alpha\in L$, let $\underline{\alpha}$ denote the constant fuzzy subset of X with value α. All the other notations and the other definitions are standard in fuzzy set theory.

2. Smooth ideals

Definition 2.1 A nonzero function $D: L^X \to L$ is called a smooth ideal (S-ideal, for short) on X if it satisfying the following conditions:

(SI1) $D(\underline{1})=0$.

(SI2) If $\lambda\le\mu$, then $D(\lambda)\ge D(\mu)$, for each $\lambda,\mu\in L^X$.

(SI3) $D(\lambda)\wedge D(\mu)\le D(\lambda\vee\mu)$, for each $\lambda,\mu\in L^X$.

Remark 2.2 (1) The conditions (SI2) and (SI3) are equivalent to the following condition :

(SI) $D(\lambda)\wedge D(\mu)=D(\lambda\vee\mu)$, for each $\lambda,\mu\in L^X$.

(2) Since a smooth ideal D is a nonzero function, by (SI2), $D(\underline{0})>0$.

Definition 2.3 An S-ideal $D: L^X\to L$ is called prime if $D(\lambda)\vee D(\mu)\ge D(\lambda\wedge\mu)$, for each $\lambda,\mu\in L^X$.

Notation 2.4 For a function $B: L^X\to L$ and $\lambda\in L^X$, we denote $\langle B\rangle(\lambda)=\bigvee_{\lambda\le\mu} B(\mu)$.

Definition 2.5 A nonzero function $B: L^X\to L$ is called a smooth ideal base (S-ideal base, for short) on X if it satisfying the following conditions:

(SB1) $B(\underline{1})=0$.

(SB2) $B(\lambda)\wedge B(\mu)\le\langle B\rangle(\lambda\vee\mu)$, for each $\lambda,\mu\in L^X$.

Naturally, an S-ideal is an S-ideal base.

Theorem 2.6 If a function $B: L^X\to L$ is an S-ideal base, then $\langle B\rangle$ is an S-ideal .

Proof The conditions (SI1) and (SI2) are easily checked.

(SI3) $\langle B\rangle(\lambda)\wedge\langle B\rangle(\mu)=(\bigvee_{\lambda\le\lambda_1} B(\lambda_1))\wedge(\bigvee_{\mu\le\mu_1} B(\mu_1))$

(Since L is a completely distributive lattice (ref.[5]))

$$=\bigvee_{\lambda\le\lambda_1}\bigvee_{\mu\le\mu_1}(B(\lambda_1)\wedge B(\mu_1))$$

(Since $\lambda\le\lambda_1,\mu\le\mu_1$ implies $\lambda\vee\mu\le\lambda_1\vee\mu_1$)

$$\le\bigvee_{\lambda\vee\mu\le\lambda_1\vee\mu_1}\langle B\rangle(\lambda_1\vee\mu_1)$$

$$=\langle B\rangle(\lambda\vee\mu).\ (\text{by(SI2)})$$

Thus $\langle B\rangle$ is an S-ideal.

Definition 2.7 An S-ideal D is said to be stratified if, for every $\lambda\in L^X$,

$$D(\lambda)=\bigvee_{\alpha\in L}(\alpha\wedge D(1_{\lambda_\alpha}))$$

where $\lambda_\alpha=\{x\in X \mid \lambda(x)\ge\alpha\}$.

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3. Generalized ideals

Definition 3.1 A nonzero function $d: 2^X \to L$ is called a generalized ideal (g-ideal, for short) on X if it satisfying the following conditions:

(GI1) $d(X)=0$.

(GI2) If $A \subset B$, then $d(A) \geq d(B)$ for each $A, B \in 2^X$.

(GI3) $d(A) \wedge d(B) \leq d(A \cup B)$, for each $A, B \in 2^X$.

Remark 3.2 (1) The conditions (GI2) and (GI3) are equivalent to the following condition:

(GI) $d(A) \wedge d(B) = d(A \cup B)$, for each $A, B \in 2^X$.

