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Diagnostics for Regression with Finite-Order Autoregressive Disturbances†

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ABSTRACT

Motivated by Cook's (1986) assessment of local influence by investigating the curvature of a surface associated with the overall discrepancy measure, this paper extends this idea to the linear regression model with AR(p) disturbances. Diagnostic for the linear regression models with AR(p) disturbances are discussed when simultaneous perturbations of the response vector are allowed. For the derived criterion, numerical studies demonstrate routine application of this work.

Keywords. Direction of maximum curvature, likelihood displacement, linear regression model, observed Hessian matrix, observed information matrix.
AMS 2000 subject classifications. Primary 62J20; Secondary 62M10.

1. Introduction

Regression models including time series data can often be applied to economics, business, and some fields of engineering. Usually the errors in such time series data exhibit serial correlation. For example, such data could be easily collected for a single economic unit or they could be aggregate quantities for a whole region or an economy.

Yet in the autocorrelated regression models, not much progress has been made for assessing the influence of local departures from model assumptions. Of course, it is not appropriate to apply and discuss case-deletion diagnosis in linear regression model in which the errors are autocorrelated. It is due to the fact that the dependency structure of the autoregressive model will not be valid after deleting a single observation from the data, except the last observation. To overcome this

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difficulty for case-deletion diagnosis, in this paper we examine the local change in the autocorrelated parameter estimate caused by small perturbations.

Cook (1986) proposed a general method for assessing the influence of local departures from the assumptions underlying the statistical model. A distinguishing character of this method is using log-likelihood contours to measure the local influence. Kim and Huggins (1998) discussed the local influence approach to the linear regression model with AR(1) errors. This paper treats the effects of simultaneous perturbations of the response vector on all the parameters in the linear regression model with AR(p) errors. In Section 2 the model is formulated and diagnostics for the linear regression model with AR(p) errors are proposed. Furthermore, the direction of maximum curvature of local influence analysis is obtained. Section 3 presents illustrative examples. Finally, some concluding remarks are given in Section 4.

2. Local Influence Approach for the Linear Regression Models with AR(p) Errors

2.1. Model

We consider the following linear regression model

$$y = X\beta + \varepsilon,$$

where y is an $n \times 1$ observable random vector, X is an $n \times (k+1)$ nonstochastic matrix of explanatory variables, β is a $(k+1) \times 1$ vector of parameters to be estimated, and ε is an $n \times 1$ unobservable random vector with

$$E(\varepsilon) = 0 \text{ and } E(\varepsilon\varepsilon') = \sigma^2\Psi.$$

The disturbances are assumed to follow a pth-order autoregressive process

$$\varepsilon_t = \rho_1\varepsilon_{t-1} + \ldots + \rho_p\,\varepsilon_{t-p} + a_t,$$

where $E(a_t) = 0$, $E(a_t a_s) = 0$ for $t \neq s$, and $E(a_t^2) = \sigma^2$. We assume that the roots of $A(m) = m^p - \rho_1 m^{p-1} - \ldots - \rho_p = 0$ lie inside the unit circle.

Using the method given by Wise (1955), the general form of the inverse of Ψ can be obtained as

$$\begin{aligned}\Psi^{-1} = {} & I - \sum_{k=1}^{p} \rho_k \left\{ (U^k)' + U^k \right\} + \sum_{k=1}^{p} \rho_k^2\, I_{-k} \\ & + \sum_{1 \le k < l \le p} \rho_k\,\rho_l \left[\left\{ (U^{l-k})_{-k} \right\}' + (U^{l-k})_{-k} \right],\end{aligned} \tag{2.1}$$

where I is the identity matrix of order n, I_{-k} is an $n \times n$ matrix obtained by replacing the first k and the last k diagonal elements of I with zeros, U is an $n \times n$ matrix with (i,j)th element $\delta_{i+1,j}$ (the Kronecker's delta), and so U^k is an $n \times n$ matrix with (i,j)th element $\delta_{i+k,j}$. The matrix $(U^{l-k})_{-k}$ is obtained by replacing the first k rows and the last k columns of U^{l-k} with zero vectors.

2.2. Local influence

We develop diagnostics for the linear regression models with AR(p) errors by using the local influence approach introduced by Cook (1986). Cook presents a general method for assessing the local influence of minor perturbations of a statistical model. The method relies on a well-behaved likelihood and certain elementary ideas from differential geometry.

