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Lagged Unstable Regressor Models and Asymptotic Efficiency of the Ordinary Least Squares Estimator†

Dong-Wan Shin[1] and Man-Suk Oh[1]

ABSTRACT

Lagged regressor models with general stationary errors independent of the regressors are considered. The regressor process is unstable having characteristic roots on the unit circle. If the order of the lag matches the number of roots on the unit circle, the ordinary least squares estimator (OLSE) is asymptotically efficient in that it has the same limiting distribution as the generalized least squares estimator (GLSE) under the same normalization. This result extends the well-known result of Grenander and Rosenblatt (1957) for asymptotic efficiency of the OLSE in deterministic polynomial and/or trigonometric regressor models to a class of models with stochastic regressors.

Keywords. Efficiency, GLSE, stochastic regressors.
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1. Introduction

Consider a time series regression model

$$y_t = \beta_1 x_t + \ldots + \beta_p x_{t-p+1} + z_t, \;\; t = 1, \ldots, n, \tag{1}$$

where $\{y_t,\, t = 1, \ldots, n\}$ is a set of observations, x_t is a sequence of unstable time series such as $I(p)$ processes, seasonally integrated processes, polynomial time trends, or trigonometric time trends. The process z_t is a sequence of unobservable stationary errors independent of x_t, and $\beta = (\beta_1, \ldots, \beta_p)'$ is a $p \times 1$ vector of unknown parameters of our interest. The process x_t is an unstable autoregression $\phi(B)x_t = u_t$, where $\phi(B)$ is of the form

$$\phi(B) = (1-B)^a (1+B)^b \prod_{\ell=1}^{L} (1 - 2B\cos\theta_\ell + B^2)^{d_\ell}, \tag{2}$$

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[1]Department of Statistics, Ewha Womans University, Seoul 120-750, Korea

for some nonnegative integers a, b, d_ℓ, and real numbers θ_ℓ, $\ell = 1, \ldots, L$, B is the back-shift operator such that $Bx_t = x_{t-1}$ and the process u_t is either a zero-mean stationary process or zero. Note that all the roots of $\phi(B)$ lie on the unit circle.

Our object is to prove asymptotic efficiency of the OLSE, $\hat{\beta}_O = (X'X)^{-1}X'Y$, in the sense that it has the same limiting distribution as the GLSE, $\hat{\beta}_G = (X'\Gamma^{-1}X)^{-1}X'\Gamma^{-1}Y$, under the same normalization, where $X = (X_1 | \ldots | X_p)$, $X_i = (x_{1-i+1}, \ldots, x_{n-i+1})', i = 1, \ldots, p$, $Y = (y_1, \ldots, y_n)'$, $\Gamma = var(Z)$, and $Z = (z_1, \ldots, z_n)'$.

For a general linear model $Y = X\beta + Z$, it is well known that a necessary and sufficient condition for numerical equivalence of $\hat{\beta}_O$ and $\hat{\beta}_G$ is $\Gamma X = XC$ for some constant matrix C. See Zyskind (1967), Kruskal (1968), and the review paper by Puntanen and Styan (1989). Asymptotic efficiency of the OLSE was studied in some time series literature such as Grenander and Rosenblatt (1957) for deterministic time trend regressor models, Kramer (1986) and Phillips and Park (1988) for regression with integrated regressors, and Kramer and Hassler (1998) for fractionally integrated regressor models. Especially, Shin and Oh (2002) considered the model (1) with $p = 1$ and showed that the OLSE is asymptotically efficient if one of $\{a, b, d_1, \ldots, d_L\}$ is strictly greater than all the other values.

In the remainder of this paper, conditions and results are stated in Section 2 and proofs of theoretical results are provided in the Appendix.

2. Conditions and Results

Let $\gamma_h = E(z_0 z_h)$ and let

$$f(\theta) = (2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma_h \exp(-\iota\theta h),\ \iota^2 = -1$$

be the spectral density of z_t. Let $\lambda_m(M)$ denote the minimum eigenvalue of a symmetric matrix M. For an $m \times 1$ vector $a = (a_1, \ldots, a_m)'$, let $\|a\|$ denote the Euclidean norm such that $\|a\|^2 = \sum_{i=1}^m a_i^2$. We state conditions required for our analysis.