(2) Since d is a nonzero function, by (GI2), $d(\emptyset) > 0$.

Notation 3.3 For a function $b: 2^X \to L$ and $A \subset X$, we denote $\langle b \rangle(A) = \bigvee_{A \subset B} b(B)$.

Definition 3.4 A nonzero function $b: 2^X \to L$ is called a generalized ideal base (g-ideal base, for short) on X if it satisfying the following conditions:

(GB1) $b(X)=0$.

(GB2) $b(A) \wedge b(B) \leq \langle b \rangle (A \cup B)$, for each $A, B \in 2^X$.

Naturally, a g-ideal is a g-ideal base.

Corollary 3.5 If a function $b: 2^X \to L$ is a g-ideal base, then $\langle b \rangle$ is a g-ideal.

Proof Similar to the proof of Theorem (2.6).

Definition 3.6 A g-ideal $d: 2^X \to L$ is called prime if $d(A \cap B) \leq d(A) \vee d(B)$, for each $A, B \in 2^X$.

Example 3.7 Let $X=\{x, y, z, w\}$ be a set. Since the unit interval I is a completely distributive lattice (ref.[5]), we define a function $b: 2^X \to I$ as follows:

$$b(A) = \begin{cases} 1, & \text{if } A = \emptyset \\ \frac{1}{3} & \text{if } A \in \{\{x\}, \{y\}\}, \\ \frac{1}{2} & \text{if } A = \{x, y, z\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $b(\{x\}), b(\{y\}) < b(\{x, y, z\})$ and

$$\frac{1}{3} = b(\{x\}) \wedge b(\{y\}) > b(\{x, y\}) = 0,$$

b is not a g-ideal on X. But b is a g-ideal base on X because

$$\frac{1}{3} = b(\{x\}) \wedge b(\{y\}) \leq \langle b \rangle (\{x, y\}) = \frac{1}{2}.$$

Since

$$\langle b \rangle (A) = \begin{cases} 1, & \text{if } A = \emptyset \\ \frac{1}{2} & \text{if } A \in P(\{x, y, z\}) \setminus \{\emptyset\}, \\ 0 & \text{otherwise} \end{cases}$$

where $P(\{x, y, z\})$ is the family of all subsets of $\{x, y, z\}$. $\langle b \rangle$ a g-ideal on X, but not prime because

$$1 = \langle b \rangle (\{x\} \cap \{y\}) \nleq \langle b \rangle (\{x\}) \vee \langle b \rangle (\{y\}) = \frac{1}{2}.$$

Theorem 3.8 Let X be a set and let $G(X)$ denote the collection of g-ideals on X and let $S(X)$ denote the collection of stratified S-ideals on X. For $d \in G(X)$, let $D^d: L^X \to L$ be defined by

$$D^d(\mu) = \bigvee_{\alpha \in L} (\alpha \wedge d(\mu_\alpha)).$$

For $D \in S(X)$, let $d^D: L^X \to L$ be defined by $d^D(A) = D(1_A)$.

Let $\psi: G(X) \to S(X)$ be a mapping with $\psi(d) = D^d$ and $\phi: S(X) \to G(X)$ be a mapping with $\phi(D) = d^D$. Then we have the following properties.

(1) $D^d \in S(X)$.

(2) $d^D \in G(X)$.

(3) $\psi \circ \phi = 1_{S(X)}$, that is, $D^{d^D} = D$.

(4) $\phi \circ \psi = 1_{G(X)}$, that is, $d^{D^d} = d$.

(5) ψ is a bijective function.

(6) d^D is a prime iff D^d is a prime.