To assess the influence of varying the $q \times 1$ vector ω throughout some open subset Ω of R^q, Cook (1986) proposed the likelihood displacement

$$LD(\omega) = 2[L(\hat{\theta}) - L(\hat{\theta}_\omega)] \tag{2.2}$$

and the associated influence graph

$$\alpha(\omega) = \begin{pmatrix} \omega \\ LD(\omega) \end{pmatrix}, \tag{2.3}$$

where L denotes the log-likelihood corresponding to the postulated model, $\hat{\theta}$ and $\hat{\theta}_\omega$ are the maximum likelihood estimates for θ in the unperturbed and the perturbed model, respectively, and θ is a $p \times 1$ vector of unknown parameters. Cook proposed to assess the local influence of ω at the postulated model ($\omega = \omega_0$) by the curvature

$$C_l = 2\left|l'\ddot{F}l\right| = 2\left|l'\Delta'\ddot{L}^{-1}\Delta l\right| \tag{2.4}$$

of the influence graph (2.3) in direction $l \in R^q$, where l is a fixed vector of unit length, $\ddot{F}$ is the $q \times q$ matrix with (k,j)th element $\frac{\partial^2 L(\hat{\theta}_\omega)}{\partial\omega_k\,\partial\omega_j}$, Δ is the $p \times q$ matrix with elements

$$\Delta_{ij} = \left.\frac{\partial^2 L(\theta|\,\omega)}{\partial\theta_i\,\partial\omega_j}\right|_{\hat{\theta},\omega_0}, \qquad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q,$$

$L(\theta|\,\omega)$ denotes the log-likelihood corresponding to the perturbed model for a given ω in Ω, and $-\ddot{L}$ is the $p \times p$ observed information matrix for the postulated model ($\omega = \omega_0$), where

$$\ddot{L} = \left.\left(\frac{\partial^2 L(\theta)}{\partial\theta_i\,\partial\theta_j}\right)\right|_{\hat{\theta}}.$$

The vector ω_0 is called the null vector or null perturbation. The direction of maximum curvature of the likelihood displacement surface is used as the main diagnostic tool in the local influence method.

2.3. Diagnostic for finite-order autoregressive models

Now suppose that the response vector y is perturbed according to

$$\tilde{y} = y + \omega = y + al,$$

where $l = (l_1, \ldots, l_n)'$ denotes a directional vector with unit length and the quantity a measures the distance from y along the direction l. Therefore, this perturbation scheme produces an n-dimensional space, referred to as the ω-space. The likelihood displacement $LD(\omega)$ is a function on the ω-space, and it forms a surface $\alpha(\omega)$ in an $n+1$ dimensional space :

$$\alpha(\omega) = \begin{pmatrix} \omega \\ LD(\omega) \end{pmatrix}.$$

The relevant parts of the log-likelihood function for the unperturbed and the perturbed model are, respectively,

$$L(\rho, \sigma^2, \beta) = -\frac{n}{2}\log\sigma^2 + \frac{1}{2}\log|\Psi^{-1}| - \frac{(y - X\beta)'\Psi^{-1}(y - X\beta)}{2\sigma^2}, \qquad (2.5)$$

$$L(\rho, \sigma^2, \beta|\,\omega) = -\frac{n}{2}\log\sigma^2 + \frac{1}{2}\log|\Psi^{-1}| - \frac{(\tilde{y} - X\beta)'\Psi^{-1}(\tilde{y} - X\beta)}{2\sigma^2} \qquad (2.6)$$

where $\rho = (\rho_1, \ldots, \rho_p)'$ is the vector of autoregressive coefficients, Ψ^{-1} is given by (2.1), and $|\Psi^{-1}|$ is, using the Gram-Schmidt orthogonalization (see Fuller, 1976, p. 423), for example

(a) $p = 1 : |\Psi^{-1}| = 1 - \rho_1^2$;

(b) $p = 2 : |\Psi^{-1}| = (1 + \rho_2)^2 \left\{(1 - \rho_2)^2 - \rho_1^2\right\}$;

(c) $p = 3 : |\Psi^{-1}| = \left\{(1 - \rho_2)^2 - (\rho_1 + \rho_3)^2\right\}(\rho_1\rho_3 - \rho_3^2 + \rho_2 + 1)^2$.