A1. $\phi(B)$ is a p-th order polynomial having all roots on the unit circle and is of the form (2) for some integers a, b, $d_\ell \geq 0, \ell = 1, \ldots, L, p = a+b+\sum_{\ell=1}^L d_\ell \geq 1$, and θ_ℓ's in $(0, \pi)$.

A2. The process $u_t = \phi(B)x_t$ is either zero or a zero mean stationary process having positive variance. Let $v_t = (u_t, z_t)'$ if u_t is a stationary random process and let $v_t = z_t$ if $u_t = 0$. The process v_t satisfies the invariance principle that $n^{-1/2} \sum_{t=1}^{[ns]} v_t \xrightarrow{d} W(s)$ for some Brownian motion $W(s)$ with positive definite variance matrix, where $\xrightarrow{d}$ denotes convergence in distribution and $[ns]$ is the integer part of $ns, 0 \leq s \leq 1$.

A3. There is a sequence of square matrices A_n such that $n^{-1} A_n'^{-1} X' X A_n^{-1}$ converges in distribution to an almost surely ($a.s.$) positive definite and possibly random matrix.

A4. $\inf_n \lambda_m(\Gamma) > 0$.

A5. $\sum_{h=0}^{\infty} |\gamma_h| < \infty$. If u_t is a random process, $\sum_{h=0}^{\infty} |E(u_0 u_h)| < \infty$.

A6. For each θ associated with the characteristic roots of $\phi(B)$, we have $f(\theta) \neq 0$ and

$$\sum_{i=1}^{n} \left| \sum_{h=i}^{\infty} \gamma_h e^{\iota \theta h} \right|^2 = o(n).$$

A7. $\{x_t\}$ and $\{z_t\}$ are independent.

The first condition A1 states that every element of x_t is unstable. Special simple cases of regressors which satisfy A1 and are frequently encountered in practice are $I(d)$, $I_D(1)$ and $I(1) \times I_D(1)$ defined by $(1-B)^d x_t = u_t$, $(1-B^D)x_t = u_t$, $(1-B)(1-B^D)x_t = u_t$, respectively, where d and $D \geq 1$ are integers.

Note that each element of x_t can be not only a random process but also a deterministic general trigonometric function which is a linear combination of $\{t^j, j = 0, \ldots, a-1\}$, $\{(-1)^t t^j, j = 0, \ldots, b-1\}$, $\{t^j \cos(t\theta_\ell), t^j \sin(t\theta_\ell), j = 0, \ldots, d_\ell - 1\}$, $\ell = 1, \ldots, L$, for some integers a, b, d_ℓ and real numbers $\theta_\ell \in (0, \pi)$. In this case, $u_t = \phi(B) x_t = 0$ for some $\phi(B)$ of the form (2) and A1 is also satisfied. Therefore, the results that will be established below are valid for the cases in which the set of regressors contains general trigonometric functions as well as random processes satisfying A1.

The other conditions A2–A7 are very mild and are satisfied by general stationary time series $\{u_t, z_t\}$ as discussed in Shin and Oh (2002).

Lemma 2.1. *Consider model (1). Under A1-A7, for some $n \times p$ matrix V and $p \times p$ nonsingular matrix Q,*

$$\Gamma X = X\ QFQ^{-1} + R \tag{3}$$

and

$$\|R\|/\|X\| = o_p(1), \tag{4}$$

where

$$F = diag[2\pi f(\theta_1),\ \ldots,\ 2\pi f(\theta_p)],$$

and $\{\exp(\iota\theta_1),\ \ldots,\ \exp(\iota\theta_p)\}$ *are the characteristic roots of* $\phi(B)$.