Proof (1) For all $A \subset X$ and for all $\alpha \in L$, we have

$$\begin{aligned} D^d(\alpha 1_A) &= \bigvee_{\beta \in L} (\beta \wedge d((\alpha 1_A)_\beta)) \\ &= \bigvee_{\beta \leq \alpha} (\beta \wedge d(A)) \qquad \text{(A)} \\ &= \alpha \wedge d(A). \end{aligned}$$

(SI1) From (A), $D^d(1_X) = 1 \wedge d(X) = 0$.

(SI2) If $\mu_1 \leq \mu_2$ for $\mu_1, \mu_2 \in L^X$, we have

$$\begin{aligned} D^d(\mu_1) &= \bigvee_{\beta \in L} (\beta \wedge d((\mu_1)_\beta)) \\ &\geq \bigvee_{\beta \in L} (\beta \wedge d((\mu_2)_\beta)) \\ &= D^d(\mu_2) \end{aligned}$$

because $(\mu_1)_\beta \subset (\mu_2)_\beta$.

(SI3) For $\mu, \nu \in L^X$,

$$\begin{aligned} D^d(\mu \vee \nu) &= \bigvee_{\beta \in L} (\beta \wedge d((\mu \vee \nu)_\beta)) \\ &= \bigvee_{\beta \in L} (\beta \wedge d((\mu_\beta \vee \nu_\beta)) \\ &\geq \bigvee_{\beta \in L} (\beta \wedge d(\mu_\beta) \wedge d(\nu_\beta)) \\ &= (\bigvee_{\beta \in L} (\beta \wedge d(\mu_\beta))) \wedge (\bigvee_{\beta \in L} (\beta \wedge d(\nu_\beta))) \\ &= D^d(\mu) \wedge D^d(\nu). \end{aligned}$$

From (A), since $D^d(1_{\mu_\alpha}) = d(\mu_\alpha)$ for each $\mu \in L^X$,

$$\begin{aligned} D^d(\mu) &= \bigvee_{\alpha \in L} (\alpha \wedge d((\mu)_\alpha)) \\ &= \bigvee_{\alpha \in L} (\alpha \wedge D^d(1_{\mu_\alpha}). \end{aligned}$$

From Definition 2.7, D^d is stratified. Thus, $D^d \in S(X)$.

(2) The axioms (GI1) and (GI2) follows from (SI1) and (SI2), respectively. Axioms (GI3) follows from (SI3) because

$$\begin{aligned} d^D(A_1)\wedge d^D(A_2) &= D(1_{A_1})\wedge D(1_{A_2}) \\ &\le D(1_{A_1}\vee 1_{A_2}) \\ &= D(1_{A_1\cup A_2}) \\ &= d^D(A_1\cup A_2). \end{aligned}$$

Hence $d^D\in G(X)$.

(3)

$$\begin{aligned} D^{d^D}(\mu) &= \bigvee_{a\in L}(a\wedge d^D(\mu_a)) \\ &= \bigvee_{a\in L}(a\wedge D(1_{\mu_a})) \\ &= D(\mu). \end{aligned}$$

because D is stratified.

(4) If d is a g-ideal on X, then

$$d^{D^d}(A)=D^d(1_A)=\bigvee_{a\in L}(a\wedge d((1_A)_a)=d(A).$$

(5) follows from (3) and (4).

(6) follows from the definitions of D^d and d^D.

Definition 3.9 If $b:2^X\to L$ is a g-ideal base on X, we define the characteristic, denoted by $c(b)$, of b by

$$c(b)=\bigvee_{A\in 2^X} b(A).$$

Lemma 3.10 If $b:2^X\to L$ is a g-ideal base on X, then $c(\langle b\rangle)=c(b)$.

Proof

$$\begin{aligned} c(\langle b\rangle) &= \bigvee_{A\in 2^X}\langle b\rangle(A) \\ &= \bigvee_{A\in 2^X}(\bigvee_{A\subset B} b(B)) \\ &= \bigvee_{B\in 2^X} b(B)=c(b). \end{aligned}$$

Definition 3.11 If $d:2^X\to I$ is a g-ideal (base) on X with $c(d)=k$, then ,for $0\le a< k$, we define the (upper) a-level g-ideal (base), denoted by d^a, associated with d by $d^a=\{A\in 2^X \mid d(A)> a\}$ and, for $0< a\le k$, we define the (lower) a-level g-ideal (base), denoted by d_a, associated with d by $d_a=\{A\in 2^X \mid d(A)\ge a\}$.