The derivation of $|\Psi^{-1}|$ for $p = 3$ by the authors is given in the appendix.

Remark. The expression for $|\Psi^{-1}|$ becomes progressively more complicated as the value of p increases.

Then, Cook's (1986) likelihood displacement is defined as

$$LD(\omega) = 2[L(\hat{\theta}) - L(\hat{\theta}_\omega)],$$

where $\hat{\theta}$ and $\hat{\theta}_\omega$ are the maximum likelihood estimates for $\theta = (\rho', \sigma^2, \beta')'$ in the unperturbed and the perturbed model, respectively.

Differentiating (2.6) with respect to θ and ω, and evaluating at $\hat{\theta}$ and $\omega_0 = 0$, we find

$$\Delta = \left.\begin{pmatrix} -(y - X\beta)'\left(\frac{\partial}{\partial\rho_1}\Psi^{-1}\right)\Big/\sigma^2 \\ \vdots \\ -(y - X\beta)'\left(\frac{\partial}{\partial\rho_p}\Psi^{-1}\right)\Big/\sigma^2 \\ (y - X\beta)'\Psi^{-1}/(\sigma^2)^2 \\ X'\Psi^{-1}/\sigma^2 \end{pmatrix}\right|_{\theta=\hat{\theta}} \qquad \text{for } p = 1, 2, 3, \ldots$$

where $\hat{\beta} = (X'\Psi^{-1}X)^{-1}X'\Psi^{-1}y$. The observed information matrix, $-\ddot{L}$, may be found by taking minus the matrix of second derivatives of the log-likelihood function (2.5), with respect to θ and all derivatives are evaluated at the maximum likelihood estimates $\hat{\theta}$, but to simplify the notation we do not write the ^ hereafter. Let x_i' denote the ith row vector of X. Then,

(i) $p = 1$: (Kim and Huggins, 1998, p. 68)

$$\ddot{L} = \left.\begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & 0' \\ l_{31} & 0 & L_{33} \end{pmatrix}\right|_{\hat{\theta}}$$

where

$$\begin{aligned} l_{11} &= -\frac{1+\rho_1^2}{(1-\rho_1^2)^2} - \frac{1}{\sigma^2}\sum_{i=2}^{n-1}(y_i - x_i'\beta)^2, \\ l_{21} &= -\frac{\rho_1}{(1-\rho_1^2)\sigma^2}, \\ l_{22} &= -\frac{n}{2(\sigma^2)^2}, \\ l_{31} &= \frac{X'\left(\frac{\partial}{\partial\rho_1}\Psi^{-1}\right)(y - X\beta)}{\sigma^2}, \\ L_{33} &= -\frac{X'\Psi^{-1}X}{\sigma^2}\ ; \end{aligned}$$

(ii) $p = 2$:

$$\ddot{L} = \left.\begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & 0' \\ l_{41} & l_{42} & 0 & L_{44} \end{pmatrix}\right|_{\hat{\theta}}$$

where

$$\begin{aligned}
l_{11} &= -\frac{(1-\rho_2)^2+\rho_1^2}{\{(1-\rho_2)^2-\rho_1^2\}^2} - \frac{1}{\sigma^2}\sum_{i=2}^{n-1}(y_i - x_i'\beta)^2, \\
l_{21} &= -\frac{2(1-\rho_2)\rho_1}{\{(1-\rho_2)^2-\rho_1^2\}^2} - \frac{1}{\sigma^2}\sum_{i=2}^{n-2}(y_i - x_i'\beta)(y_{i+1} - x_{i+1}'\beta), \\
l_{22} &= -\frac{(1-\rho_2)^2+\rho_1^2}{\{(1-\rho_2)^2-\rho_1^2\}^2} - \frac{1}{(1+\rho_2)^2} - \frac{1}{\sigma^2}\sum_{i=3}^{n-2}(y_i - x_i'\beta)^2, \\
l_{31} &= -\frac{\rho_1}{(1-\rho_2)^2-\rho_1^2}\cdot\frac{1}{\sigma^2}, \\
l_{32} &= \left\{\frac{1}{1+\rho_2} - \frac{1-\rho_2}{(1-\rho_2)^2-\rho_1^2}\right\}\frac{1}{\sigma^2}, \\
l_{33} &= -\frac{n}{2(\sigma^2)^2}, \\
l_{41} &= \frac{X'\left(\frac{\partial}{\partial\rho_1}\Psi^{-1}\right)(y - X\beta)}{\sigma^2}, \\
l_{42} &= \frac{X'\left(\frac{\partial}{\partial\rho_2}\Psi^{-1}\right)(y - X\beta)}{\sigma^2}, \\
L_{44} &= -\frac{X'\Psi^{-1}X}{\sigma^2}\ ;
\end{aligned}$$