The expressions (3)–(4) state that ΓX is asymptotically in the column space of X and $X'\Gamma^{-1} \cong C'^{-1}X'$ for a nonsingular matrix $C = QFQ^{-1}$. Therefore,

$$\hat{\beta}_G = (X'\Gamma^{-1}X)^{-1}(X'\Gamma^{-1}Y) \cong (C'^{-1}X'X)^{-1}(C'^{-1}X'Y) = (X'X)^{-1}X'Y = \hat{\beta}_O$$

and hence the OLSE is asymptotically efficient. The matrix Q is defined in the proof of Lemma 2.1 and its explicit expression is found in Chan and Wei (1988).

Theorem 2.1. *Consider model (1). If A1–A7 hold, then the OLSE and the GLSE for the regression (1) have the same nontrivial limiting distribution under the same normalization.*

Remark 1. Suppose that x_t is $I(d)$ for some integer $d \geq 1$, *i.e.* $\Delta^d x_t = u_t$, where $\Delta = 1 - B$ is the difference operator. Consider the regression

$$y_t = \beta_1 x_t + \ldots + \beta_p x_{t-p+1} + z_t$$

with $p \geq 1$. Note that this model is equivalent to

$$y_t = \gamma_1 x_t + \gamma_2 \Delta x_t + \ldots + \gamma_p \Delta^{p-1} x_t + z_t$$

because there is one-to-one correspondence between $(x_t,\ \ldots,\ x_{t-p+1})'$ and $(x_t, \Delta x_t, \ldots, \Delta^{p-1}x_t)'$. Now, according to Theorem 2.1, if $p = d$ then the OLSE of $\beta = (\beta_1,\ \ldots,\ \beta_p)'$ and hence of $\gamma = (\gamma_1,\ \ldots,\ \gamma_p)'$ are asymptotically efficient. On the other hand, if $p > d$, the OLSE is not asymptotically efficient because there are lack of asymptotic efficiencies for the OLSE of γ_j for $d < j \leq p$, which correspond to stationary components $\Delta^j x_t$, $d < j \leq p$.

Remark 2. According to Theorem 2.1, when x_t is a seasonally integrated process

$$(1 - B^D)x_t = u_t,$$

the OLSE for the regression model

$$y_t = \beta_1 x_t + \beta_2 x_{t-1} + \ldots + \beta_D x_{t-D+1} + z_t$$

is asymptotically efficient under the other regularity conditions of A2–A7. On the other hand, Shin and Oh (2000) showed that, for model $y_t = \beta_1 x_t + z_t$, the OLSE is not asymptotically as efficient as the GLSE. Note that there is one-to-one correspondence between $(x_t, \ldots, x_{t-D+1})'$ and $(x^*_{0t}, \ldots, x^*_{D-1,t})'$ where

$$x^*_{it} = \sum_{j=1}^{D} \cos\big(\frac{2i}{D} j\pi\big) B^{j-1} x_t, \; i = 0, \ldots, r,$$
$$x^*_{i+r,t} = \sum_{j=1}^{D} \sin\big(\frac{2i}{D} j\pi\big) B^{j-1} x_t, \; i = 1, \ldots, r_c,$$

are the Fourier coefficients of $\{x_{t-D+1}, x_{t-D+2}, \ldots, x_t\}$, $r = [D/2]$, and $r_c = [(D-1)/2]$. For the quarterly case of $D = 4$,

$$x^*_{0t} = (1 + B + B^2 + B^3)x_t, \; x^*_{1t} = -(1 - B^2)Bx_t,$$
$$x^*_{2t} = -(1 - B + B^2 - B^3)x_t, \; x^*_{3t} = (1 - B^2)x_t$$

are the regressors considered by Hylleberg *et al.* (1990) and Shin and So (2000) in constructing tests for seasonal unit roots. Now, regression (1) is equivalent to

$$y_t = \pi_0 x^*_{0t} + \pi_1 x^*_{1t} + \ldots + \pi_{D-1} x^*_{D-1,t} + z_t. \tag{5}$$

This type of regression is encountered in the fields of seasonal unit root tests and seasonal cointegration, see Hylleberg *et al.* (1990), Engle *et al.* (1993), and Shin and So (2000). According to Theorem 2.1, the OLSE $\hat{\pi}_O$ of $\pi = (\pi_0, \ldots, \pi_{D-1})'$ in (5) is asymptotically efficient.