Theorem 3.12 If $d:2^X\to I$ is a g-ideal (base) on X with $c(d)=k$, then:

(1) for $0\le a<k$, d^a is an ideal (base) on X.

(2) for $0<a\le k$, d_a is an ideal (base) on X.

Proof (1) Let d be a g-ideal on X. $d(\emptyset)=k>a$ implies $\emptyset\in d^a$. Thus $d^a\ne\emptyset$. Since $d(X)=0$, then $X\notin d^a$. If $A,B\in d^a$ implies $a< d(A)\wedge d(B)=d(A\cup B)$, then $A\cup B\in d^a$. Finally, if $B\in d^a$ and $A\subset B$, then $d(A)\ge d(B)>a$, that is, $A\in d^a$.

The proof of (2) is clear and left to the reader.

4. Images and preimages of smooth ideals

In this section, we study the images and preimages of smooth ideals on the unit interval I instead of L.

Definition 4.1 Let $f:X\to Y$ be a function and $B:I^X\to I$ be an S-ideal base on X. The image of B is a function $f(B):I^Y\to I$ which is defined as follows:

$$f(B)(\mu)=\begin{cases}\bigvee_{f(\lambda)=\mu}B(\lambda), & \text{if } f(\lambda)=\mu,\\ 0, & \textit{otherwise}.\end{cases}$$

Theorem 4.2 Let $f:X\to Y$ be a function and $B:I^X\to I$ be an S-ideal base on X. Then $f(B)$ is an S-ideal base on Y.

Proof Since B is non-zero, there exists $\lambda\in I^X$ such that $B(\lambda)>0$ and so $f(B)(f(\lambda))\ge B(\lambda)>0$. Hence $f(B)$ is non-zero.

(SB1) From the definition of $f(B)$, $f(B)(1_Y)=0$.

(SB2) Suppose there exist $\mu_1,\mu_2\in I^Y$ such that

$$f(B)(\mu_1)\wedge f(B)(\mu_2)\not\le\langle f(B)\rangle(\mu_1\vee\mu_2).$$

Then there exists $t\in I_0=(0,1]$ such that

$$f(B)(\mu_1)\wedge f(B)(\mu_2)>t>\langle f(B)\rangle(\mu_1\vee\mu_2). \qquad \text{(B)}$$

Since $f(B)(\mu_i)>t$ for $i\in\{1,2\}$, by the definition of $f(B)$, there exists $\lambda_i\in I^X$ with $f(\lambda_i)=\mu_i$ such that

$$f(B)(\mu_1)\wedge f(B)(\mu_2)\ge B(\lambda_1)\wedge B(\lambda_2)>t.$$

Since $\langle B\rangle(\lambda_1\vee\lambda_2)\ge B(\lambda_1)\wedge B(\lambda_2)>t$, there exists $\lambda_3\in I^X$ with $\lambda_1\vee\lambda_2\le\lambda_3$ such that

$$\langle B\rangle(\lambda_1\vee\lambda_2)\ge B(\lambda_3)>t.$$

On the other hand, since $\mu_1\vee\mu_2=f(\lambda_1\vee\lambda_2)\le f(\lambda_3)$,

$$\begin{aligned} \langle f(B)\rangle(\mu_1\vee\mu_2) &= \bigvee_{\mu_1\vee\mu_2\le\mu} f(B)(\mu) \\ &\ge f(B)(f(\lambda_3)) \\ &\ge B(\lambda_3)>t. \end{aligned}$$

It is a contradiction to the equation (B). Thus,

$$f(B)(\mu_1)\wedge f(B)(\mu_2)\le\langle f(B)\rangle(\mu_1\vee\mu_2).$$

Theorem 4.3 Let $f:X\to Y$ be a function and $B:I^X\to I$ be an S-ideal base on X.

