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Fuzzy Regular Topological Spaces Over a Base Space

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Abstract

In this paper, we introduce the notion of fuzzy regular spaces over a base space and investigate some properties of these spaces.

Key words : fuzzy regular, fuzzy regular over a base space, well-behaved

1. Introduction

Since I. M. James has been promoting the fibrewise viewpoint in topology systematically [3-5], fibrewise topology has emerged as a subject in its own right. As a matter of fact interest in fibrewise theory is growing now in many directions. Many of the familiar definitions and theorems of ordinary topology can be generalized, in a natural way, so that one can develop a theory of topology over a base space.

From these points of view, it is considered to be meaningful to generalize the theory of ordinary fuzzy topology.

In this paper, we introduce the notion of fuzzy regular topological spaces over a base space and investigate some properties of these spaces.

2. Preliminaries

For the definitions of the fuzzy topology and fuzzy continuity, we refer to R. Lowen.

Let **Top** be the category of topological spaces and continuous maps and **Fuz** be the category of fuzzy topological spaces and fuzzy continuous maps.

On R we consider the topology T_r generated by $\{(a,\infty) | a\in R\}$. The topological space one obtains giving $I=[0,1]$ the induced topology is denoted by I_r.

We then define the following two functors

$$\iota : \textbf{Fuz} \rightarrow \textbf{Top}$$

by for a fuzzy topological space (X,δ), $\iota(X,\delta)=(X,\iota(\delta))$ where $\iota(\delta)$ is the initial topology on X for the family of functions in $\delta$and the topological space I_r and for a fuzzy continuous function $f: X\rightarrow Y$, $\iota(f)=f$, and

$$\omega : \textbf{Top} \rightarrow \textbf{Fuz}$$

by for a topological space $(X,\mathfrak{T})$, $\omega(X,\mathfrak{T})=(X,\omega(\mathfrak{T}))$ where $\omega(\mathfrak{T})$ is the set of all continuous functions from $(X,\mathfrak{T})$ to $I_r$and for a continuous function $f: X\rightarrow Y$, $\omega(f)=f$.

Remark [6] It is well known that $\iota\circ\omega$ is the identity functor from **Top** to **Top**, and hence ω is a left adjoint of ι. But, in general, for a fuzzy topological space (X,δ), $\delta\leq\omega\circ\iota(\delta)$. If $\delta=\omega(\mathfrak{T})$ for some topology T on X, we say that δ is topologically generated.

Proposition 2.1. [6] Let (X,δ) be a fuzzy topological space. Then δ is topologically generated if and only if $\delta=\omega\circ\iota(\delta)$.

Given a topological space B, the category $\textbf{Top}_B$ of topological spaces over B is defined as follows. A topological space over B is a pair (X,p) consisting of a topological space X and a morphism $p: X\rightarrow B$ in **Top**, called the projection; in practice X alone is usually a sufficient notation. If X and Y are topological spaces over B with projections p and q, respectively, then a morphism $f: X\rightarrow Y$ in **Top** is called a morphism

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over B if $q\circ f=p$. Compositions in **Top**$_B$ is defined according to the compositions in **Top**.

Similarly, for a given fuzzy topological space B, the category **Fuz**$_B$ is defined by the same manner.

Proposition 2.2. The category **FuzB** has an initial structure over **SetB**.

Proof. Let $\{(X_i, p_i) \mid i\in\Lambda\}$ be a family of fuzzy topological spaces over B, (X,p) a set over B and $\{f_i: X\to X_i \mid i\in\Lambda\}$ a family of maps in **Set**$_B$. Let X be endowed with the initial fuzzy topology with respect to the family $\{f_i: X\to X_i \mid i\in\Lambda\}$. Since $p_i\circ f_i=p$ and p_i is fuzzy continuous for all $i\in\Lambda$, (X,p) is an object in FuzB. And for any fuzzy topological space (Y,q) over B and for any map $f: Y\to X$ in **Set**$_B$, f is a morphism in **Fuz**$_B$ if and only if $f_i\circ f$ is a morphism in **Fuz**$_B$. Moreover such a fuzzy topology on X is obviously unique.

Proposition 2.3. *The category* **Fuz**$_B$ *has a final structure over* **Set**$_B$.