(iii) $p = 3$:

$$\ddot{L} = \left.\begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} \\ l_{41} & l_{42} & l_{43} & l_{44} & 0' \\ l_{51} & l_{52} & l_{53} & 0 & L_{55} \end{pmatrix}\right|_{\hat{\theta}}$$

where

$$l_{11} = -\frac{(1-\rho_2)^2+(\rho_1+\rho_3)^2}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} - \frac{\rho_3^2}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} - \frac{1}{\sigma^2}\sum_{i=2}^{n-1}(y_i - x_i'\beta)^2$$

$$
\begin{aligned}
l_{21} &= -\frac{2(1-\rho_2)(\rho_1+\rho_3)}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} - \frac{\rho_3}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} \\
&\quad -\frac{1}{\sigma^2}\sum_{i=2}^{n-2}(y_i-x_i'\beta)(y_{i+1}-x_{i+1}'\beta), \\
l_{22} &= -\frac{(1-\rho_2)^2+(\rho_1+\rho_3)^2}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} - \frac{1}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} \\
&\quad -\frac{1}{\sigma^2}\sum_{i=3}^{n-2}(y_i-x_i'\beta)^2, \\
l_{31} &= -\frac{(1-\rho_2)^2+(\rho_1+\rho_3)^2}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} + \frac{\rho_3^2+\rho_2+1}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} \\
&\quad -\frac{1}{\sigma^2}\sum_{i=2}^{n-3}(y_i-x_i'\beta)(y_{i+2}-x_{i+2}'\beta), \\
l_{32} &= -\frac{2(1-\rho_2)(\rho_1+\rho_3)}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} - \frac{\rho_1-2\rho_3}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} \\
&\quad -\frac{1}{\sigma^2}\sum_{i=3}^{n-3}(y_i-x_i'\beta)(y_{i+1}-x_{i+1}'\beta), \\
l_{33} &= -\frac{(1-\rho_2)^2+(\rho_1+\rho_3)^2}{\{(1-\rho_2)^2-(\rho_1+\rho_3)^2\}^2} + \frac{2(\rho_1\rho_3-\rho_3^2-\rho_2-1)-\rho_1^2}{(\rho_1\rho_3-\rho_3^2+\rho_2+1)^2} \\
&\quad -\frac{1}{\sigma^2}\sum_{i=4}^{n-3}(y_i-x_i'\beta)^2, \\
l_{41} &= \left\{\frac{\rho_3}{\rho_1\rho_3-\rho_3^2+\rho_2+1} - \frac{\rho_1+\rho_3}{(1-\rho_2)^2-(\rho_1+\rho_3)^2}\right\}\frac{1}{\sigma^2}, \\
l_{42} &= \left\{\frac{1}{\rho_1\rho_3-\rho_3^2+\rho_2+1} - \frac{1-\rho_2}{(1-\rho_2)^2-(\rho_1+\rho_3)^2}\right\}\frac{1}{\sigma^2}, \\
l_{43} &= \left\{\frac{\rho_1-2\rho_3}{\rho_1\rho_3-\rho_3^2+\rho_2+1} - \frac{\rho_1+\rho_3}{(1-\rho_2)^2-(\rho_1+\rho_3)^2}\right\}\frac{1}{\sigma^2}, \\
l_{44} &= -\frac{n}{2(\sigma^2)^2}, \\
l_{51} &= \frac{X'\left(\frac{\partial}{\partial\rho_1}\Psi^{-1}\right)(y-X\beta)}{\sigma^2}, \\
l_{52} &= \frac{X'\left(\frac{\partial}{\partial\rho_2}\Psi^{-1}\right)(y-X\beta)}{\sigma^2},
\end{aligned}
$$

$$
\begin{aligned}
l_{53} &= \frac{X'\left(\frac{\partial}{\partial\rho_3}\Psi^{-1}\right)(y - X\beta)}{\sigma^2}, \\
L_{55} &= -\frac{X'\Psi^{-1}X}{\sigma^2}.
\end{aligned}
$$