(1) If f is surjective and B is an S-ideal, then for each $\mu\in I^Y$, $B^f(\mu)\le\langle f(B)\rangle(\mu)$ where $B^f(\mu)=B(f^{-1}(\mu))$.

(2) If B is a prime S-ideal, then B^f is a prime S-ideal.

(3) $\langle f(\langle B\rangle)\rangle=\langle f(B)\rangle$.

Proof (1) Since f is surjective, $\mu=f(f^{-1}(\mu))$. Thus

$$\begin{aligned}\langle f(B)\rangle(\mu) &= \bigvee_{\mu\le\mu_1} f(B)(\mu_1)\\ &= \bigvee_{\mu\le\mu_1}\bigvee_{f(\lambda)=\mu_1} B(\lambda)\\ &= \bigvee_{\mu\le f(\lambda)} B(\mu)\\ &\ge \bigvee_{\mu=f(f^{-1}(\mu))} B(f^{-1}(\mu))\\ &\ge B(f^{-1}(\mu)) = B^f(\mu).\end{aligned}$$

(2) We easily prove that B^f is an S-ideal. For each $\mu_1,\mu_2\in I^Y$, it is prime because

$$\begin{aligned}B^f(\mu_1\wedge\mu_2) &= B(f^{-1}(\mu_1))\wedge B(f^{-1}(\mu_2))\\ &\le B(f^{-1}(\mu_1))\vee B(f^{-1}(\mu_2))\\ &= B^f(\mu_1)\vee B^f(\mu_2).\end{aligned}$$

(3)

$$\begin{aligned}\langle f(\langle B\rangle)\rangle(\mu) &= \bigvee_{\mu\le f(\lambda)}\langle B\rangle(\lambda) \quad (\text{by}(1))\\ &= \bigvee_{\mu\le f(\lambda)}\bigvee_{\lambda\le\lambda_1} B(\lambda_1)\\ &= \bigvee_{\mu\le f(\lambda_1)} B(\lambda_1)\\ &= \langle f(B)\rangle(\mu). \quad (\text{by}(1))\end{aligned}$$

Example 4.4 Let $X=\{a,b,c\}$ and $Y=\{x,y,z\}$ be sets. Let $f:X\to Y$ be a function as follows:

$f(a)=f(b)=x,\ f(c)=y.$

(1) We define a function $B:I^X\to I$ as follows:

$$B(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\\ \frac{2}{3}, & \text{if } \lambda=1_{\{a\}},\\ \frac{1}{2}, & \text{if } \lambda\in\{1_{\{b\}},1_{\{a,b\}}\},\\ 0, & otherwise,\end{cases}$$

Then B is an S-ideal base on X. We obtain

$$f(B)(\mu)=\begin{cases}1, & \text{if } \mu=\underline{0},\\ \frac{2}{3}, & \text{if } \mu=1_{\{x\}},\\ 0, & otherwise,\end{cases}$$

$$B^f(\mu)=B(f^{-1}(\mu))=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},1_{\{z\}}\}\\ \frac{1}{2}, & \text{if } \lambda\in\{1_{\{x\}},1_{\{x,z\}}\},\\ 0, & otherwise,\end{cases}$$

$$\langle f(B)\rangle(\mu)=\begin{cases}1, & \text{if } \mu=\underline{0},\\ \frac{2}{3}, & \text{if } \underline{0}<\mu\le 1_{\{x\}},\\ 0, & otherwise.\end{cases}$$

Since f is not surjective, $B^f\not\le\langle f(B)\rangle$.