Proposition 2.4. *For fuzzy topological spaces* $(X,p), (Y,q)$ *over* B, *let* $X\times_B Y$ *be the fibre product as a set endowed with the initial fuzzy topology with respect to the* $\{pr_1: X\times_B Y\to X, pr_2: X\times_B Y\to Y\}$. *Then* $(X\times_B Y, p\circ pr_1)$ *is the product of* X *and* Y *in* **Fuz**$_B$.

The catesian product of two fuzzy sets $\mu: X\to I$ and $\nu: Y\to I$ is a fuzzy set in $X\times Y$ defined by $(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}$. The product fuzzy topology on $X\times Y$ of (X,τ) and (Y,δ) is the fuzzy topology generated by $\{\mu\times\nu \mid \mu\in\tau, \nu\in\delta\}$. In fact, the product topology described in the above proposition is the same as the subspace topology on $X\times_B Y$ induced by the product topology on $X\times Y$ in **Fuz**. From now on, for fuzzy open sets μ in X and ν in Y, the basic fuzzy open set $(\mu\times\nu)\mid_{X\times_B Y}$ in $X\times_B Y$ will be denoted by $\mu\times_B\nu$.

Let (X,p) be an object in **Top**$_B$. Then $(\omega(X),p)$ is an object in **Fuzω**(B). Moreover, let (X,p) and (Y,q) be objects in **Top**$_B$ and $f: X\to Y$ be a morphism in **Top**$_B$. Then $f:\omega(X)\to\omega(Y)$ is a morphism in **Fuzω**$_{(B)}$. Hence we can define a functor

$$\overline{\omega}: \mathbf{Top}_B \to \mathbf{Fuz}\omega_{(B)}.$$

Conversely, for an object (X,p) in **Fuz**$_B$, $(\iota(X),p)$ is an object in **Topι**$_{(B)}$. And for objects (X,p) and (Y,q) in FuzB and a morphism $f: X\to Y$ in **Fuz**$_B$, $f:\iota(X)\to\iota(Y)$ is a morphism in **Topι**$_{(B)}$. Hence we can define a functor

$$\overline{\iota}: \mathbf{Fuz}_B \to \mathbf{Top}\iota_{(B)}.$$

Note that $\overline{\iota}\circ\overline{\omega}$ is a functor from **Top**$_B$ to **Top**$_B$ and $\overline{\omega}\circ\overline{\iota}$ is a functor from **Fuz**$_B$ to **Fuzω°ι**$_{(B)}$. If B is topologically generated, then $\overline{\omega}\circ\overline{\iota}$ is a functor from **Fuz**$_B$ to **Fuz**$_B$.

We have the following results.

Proposition 2.5. $\overline{\iota}\circ\overline{\omega}$ *is the identity functor from* **Top**$_B$ *to* **Top**$_B$.

Proposition 2.6. $\overline{\omega}$ *is a left adjoint of* $\overline{\iota}$.

3. The fuzzy regular topological spaces over a base space

In this section, we introduce the notion of the fuzzy regular spaces over a base space and investigate some properties of these spaces.

Definition 3.1. [7] Let x_t be a fuzzy point and μ be a fuzzy set in X. We say that μcontains x_t if $t<\mu(x)$.

Definition 3.2. [7] A fuzzy topological space (X,δ) is fuzzy regular if for each fuzzy point x_t and fuzzy open set μ containing x_t, there exists a fuzzy open set ν such that $x_t\in\nu\le cl(\nu)\le\mu$.

The following proposition can be obtained by the straightforward proof.

Proposition 3.3. *Let* X *be a topological space and* F *be a closed set in* X. *Then for any* $t\in[0,1]$, *the function* $t\chi_F$ *is fuzzy closed in* $\omega(X)$.