By that, the observed Hessian matrix $\ddot{F}$ takes the form

$$\ddot{F} = \Delta'\ddot{L}^{-1}\Delta.$$

But explicit analytic expressions of the direction of maximum curvature can not be derived for (2.4). And $c_{\max}$, the maximum value of C_l, corresponds to the maximum absolute eigenvalue of $\ddot{F}$ in (2.4) with corresponding eigenvector $l_{\max}$. Thus the maximum curvature occurs in the direction $l_{\max}$. Note that the ith diagonal element of the Hessian matrix $\ddot{F}$ becomes the curvature of the likelihood displacement (2.2), when only a single observation, say the ith, is perturbed according to $\tilde{y}_i = y_i + \omega$.

3. Examples

3.1. Real data

To illustrate how local influence in a linear regression model with finite-order error structure is assessed, we consider the car-insurance data reported in Park and Shin (1999, Section 3). These data consist of 98 cases and the regression model contains one explanatory variable, NI = (registered motorvehicles increase rate per month) and response variable, Y = (accident rate). A scatter plot of the values of NI versus $\log(Y)$ is given in Figure 1.

Park and Shin proposed an autoregression model to explain this phenomenon, *i.e.,*

$$\log(Y_t) = \alpha + \beta(NI)_t + \varepsilon_t, \qquad t = 1, \ldots, 98$$

where $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + a_t$. The values α, β, ρ_1 and ρ_2 are to be estimated -21.9, 15.977, -0.37 and -0.28, respectively.

Table 1 presents the values of the elements of $l_{\max}$. Considering the elements of $l_{\max}$ versus case, cases 2, 3, 4, 5, 6, 96 and 97 seem to be influential, and such examinations of $l_{\max}$ can provide useful diagnostic information that may be used to guide subsequent insurance analysis and future experimentation.

TABLE 1 *The values of the elements of* $l_{\max}$

case	$l_{\max}$	*case*	$l_{\max}$	*case*	$l_{\max}$
1	−0.106534	36	0.0430088	71	0.1114011
2	−0.219575	37	0.0580346	72	0.133865
3	−0.226722	38	0.0459937	73	0.1337188
4	−0.232128	39	0.0446492	74	0.1078964
5	−0.228721	40	0.0395268	75	0.0796346
6	−0.203998	41	0.015692	76	0.0711202
7	−0.176442	42	−0.018115	77	0.0763921
8	−0.170958	43	−0.03626	78	0.0734319
9	−0.170963	44	−0.024192	79	0.0762685
10	−0.158036	45	−0.021718	80	0.0714967
11	−0.129816	46	−0.027029	81	0.0489476
12	−0.084959	47	−0.01003	82	0.055432
13	−0.048825	48	0.0270328	83	0.0849337
14	−0.063124	49	0.0545308	84	0.1223755
15	−0.115531	50	0.0415157	85	0.1481787
16	−0.151211	51	0.0038703	86	0.1332545
17	−0.150038	52	−0.019361	87	0.1059106
18	−0.141041	53	−0.019156	88	0.0918163
19	−0.138721	54	−0.015962	89	0.0899947
20	−0.12955	55	−0.012948	90	0.099002
21	−0.121291	56	−0.005569	91	0.109266
22	−0.105564	57	−0.002461	92	0.0955621
23	−0.068088	58	0.0047555	93	0.0782909
24	−0.022599	59	0.0367212	94	0.0975018
25	0.0075927	60	0.0893225	95	0.1404327
26	0.0004834	61	0.1145467	96	0.1838601
27	−0.023963	62	0.1031646	97	0.1988312
28	−0.035958	63	0.0891628	98	0.0985226
29	−0.03971	64	0.0776031		
30	−0.023478	65	0.0805028		
31	0.0046734	66	0.0844544		
32	0.001135	67	0.0806506		
33	−0.019195	68	0.08302		
34	−0.017871	69	0.0759414		
35	0.006093	70	0.0824632		

NOTE : $l_{\max}$ *corresponds to the maximum absolute eigenvalue of* $\ddot{F}$.