Remark 3. Consider x_t in $(1 - B^D)(1 - B)x_t = u_t$. By Theorem 2.1, the OLSE in the regression $y_t = \beta_1 x_t + \ldots + \beta_{D+1} x_{t-D} + z_t$ is asymptotically efficient under the other regularity conditions. Also, according to Shin and Oh (2002), the OLSE in the regression model $y_t = \beta_1 x_t + z_t$ is asymptotically efficient because the multiplicity of the unit root 1 is two and is greater than multiplicities of any other roots, which are all one.

Remark 4. Theorem 2.1 can be applied to give the result of Grenander and Rosenblatt (1957, Section 7.5), which states asymptotic efficiency of the OLSE in the trigonometric regressions consisting of regressors

$$x^*_{0t} = [1, t, \ldots, t^{d_0 - 1}]', \quad x^*_{rt} = [(-1)^t, (-1)^t t, \ldots, (-1)^t t^{d_r - 1}]',$$

$$x^*_{\ell t} = [\cos(t\theta_\ell), t\cos(t\theta_\ell), \ldots, t^{d_\ell - 1}\cos(t\theta_\ell), \sin(t\theta_\ell), t\sin(t\theta_\ell), \ldots, t^{d_\ell - 1}\sin(t\theta_\ell)]',$$

$\ell = 1, 2, \ldots, r-1$, with $d_\ell \geq 0$ and $0 = \theta_0 < \theta_1 < \ldots < \theta_{r-1} < \theta_r = \pi$. Let

$$x_t = \sum_{\ell=0}^{r} t^{d_\ell - 1}\{\cos(t\theta_\ell) + \sin(t\theta_\ell)\}. \tag{6}$$

Note that, for any nonnegative integer d, $(1-B)^d t^{d-1} = 0$, $(1+B)^d(-1)^t t^{d-1} = 0$, and $(1 - 2B\cos\theta + B^2)^d t^{d-1}\cos(t\theta) = 0$. Hence, in general, we have

$$(1-B)^{d_0}(1+B)^{d_r}\prod_{\ell=1}^{r-1}(1 - 2B\cos\theta_\ell + B^2)^{d_\ell} x_t = 0$$

and x_t satisfies A1. We now have $p = d_0 + 2\sum_{\ell=1}^{r-1} d_\ell + d_r$. Also, we can easily show that there is one-to-one correspondence between $(x_t,\, x_{t-1},\, \ldots,\, x_{t-p})'$ and $(x_{0t}^{*'},\, \ldots,\, x_{rt}^{*'})'$. Therefore, the regression considered by Grenander and Rosenblatt (1957) is equivalent to regression (1) with x_t in (6). According to Theorem 2.1, under the other regularity conditions of A2–A7, the OLSE is asymptotically as efficient as the GLSE, yielding the result of Grenander and Rosenblatt (1957, Section 7.5). Therefore, Theorem 2.1 extends the result of Grenander and Rosenblatt (1957) to a class of models with stochastic regressors.

Appendix : Proofs

Proof of Lemma 2.1. Let

$$u_t(j) = (1-B)^{-j}u_t,\ j = 1,\ \ldots,\ a, \tag{A.1}$$

$$v_t(j) = (1+B)^{-j}u_t,\ j = 1,\ \ldots,\ b, \tag{A.2}$$

$$x_{\ell t}(j) = (1 - 2B\cos\theta_\ell + B^2)^{-j}u_t, j = 1, \ldots, d_\ell, \ell = 1, \ldots, L. \tag{A.3}$$

Let

$$\begin{aligned} w_t = [&u_t(1),\ \ldots,\ u_t(a),\ v_t(1),\ \ldots,\ v_t(b), x_{1t}(1),\ x_{1,t-1}(1),\ \ldots,\\ &x_{1t}(d_1),\ x_{1,t-1}(d_1),\ \ldots, x_{Lt}(1),\ x_{L,t-1}(1),\ \ldots,\ x_{Lt}(d_L),\ x_{L,t-1}(d_L)]'. \end{aligned}$$