Proposition 3.4. *Let* X *be a topological space. If* X *is regular, then* $\omega(X)$ *is fuzzy regular.*

Proof. Suppose X is a regular topological space and let x_t $(t<1)$ be a fuzzy point and μ be a fuzzy open set containing x_t in $\omega(X)$. Then $t<\mu(x)$. Since $\mu: X\to I_r$ is continuous, $\mu^{-1}((t+\varepsilon,1])$ $(0<\varepsilon<\mu(x)-t)$ is an open set in X containing x. Since X is regular, there exists an open set V such that $x\in V\subseteq cl(V)\subseteq\mu^{-1}((t+\varepsilon,1])$. Define $\nu:X\to I_r$ by $\nu=(t+\varepsilon)\chi_V$. Then trivially, ν is fuzzy open in $\omega(X)$ containing x_t. By the above proposition, $(t+\varepsilon)\chi_{cl(V)}$ is a fuzzy closed set in $\omega(X)$ such that $(t+\varepsilon)\chi_V\leq(t+\varepsilon)\chi_{cl(V)}$. Thus $cl(\nu)\leq(t+\varepsilon)\chi_{cl(V)}$. So if $x\in cl(V)$, then $cl(\nu)(x)\leq t+\varepsilon\leq\mu(x)$, and if $x\notin cl(V)$, then $cl(\nu)(x)=0\leq\mu(x)$. Hence $cl(\nu)\leq\mu$. Therefore, $\omega(X)$ is fuzzy regular.

Now we will extend a property of fuzzy topological spaces to fuzzy topological spaces over a base space, in a natural way. Specifically we aim to define, for a fuzzy topological space B, a property P_B of fuzzy topological spaces over B such that the following three conditions are satisfied :

(Condition 1) If X and Y are isomorphic fuzzy topological spaces over B and if X has property P_B then so does Y.

(Condition 2) A fuzzy topological space X has property P if and only if the fuzzy topological space X over the point $*$ has property P_*.

(Condition 3) If a fuzzy topological space X over B has property P_B then the fuzzy topological space ξ^*X over A has property P_A for each fuzzy topological space A and continuous map $\xi:A\to B$, where ξ^*X is the fuzzy topological space $(A\times_B X, pr_1)$ over A.

In this case, the property P_B of fuzzy topological spaces over B is said to be well-behaved [4].

Definition 3.5. A fuzzy topological space (X,δ) over B is fuzzy regular over B if for each $b\in B$ and for each fuzzy point x_t with $x\in X_b$ and fuzzy open set μ containing x_t, there exists a fuzzy open set ν containing x_t such that $cl_b(\nu)\leq\mu$, where $cl_b(\nu)=cl(\nu)\wedge\chi_{X_b}$.

Proposition 3.6. *The property 'fuzzy regular over B' is well-behaved.*

Proof. (1) Suppose X and Y are isomorphic fuzzy topological spaces over B with an isomorphism $f:X\to Y$ and Y is fuzzy regular over B. Let x_t be a fuzzy point with $x\in X_b$ and μ be a fuzzy open set in X such that $x_t\in\mu$. Note that $f(x_t)=f(x)_t$ with $f(x)\in Y_b$ and $f(\mu)$ is a fuzzy open set in Y with $f(x)_t\in f(\mu)$. Since Y is fuzzy regular over B, there exists a fuzzy open set ν in Y containing $f(x)_t$ such that $cl_b(\nu)\leq f(\nu)$. Then $f^{-1}(\nu)=\nu\circ f$ is a fuzzy open set with $x_t\in f^{-1}(\nu)$ and $f^{-1}(cl_b(\nu))\leq\mu$. We want to show that $cl_b(f^{-1}(\nu))\leq f^{-1}(cl_b(\nu))$. If $x\notin X_b$, $cl_b(f^{-1}(\nu))(x)=cl(f^{-1}(\nu))(x)\wedge\chi_{X_b}(x)=0$. Hence $cl_b(f^{-1}(\nu))(x)\leq f^{-1}(cl_b(\nu))(x)$ for all $x\notin X_b$. Suppose $x\in X_b$. Then $cl_b(f^{-1}(\nu))(x)=cl(f^{-1}(\nu))(x)$. And,

$$\begin{aligned}f^{-1}(cl_b(\nu))&=f^{-1}(cl(\nu)\wedge\chi_{Y_b})(x)\\&=(cl(\nu)\wedge\chi_{Y_b})(f(x))\\&=cl(\nu)(f(x))\\&=f^{-1}(cl(\nu))(x)\\&\geq f^{-1}(\nu)(x).\end{aligned}$$

But, since $f^{-1}(cl(\nu))$ is a fuzzy closed set with $f^{-1}(\nu)\leq f^{-1}(cl(\nu))$, $cl_b(f^{-1}(\nu))(x)\leq f^{-1}(cl_b(\nu))(x)$ for all $x\in X_b$. In all, X is fuzzy regular over B.