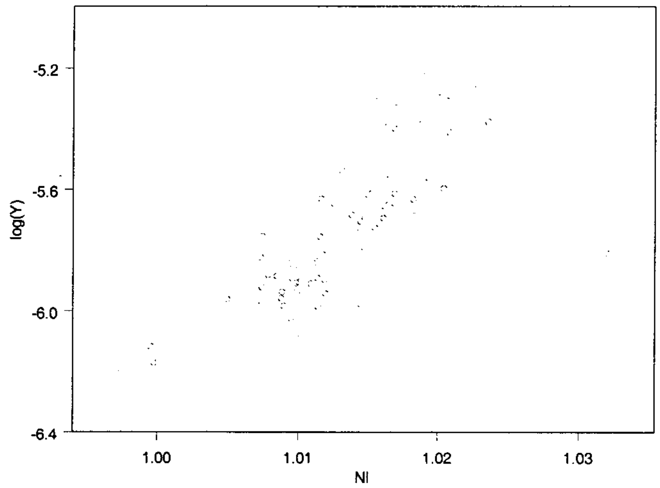

FIGURE 1 *Scatter plot of NI versus* $\log(Y)$

3.2. Artificial data

As a second numerical illustration, we generate 30 samples of y for the model

$$y_t = \alpha + \beta x_t + \varepsilon_t,$$

where $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + \rho_3\varepsilon_{t-3} + a_t$ and a_t are independent standard normal random variables. The normal random numbers $\{a_t\}$ were generated by the subroutine DRNNOA of the IMSL subroutine library. The value of α was set to -2.0, β to -0.45, ρ_1 to 0.9100, ρ_2 to 0.1785 and ρ_3 to -0.1950, respectively. Of course, the design matrix and observation vector have indicated that the errors exhibit third-order autoregressive structure.

The diagnostics of local influence, $l_{\max}$, on all the parameters in AR(3) model are given in Table 2, in case the response vector y is perturbed $\tilde{y}$. Note that the ith diagonal element of the Hessian matrix $\ddot{F}$ becomes the curvature of the likelihood displacement (2.2), when only a single observation, as a special case, say the ith, is perturbed according to $\tilde{y}_i = y_i + \omega$.

Consider the joint influence provided by $l_{\max}$, when both y_7 and y_{27} are perturbed $\tilde{y}_7 = y_7 - \delta$ and $\tilde{y}_{27} = y_{27} - \delta$, respectively. We can find the fact that the elements of $l_{\max}$ corresponding to $\tilde{y}_7$ and $\tilde{y}_{27}$ are the largest among those of $l_{\max}$. Three stages before and after the perturbed observations, $((y_4,\, y_{10}),\, (y_{24},\, y_{30}))$

have larger influential effects than any other elements. When there is only one perturbed observation, the same results can be obtained.

4. Concluding Remarks

In this paper local influence approach to the linear regression model with finite-order AR disturbances is discussed. The direction of maximum curvature $l_{\max}$ can be used to assess the effect of joint perturbations of the response vector on the parameter estimates. The advantage of using this approach lies in the fact that it avoids inappropriate case-deletion diagnostic in AR model and also allows simultaneous perturbations on all responses and thus leads to the identification of relevant and perhaps unexpected traits of the data as seen in the previous two examples.

Appendix : Derivation of $|\Psi^{-1}|$ for $p = 3$

The Gram-Schmidt orthogonalization leads one to the transformed variables (see Fuller, 1977, p. 423)

$$\begin{aligned}
b_1 &= d_{11}\varepsilon_1, \\
b_2 &= d_{22}\varepsilon_2 - d_{21}\varepsilon_1, \\
&\vdots \\
b_p &= d_{pp}\,\varepsilon_p - \sum_{j=1}^{p-1} d_{p,p-j}\,\varepsilon_{p-j}, \\
b_t &= a_t = \varepsilon_t - \sum_{j=1}^{p} \rho_j\,\varepsilon_{t-j}, \qquad t = p+1, p+2, \ldots, n,
\end{aligned}$$

where

$$\begin{aligned}
d_{11} &= \frac{\sigma}{\sqrt{\gamma_\varepsilon(0)}}, \\
d_{22} &= \frac{\sigma}{\sqrt{\gamma_\varepsilon(0)\left[1 - \{\lambda_\varepsilon(1)\}^2\right]}}, \\
d_{33} &= \frac{\sqrt{1 - \{\lambda_\varepsilon(1)\}^2}}{\sqrt{\gamma_\varepsilon(0)\{1 - \lambda_\varepsilon(2)\}\left[1 + \lambda_\varepsilon(2) - 2\{\lambda_\varepsilon(1)\}^2\right]}}\sigma, \ \textit{etc.},
\end{aligned}$$