By (3.2), (3.1.3), (3.2.2), (3.3.2) of Chan and Wei (1988), there is a nonsingular matrix Q such that

$$(x_t,\ x_{t-1},\ \ldots,\ x_{t-p+1})Q = w_t. \tag{A.4}$$

The fact that the regression order p exactly matches the number of roots on the unit circle is essential for nonsingularity of Q. Let $U(j) = [u_1(j), \ldots, u_n(j)]'$. Define similarly $V(j)$ and $X_\ell(j)$ from $v_t(j)$ and $x_{\ell t}(j)$ in (A.2)–(A.3). Let

$$W = [U(1)|\ U(2)|\ \ldots|\ X_L(d_L)].$$

Then, from (A.4), $X = WQ^{-1}$. Since each element of $u_t(j)$, $v_t(j)$, and $x_{\ell t}(j)$ has only one root on the unit circle, Proposition 1 and Proposition 2 of Shin and Oh (2002) are applicable to give us

$$\Gamma U(j) = 2\pi f(0)U(j) + R_u(j), \tag{A.5}$$

$$\Gamma V(j) = 2\pi f(\pi)V(j) + R_v(j), \tag{A.6}$$

$$\Gamma X_\ell(j) = 2\pi f(\theta_\ell)X_\ell(j) + R_{x\ell}(j), \tag{A.7}$$

$j = 1, 2, \ldots$, for some n-vectors $R_u(j)$, $R_v(j)$, $R_{x\ell}(j)$ such that

$$\begin{aligned} &\|R_u(j)\|/\|U(j)\| = o_p(1), \|R_v(j)\|/\|V(j)\| = o_p(1), \\ &\|R_{x\ell}(j)\|/\|X_\ell(j)\| = o_p(1). \end{aligned} \tag{A.8}$$

Combining (A.5)–(A.7), we get

$$\Gamma W = WF + [R_u(1)|\ \ldots|\ R_{xL}(d_L)]$$

and

$$\Gamma X = \Gamma W Q^{-1} = XQFQ^{-1} + R,$$

where $R = [R_u(1)|\ \ldots|\ R_{xL}(d_L)]Q^{-1}$. Noting that $\|X\|$ is of the same order as the maximum ordered one among $\{\|U(1)\|, \ldots, \|X_L(d_L)\|\}$, we get (4) from (A.8). □

Proof of Theorem 2.1. From (3) of Lemma 2.1 and nonsingularity of $K = QFQ^{-1}$, we have

$$\Gamma^{-1}X = XK^{-1} - \Gamma^{-1}RK^{-1}.$$

We thus have

$$X'\Gamma^{-1}X = X'XK^{-1} - X'\Gamma^{-1}RK^{-1}$$

and

$$Z'\Gamma^{-1}X = Z'XK^{-1} - Z'\Gamma^{-1}RK^{-1}.$$

Now, according to (4) of Lemma 2.1, the last terms $X'\Gamma^{-1}RK^{-1}$ and $Z'\Gamma^{-1}RK^{-1}$ are negligible compared with the first terms $X'XK^{-1}$ and $Z'XK^{-1}$, respectively.

Therefore, the arguments of Theorem 2 and Lemma 3 of Shin and Oh (2002) are applicable to show that

$$[n^{1/2}A_n'(\hat{\beta}_G - \beta)]' = n^{-1/2}Z'\Gamma^{-1}X(X'\Gamma^{-1}X)^{-1}A_n$$

has the same limiting distribution as

$$\begin{aligned}[n^{1/2}A_n'(\hat{\beta}_O - \beta)]' &= n^{-1/2}Z'XK^{-1}(X'XK^{-1})^{-1}A_n \\ &= n^{1/2}Z'XA_n^{-1}(A_n^{-1}X'XA_n^{-1})^{-1}\end{aligned}$$

and the result follows. □

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