(2) It is obvious.

(3) Suppose X is fuzzy regular over B. Let A be a fuzzy topological space and $\xi:A\to B$ a fuzzy continuous map. We need to show that $A\times_B X$ is fuzzy regular over A. Let $(a,x)_t$ be a fuzzy point in $A\times_B X$ and $\mu_1\times_B\mu_2$ be a basic fuzzy open set in $A\times_B X$ such that $(a,x)_t\in\mu_1\times_B\mu_2$. Note that $t<\mu_1(a)$ and $t<\mu_2(x)$. Then $x\in X_{\xi(a)}$ and μ_2 is a fuzzy open set in X containing a fuzzy point x_t. Since X is fuzzy regular over B, there exists a fuzzy open set ν in X containing x_t such that $cl_{\xi(a)}(\nu)\leq\mu_2$. Consider the fuzzy open set $c_{\mu_1(a)}\wedge pr_2^{-1}(\nu)$ in $A\times_B X$, where $c_{\mu_1(a)}$ is the constant function on $A\times_B X$ with the value $\mu_1(a)$. Then

$$\begin{aligned}(c_{\mu_1(a)}\wedge pr_2^{-1}(\nu))(a,x)&=\mu_1(a)\wedge pr_2^{-1}(\nu)(a,x)\\&=\mu_1(a)\wedge(\nu\circ pr_2)(a,x)\\&=\mu_1(a)\wedge\nu(x)\\&>t.\end{aligned}$$

Hence $c_{\mu_1(a)}\wedge pr_2^{-1}(\nu)$ contains $(a,x)_t$. It remains to show that $cl_a(c_{\mu_1(a)}\wedge pr_2^{-1}(\nu))\leq\mu_1\times_B\mu_2$. For $(a',x)\notin(A\times_B X)_a$,

$cl_a(c_{\mu_1(a)}\wedge pr_2^{-1}(\nu))(a',x)$

$$= cl(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a', x) \wedge \chi_{(A\times_B X)_a}(a', x)$$
$$= 0.$$

So $cl_a(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a', x) \le (\mu_1 \times_B \mu_2)(a', x)$ for all $(a', x) \notin (A\times_B X)_a$.

Let $(a, x) \in (A\times_B X)_a$. Then

$$cl_a(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a, x)$$
$$= cl(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a, x) \wedge \chi_{(A\times_B X)_a}(a, x)$$
$$= cl(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a, x).$$

Since $c_{\mu_1(a)} \wedge pr_2^{-1}(cl(\nu))$ is a fuzzy closed set with $c_{\mu_1(a)} \wedge pr_2^{-1}(\nu) \le c_{\mu_1(a)} \wedge pr_2^{-1}(cl(\nu))$, we have

$$cl\big(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu)\big)(a, x) \le \big(c_{\mu_1(a)} \wedge pr_2^{-1}(cl(\nu))\big)(a, x).$$

But

$$\big(c_{\mu_1(a)} \wedge pr_2^{-1}(cl(\nu))\big)(a, x)$$
$$= \mu_1(a) \wedge (cl(\nu) \circ pr_2)(a, x)$$
$$= \mu_1(a) \wedge cl(\nu)(x)$$
$$= \mu_1(a) \wedge cl_{\xi(a)}(\nu)(x) \quad (\text{since } x \in \xi(a))$$
$$\le \mu_1(a) \wedge \mu_2(x) \quad (\text{since } cl_{\xi(a)}(\nu) \le \mu_2)$$
$$= (\mu_1 \times_B \mu_2)(a, x).$$

So $cl(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a, x) \le (\mu_1 \times_B \mu_2)(a, x)$ and hence

$$cl_a(c_{\mu_1(a)} \wedge pr_2^{-1}(\nu))(a, x) \le (\mu_1 \times_B \mu_2)(a, x)$$

for all $(a, x) \in (A\times_B X)_a$. Therefore $A\times_B X$ is fuzzy regular over A.