TABLE 2 *Local influence of observation(s)*

	$\tilde{y}_7 = y_7 - \delta$		$\tilde{y}_{27} = y_{27} - \delta$		$\tilde{y}_7 = y_7 - \delta$, $\tilde{y}_{27} = y_{27} - \delta$	
case	f_{ii}	$l_{\max}$	f_{ii}	$l_{\max}$	f_{ii}	$l_{\max}$
1	−0.045820	0.1426289	−0.058874	−0.1390740	−0.023705	−0.0987630
2	−0.063658	0.1533339	−0.052243	−0.0673310	−0.023433	−0.1281220
3	−0.054492	0.0707273	−0.021435	−0.0387880	−0.002880	−0.0074250
4	−0.255825	−0.4487450	−0.059790	0.1527009	−0.171569	0.3487256
5	−0.190271	−0.2888870	−0.074238	0.1628182	−0.170661	0.3518245
6	−0.070975	0.0708815	−0.035640	0.0276291	−0.045000	−0.1309480
7	−0.404094	0.6309956	−0.038000	0.0123715	−0.255113	−0.4669350
8	−0.095633	0.1901450	−0.043399	−0.1234840	−0.073162	−0.2308370
9	−0.104082	−0.1259360	−0.019776	0.0352318	−0.074121	0.1770820
10	−0.120229	−0.2787510	−0.018201	−0.0563500	−0.073041	0.2147698
11	−0.026213	−0.0424540	−0.029972	0.1226250	−0.007585	0.0510144
12	−0.018525	−0.0693630	−0.019929	0.0485469	−0.009463	0.0921424
13	−0.015400	−0.0824480	−0.019718	0.0780422	−0.003868	0.0576488
14	−0.017935	0.0692104	−0.023587	−0.0793110	−0.006482	−0.0741270
15	−0.039804	0.1413830	−0.053841	−0.1221600	−0.026551	−0.1594670
16	−0.032432	0.0229225	−0.032145	−0.0347620	−0.006208	−0.0757550
17	−0.045563	−0.1070770	−0.042489	0.0028974	−0.005407	0.0549959
18	−0.068636	−0.1843920	−0.062252	0.1462962	−0.009592	0.0796829
19	−0.035229	−0.0518640	−0.026565	−0.0345550	−0.002031	0.0137132
20	−0.044662	−0.0453130	−0.025522	0.0435325	−0.004731	−0.0388080
21	−0.037944	0.0070987	−0.031319	−0.1022920	−0.004658	−0.0340620
22	−0.037912	−0.0232190	−0.016445	−0.0142410	−0.002011	−0.0371270
23	−0.035400	0.1057311	−0.052900	−0.1619690	−0.013517	−0.0960300
24	−0.024672	0.0773092	−0.196451	0.3334009	−0.071148	0.1286855
25	−0.022247	0.0267849	−0.150838	−0.1647770	−0.113686	0.2131234
26	−0.040260	−0.1060880	−0.149602	0.1986025	−0.049168	−0.0532740
27	−0.059060	0.0165365	−0.597490	−0.6114820	−0.178579	−0.3352950
28	−0.075634	0.0365385	−0.244128	0.1724178	−0.095762	−0.1953210
29	−0.097230	0.1141908	−0.208697	−0.2007220	−0.115174	0.1678422
30	−0.043940	−0.0113350	−0.303605	0.4186013	−0.094796	0.2120557

NOTE : $\delta = 10$, f_{ii} = *the ith diagonal element of matrix* $\ddot{F}$ *and* $l_{\max}$ *corresponds to the maximum absolute eigenvalue of* $\ddot{F}$.