(2) We define a function $D:I^X\to I$ as follows:

$$D(\lambda)=\begin{cases}1, & \text{if } \underline{0}\le\lambda\le 1_{\{a,b\}},\\ 0, & otherwise,\end{cases}$$

Then D is a prime S-ideal on X. Since

$$D^f(\mu)=\begin{cases}1, & \text{if } \underline{0}\le\mu\le 1_{\{x,z\}},\\ 0, & otherwise,\end{cases}$$

D^f is a prime S-ideal on X.

$$f(D)(\mu)=\begin{cases}1, & \text{if } \underline{0}\le\mu\le 1_{\{x\}},\\ 0, & otherwise.\end{cases}$$

Since $1=f(D)(1_{\{y\}}\wedge 1_{\{z\}})\not\le f(D)(1_{\{y\}})\vee f(D)(1_{\{z\}})=0$, $f(D)$ is not a prime S-ideal on Y.

Definition 4.5 Let $f:X\to Y$ be a function and $B:I^Y\to I$ be an S-ideal base on Y. The preimage of B is a function $f^{-1}(B):I^X\to I$ which is defined as follows:

$$f^{-1}(B)(\lambda)=\begin{cases}\bigvee_{\lambda=f^{-1}(\mu)} B(\mu), & \text{if } \lambda=f^{-1}(\mu),\\ 0, & otherwise.\end{cases}$$

Theorem 4.6 Let $f:X\to Y$ be a function and $B:I^Y\to I$ be an S-ideal base on Y. Then $f^{-1}(B)$ is an S-ideal base on X iff $B(\mu)=0$, for all $\mu\in I^Y$ such that $f^{-1}(\mu)=1_X$.

Proof ($\Rightarrow$) Since $f^{-1}(B)$ is an S-ideal base on X, $\underline{0}=f^{-1}(B)(1_X)=\bigvee_{1_X=f^{-1}(\mu)} d(\mu)$.

So, for all $\mu\in I^Y$ with $f^{-1}(\mu)=1_X$ implies $B(\mu)=0$.

($\Leftarrow$) Since $B\ne 0$, there exists $\rho\in I^Y$ with $B(\rho)>0$, $f^{-1}(B)(f^{-1}(\rho))=\bigvee_{f^{-1}(\mu)=f^{-1}(\rho)} d(\mu)\ge B(\rho)>0$.

Thus, $f^{-1}(B)$ is non-zero.

(SB1) From the condition,

$f^{-1}(B)(1_X)=\bigvee_{1_X=f^{-1}(\mu)} B(\mu)=0.$

(SB2) Suppose there exist $\lambda_1,\lambda_2\in I^X$ and $t\in I_0$ such that

$f^{-1}(B)(\lambda_1)\wedge f^{-1}(B)(\lambda_2)>t>\langle f^{-1}(B)\rangle(\lambda_1\vee\lambda_2).$ (C)

Since $f^{-1}(B)(\lambda_i)>t$ for $i\in\{1,2\}$, by the definition of $f^{-1}(B)$, there exists $\mu_i\in I^Y$ with $f^{-1}(\mu_i)=\lambda_i$ such that

$f^{-1}(B)(\lambda_1)\wedge f^{-1}(B)(\lambda_2)\ge B(\mu_1)\wedge B(\mu_2)>t.$

Since $\langle B\rangle(\mu_1\vee\mu_2)\ge B(\mu_1)\wedge B(\mu_2)>t$,

there exists $\mu_3\in I^Y$ with $\mu_1\vee\mu_2\le\mu_3$ such that

$\langle B\rangle(\mu_1\vee\mu_2)\ge B(\mu_3)>t.$

On the other hand, since $\lambda_1\vee\lambda_2\le f^{-1}(\mu_3)$,

$$\begin{aligned}\langle f^{-1}(B)\rangle(\lambda_1\vee\lambda_2) &= \bigvee_{\lambda_1\vee\lambda_2\le\lambda} f^{-1}(B)(\lambda)\\ &\ge f^{-1}(B)(f^{-1}(\mu_3))\\ &\ge B(\mu_3)>t.\end{aligned}$$

It is a contradiction to the equation (C). Thus,

$f^{-1}(B)(\lambda_1)\wedge f^{-1}(B)(\lambda_2)\le\langle f^{-1}(B)\rangle(\lambda_1\vee\lambda_2).$

Hence $f^{-1}(B)$ is an S-ideal base.