Proposition 3.7. Let $\{f_i : X \to Y_i \mid i \in \Lambda\}$ be an initial family in FuzB such that Y_i is fuzzy regular over B for all $i \in \Lambda$. Then X is fuzzy regular over B.

Proof. Let $\{f_i : X \to Y_i \mid i \in \Lambda\}$ be an initial family in **Fuz**$_B$ and suppose Y_i is fuzzy regular over B for all $i \in \Lambda$. Let x_t be a fuzzy point with $x \in X_b$ and μ be a fuzzy open set in X containing x_t. Then $x_t \in \wedge_{k=i_1}^{i_n} f_k^{-1}(\nu_k) \le \mu$ for some fuzzy open sets ν_k in Y_k. So $x_t \in f_k^{-1}(\nu_k)$ for each $k = i_1, i_2, \cdots, i_n$. Hence $f_k(x_t) = (f_k(x))_t \in \nu_k$ in Y_k and $f_k(x) \in Y_b$. Since Y_k is fuzzy regular over B, there exists a fuzzy open set λ_k in Y_k containing $(f_k(x))_t$ such that $cl_b(\lambda_k) \le \nu_k$. Note that $f_k^{-1}(\lambda_k)$ is a fuzzy open set in X containing x_t for each $k = i_1, i_2, \cdots, i_n$. Hence $\wedge_{k=i_1}^{i_n} f_k^{-1}(\lambda_k)$ is a fuzzy open set in X containing x_t. It remains to show that $cl_b\big(\wedge_{k=i_1}^{i_n} f_k^{-1}(\lambda_k)\big) \le \mu$. But since $cl_b\big(\wedge_{k=i_1}^{i_n} f_k^{-1}(\lambda_k)\big) \le \wedge_{k=i_1}^{i_n} cl_b(f_k^{-1}(\lambda_k))$, it is enough to show that $\wedge_{k=i_1}^{i_n} cl_b(f_k^{-1}(\lambda_k)) \le \mu$. If $x \notin X_b$, $\wedge_{k=i_1}^{i_n} cl_b(f_k^{-1}(\lambda_k))(x) = 0$ and if $x \in X_b$, $\wedge_{k=i_1}^{i_n} cl_b(f_k^{-1}(\lambda_k))(x) = \wedge_{k=i_1}^{i_n} cl(f_k^{-1}(\lambda_k))(x)$.

Since $cl_b(\lambda_k) \le \nu_k$, $f_k^{-1}(cl_b(\lambda_k)) \le f_k^{-1}(\nu_k)$. But, for $x \in X_b$,

$$f_k^{-1}(cl_b(\lambda_k))(x)$$
$$= f_k^{-1}(cl(\lambda_k) \wedge \chi_{(Y_k)_b})(x)$$
$$= f_k^{-1}(cl(\lambda_k))(x) \wedge f_k^{-1}(\chi_{(Y_k)_b})(x)$$
$$= f_k^{-1}(cl(\lambda_k))(x) \wedge \chi_{(Y_k)_b}(f_k(x))$$
$$= f_k^{-1}(cl(\lambda_k))(x) \quad (\text{since } f_k(x) \in (Y_k)_b).$$

Hence $f_k^{-1}(cl(\lambda_k)) \le f_k^{-1}(\nu_k)$ on X_b. Note that $cl(f_k^{-1}(\lambda_k)) \le f_k^{-1}(cl(\lambda_k))$. So

$$\wedge_{k=i_1}^{i_n} cl_b(f_k^{-1}(\lambda_k)) = \wedge_{k=i_1}^{i_n} cl(f_k^{-1}(\lambda_k))$$
$$\le \wedge_{k=i_1}^{i_n} f_k^{-1}(cl(\lambda_k))$$
$$\le \wedge_{k=i_1}^{i_n} f_k^{-1}(\nu_k)$$
$$\le \mu$$

on X_b. Therefore, $\wedge_{k=i_1}^{i_n} f_k^{-1}(cl(\lambda_k)) \le \mu$. In all, X is fuzzy regular over B.

Corollary 3.8. *A subspace of a fuzzy regular space over B is fuzzy regular over B.*

Corollary 3.9. *The product of fuzzy regular spaces over B in* **Fuz**$_B$ *is fuzzy regular over B.*

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