$\gamma_\varepsilon(\cdot)$ and $\lambda_\varepsilon(\cdot)$ are the autocovariance and autocorrelation functions of ε_t, respectively. The b_t are uncorrelated with constant variance σ^2.

Let $b = (b_1, \ldots, b_n)'$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$. Then $b = Q\varepsilon$ where

$$Q = \begin{pmatrix} d_{11} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ -d_{21} & d_{22} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ -d_{31} & -d_{32} & d_{33} & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ -d_{p1} & -d_{p2} & -d_{p3} & \cdots & d_{pp} & 0 & 0 & \cdots \\ -\rho_p & -\rho_{p-1} & -\rho_{p-2} & \cdots & -\rho_1 & 1 & 0 & \cdots \\ 0 & -\rho_p & -\rho_{p-1} & \cdots & -\rho_2 & -\rho_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So $\Psi^{-1} = Q'Q$ and $|\Psi^{-1}| = |Q|^2 = d_{11}^2 d_{22}^2 \cdots d_{pp}^2$. For $p = 3$

$$\begin{aligned} |\Psi^{-1}| &= d_{11}^2 d_{22}^2 d_{33}^2 \\ &= \left\{ \frac{\sigma^2}{\gamma_\varepsilon(0)} \right\}^3 \cdot \frac{1}{1 - \lambda_\varepsilon(2)} \cdot \frac{1}{1 + \lambda_\varepsilon(2) - 2\{\lambda_\varepsilon(1)\}^2}. \end{aligned}$$

Solving the Yule-Walker equation for $p = 3$, we get

$$\lambda_\varepsilon(1) = \frac{\rho_1 + \rho_2\rho_3}{(1 - \rho_3^2) - (\rho_2 + \rho_1\rho_3)}, \quad \lambda_\varepsilon(2) = \frac{(\rho_2 + \rho_1\rho_3) + (\rho_1^2 - \rho_2^2)}{(1 - \rho_3^2) - (\rho_2 + \rho_1\rho_3)},$$

$$\frac{\sigma^2}{\gamma_\varepsilon(0)} = \frac{f(\rho)}{(1 - \rho_3^2) - (\rho_2 + \rho_1\rho_3)},$$

where $\rho = (\rho_1, \rho_2, \rho_3)'$ and

$$f(\rho) = (1 - \rho_3^2)^2 - (\rho_1 + \rho_2\rho_3)^2 - (\rho_2 + \rho_1\rho_3)^2 - (\rho_2 + \rho_1\rho_3)(1 + \rho_1^2 - \rho_2^2 - \rho_3^2).$$

Hence for $p = 3$

$$|\Psi^{-1}| = \frac{\{f(\rho)\}^3}{g(\rho)h(\rho)},$$

where

$$g(\rho) = (1 - \rho_2)^2 - (\rho_1 + \rho_3)^2,$$

$$h(\rho) = (1 + \rho_1^2 - \rho_2^2 - \rho_3^2)\left\{(1 - \rho_3^2) - (\rho_2 + \rho_1\rho_3)\right\} - 2(\rho_1 + \rho_2\rho_3)^2.$$

It can be shown that $h(\rho) = f(\rho)$ and $f(\rho) = (\rho_1\rho_3 - \rho_3^2 + \rho_2 + 1)g(\rho)$. As a result, we obtain

$$|\Psi^{-1}| = \left\{(1 - \rho_2)^2 - (\rho_1 + \rho_3)^2\right\}(\rho_1\rho_3 - \rho_3^2 + \rho_2 + 1)^2$$

for $p = 3$, which completes the derivation.

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References

Cook, R. D. (1986). "Assessment of local analysis", *Journal of the Royal Statistical Society*, **B48**, 133–169.

Fuller, W. A. (1976). *Introduction to Statistical Time Series*, Wiley, New York.

Kim, S. K. and Huggins, R. (1998). "Diagnostics for autocorrelated regression models", *Australian & New Zealand Journal of Statistics*, **40**, 65–71.

Park, I. Y. and Shin, K. I. (1999). "Analysis of the effect of the limit of automobile liability insurance on the occurrence of motoring accidents", *Proceedings of the Spring Conference in Korean Statistical Society*, 234–239.

Wise, J. (1955). "The autocorrelation function and the spectral density function", *Biometrika*, **42**, 151–159.