Theorem 4.7 Let $f: X \to Y$ be a function and $B: I^Y \to I$ be an S-ideal base on Y. Then:

(1) If f is surjective, $f^{-1}(B)$ is an S-ideal base on X.

(2) If B is a (prime) S-ideal with $B(\mu)=0$, for all $\mu \in I^Y$ such that $f^{-1}(\mu)=1_X$ and f is injective, then $f^{-1}(B)$ be a (prime) S-ideal on X.

Proof (1) Since f is surjective iff $f^{-1}(1_Y)=1_X$ and $B(1_Y)=0$, by Theorem 4.6, $f^{-1}(B)$ is an S-ideal base on X.

(2)

$$\begin{aligned}\langle f^{-1}(B)\rangle(\lambda) &= \bigvee_{\lambda \le \lambda_1} f^{-1}(B)(\lambda_1)\\ &= \bigvee_{\lambda\le\lambda_1}\ \bigvee_{\lambda_1 = f^{-1}(\mu_1)} B(\mu_1)\\ &= \bigvee_{\lambda \le f^{-1}(\mu_1)} B(\mu_1)\\ &\ge f^{-1}(B)(\lambda). \quad (\text{by(SI2)})\end{aligned}$$

Suppose $\langle f^{-1}(B)\rangle \not\le f^{-1}(B)$. Then there exist $\lambda \in I^X$ and $r \in (0,1)$ such that

$\langle f^{-1}(B)\rangle(\lambda) > r > f^{-1}(B)(\lambda)$.

Since $\langle f^{-1}(B)\rangle(\lambda) > r$, there exists $\mu \in I^Y$ with $\lambda \le f^{-1}(\mu)$ such that

$$\langle f^{-1}(B)\rangle(\lambda) \ge B(\mu) > r > f^{-1}(B)(\lambda). \quad \text{(D)}$$

On the other hand, put $\rho = f(\lambda)\wedge\mu$. Since f is injective,

$f^{-1}(\rho) = f^{-1}(f(\lambda)) \wedge f^{-1}(\mu) = \lambda \wedge f^{-1}(\mu) = \lambda$.

Since B is an S-ideal and $\rho \le \mu$, $B(\rho) \ge B(\mu) > r$.

Hence $f^{-1}(B)(\lambda) \ge B(\rho) > r$. It is a contradiction for the equation (D). Thus, $\langle f^{-1}(B)\rangle = f^{-1}(B)$, by Theorem 2.6, $f^{-1}(B)$ is an S-ideal.

If B is prime, we will show that $f^{-1}(B)$ is prime.

Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that

$$f^{-1}(B)(\lambda_1 \wedge \lambda_2) > r > f^{-1}(B)(\lambda_1) \vee f^{-1}(B)(\lambda_2). \quad \text{(E)}$$

Since $f^{-1}(B)(\lambda_1 \wedge \lambda_2) > r$, there exists $\mu \in I^Y$ with $\lambda_1 \wedge \lambda_2 \le f^{-1}(\mu)$ such that

$f^{-1}(B)(\lambda_1 \wedge \lambda_2) \ge B(\mu) > r$.

Let $\mu_i = f(\lambda_i) \vee \mu$. Since f is injective,

$$f^{-1}(\mu_i) = f^{-1}(f(\lambda_i)) \vee f^{-1}(\mu) \ge \lambda_i. \quad \text{(F)}$$

Since $f(\lambda_1 \wedge \lambda_2) = f(\lambda_1) \wedge f(\lambda_2) \le f(f^{-1}(\mu)) \le \mu$,

$\mu_1 \wedge \mu_2 = (f(\lambda_1)\vee\mu) \wedge (f(\lambda_2)\vee\mu) = \mu$.

Since B is prime,

$r < B(\mu) = B(\mu_1 \wedge \mu_2) \le B(\mu_1) \vee B(\mu_2)$.

It implies, by (F) ,

$f^{-1}(B)(\lambda_1) \vee f^{-1}(B)(\lambda_2) \ge B(\mu_1) \vee B(\mu_2) > r$.

It is a contradiction to the equation (E).

Example 4.8 Let $X=\{a,b\}$ and $Y=\{x,y,z\}$ be sets. We define a function $B: I^Y \to I$ as follows:

$$B(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ \frac{1}{2}, & \text{if } \mu \in \{1_{\{x\}}, 1_{\{y\}}, 1_{\{x,y\}}\}, \\ 0, & \textit{otherwise}, \end{cases}$$

Then B is an S-ideal base on Y.

Let $f: X \to Y$ be a function as follows:

$$f(a) = x, f(b) = y.$$

Since $f^{-1}(B)(1_X) = B(1_{\{x,y\}}) = \frac{1}{2} \ne 0$, by Theorem 4.6, $f^{-1}(B)$ is not an S-ideal base on X.

Let $Z=\{a,b,c,d\}$ and $g: Z \to Y$ be defined by $g(a)=g(b)=x,\ g(c)=y, g(d)=z$.

By Theorem 2.6, $\langle B\rangle$ is an S-ideal on Y as bellow:

$$\langle B\rangle(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} \ne \mu \le 1_{\{x,y\}}, \\ 0, & \textit{otherwise}. \end{cases}$$

By Theorem 4.7(1), we obtain that $g^{-1}(\langle B\rangle)$ is an S-ideal base on Z as follows: .

$$g^{-1}(\langle B\rangle)(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \\ \frac{1}{2}, & \text{if } \lambda \in \{\alpha 1_{\{a,b\}}, \beta 1_{\{c\}}, \gamma 1_{\{a,b,c,\}}\}, \\ \frac{1}{2}, & \text{if } \lambda = \alpha 1_{\{a,b\}} \vee \beta 1_{\{c\}}, \\ 0, & \textit{otherwise}. \end{cases}$$

Since

$\frac{1}{2} = g^{-1}(\langle B\rangle)(1_{\{a,b\}})$

$\not\le g^{-1}(\langle B\rangle)(1_{\{a\}}) = g^{-1}(\langle B\rangle)(1_{\{b\}}) = 0$,

$g^{-1}(\langle B\rangle)$ is not an S-ideal on X.

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저 자 소 개

A.A. Ramadan
1979 : The Dept. of Math., Faculty of Sci., Assuit Univ., (B.S)
1982 : The Dept. of Math., Faculty of Sci., Assuit Univ., (M.S)
1987 : The Dept. of Math., Faculty of Sci., Assuit Univ., (Ph.D)
1994～: Professor Dept. of Math., Faculty of Sci, Cairo Univ,
Beni-Suef, Egypt.

Y.C.Kim
1982 : The Dept. of Math., Faculty of Sci., Yonsei Univ., (B.S)
1984 : The Dept. of Math., Faculty of Sci., Yonsei Univ., (M.S)
1991 : The Dept. of Math., Faculty of Sci., Yonsei Univ., (Ph.D)
1991～: Associate professor Dept. of Math., Kangnung Univ.

M.K. El-Gayyar
1979 : The Dept. of Math., Faculty of Sci., Assuit Univ., (B.S)
1984 : The Dept. of Math., Faculty of Sci., Assuit Univ., (M.S)
1991 : The Dept. of Math., Faculty of Eng., Suez-Canal Univ., Port Said, Egypt (Ph.D)