Two-dimensional Elastic Analysis of Doubly Periodic Circular Holes in Infinite Plane

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Two-dimensional elastic analysis of doubly periodic circular holes in an infinite plane is given in this paper. Two cases of loading, remote tension and remote shear, are considered. A rectangular cell is cut from the infinite plane. In both cases, the boundary value problem can be reduced to a complex mixed one. It is found that the eigenfunction expansion variational method is efficient to solve the problem. Based on the deformation response under certain loading, the notched medium could be modeled by an orthotropic medium without holes. Elastic properties for the equivalent orthotropic medium are investigated, and the stress concentration along the hole contour is studied. Finally, numerical examples and results are given.

Key Words: Notch Problem, Complex Mixed Boundary Value Problem, Effective Elastic Properties

1. Introduction

Notch problems were considered by many investigators. Methods for solving the problems were proposed and many interesting results were collected (Neuber, 1946; Savin, 1961; Lekhnitsky, 1963; Sih, 1978). For the plane problem case, the previous investigations were limited to the cases where several notches were involved in an infinite plate. For example, the interaction of two circular holes was studied by using the Airy's stress function in the bipolar coordinates (Savin, 1961). Stress analysis in a strip with two equal circular holes under tension was carried out by a series presentation of Airy's stress function (Atsumi, 1956). Also, the body force method and conformal mapping technique were developed to study the notch problem (Sih, 1978). An iterative method based on the Airy's stress function was used to solve a multiple circular hole problem (Ting et al., 1999). All those investigations are limited to the stress concentration problem for the relevant cases.

Recently, the stress analysis for arrays of arbitrarily located holes and cracks has been conducted (Hu et al., 1993). Elastic interactions between elliptic holes were investigated (Tsukrov et al., 1997). The just mentioned investigation mainly depends on the fundamental solution, for example, a solution of an elliptical hole with a concentrated force applied on the hole contour in an infinite plate. Clearly, it is not easy to use the mentioned method to the present case, i.e. the doubly periodic holes case.

The two-dimensional elastic analysis for doubly periodic circular holes in an infinite plate was performed using the complex variable function method (Fil'stinsky, 1964). In the paper, the complex potentials were expressed by elliptical functions. The solution was derived from a solution of an algebraic equation with infinite unknowns. No description for truncating the

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number of infinite unknowns was reported.

In this paper, the problem for two-dimensional analysis of doubly periodic circular holes in an infinite plane is investigated by using a quite different method. A cell element is cut from the infinite plate with the doubly periodic circular holes. The problem is reduced to a complex mixed boundary value problem of the cell element. Furthermore, the eigenfunction expansion variational method (EEVM) is developed to solve the problem (Chen, 1983). In fact, the EEVM belongs to a type of Trefftz method. In the method, the solution satisfies the governing equation of elasticity and a part of the boundary conditions. In this paper, the elastic response for the problem is studied in more detail. For example, equivalence of the mentioned structure to an orthotropic plate without holes is studied.

Two cases of loading, remote tension and shear, are considered. In both cases, the boundary value problems can be reduced to complex mixed ones. It is found that the EEVM is efficient to solve the problems. The problems are solved by analyzing a rectangular cell with a hole cut from the infinite plate. After the boundary value problems are solved, the average stresses and strains are also evaluated. Furthermore, from the obtained results the notched mediums are made to be equivalent to orthotropic mediums without holes. The elastic properties for the equivalent orthotropic mediums are investigated, and the stress concentration along the hole contour is also studied. Finally, numerical examples and results are given.

2. Analysis

2.1 The eigenfunction expansion variational method (EEVM)

A finite plate with a circular hole is considered for the formulation (Fig. 1(a)). The assumed boundary conditions are

$\sigma_{ij}n_j=0$ (on the circular boundary C_R) (1a)

$\sigma_{ij}n_j=\bar{p}_i$ (on the outer boundary C_p) (1b)

$u_i=\bar{u}_i$ (on the outer boundary C_u) (1c)

where σ_{ij} denotes the stress components, n_j the direction cosines, and $\bar{p}_i$ the given tractions on the boundary C_p. Also, u_i denotes the displacement components, and $\bar{u}_i$ the given displacements on the boundary C_u.

Suppose that there is a two-dimensional field with the displacements u_i, strains e_{ij} and stresses σ_{ij}, and that they satisfy all the governing equations of plane elasticity. In the case, the following functional can be defined (Hu, 1995; Washizu, 1975)

$$\Pi=\iint_{\Sigma}E(e_{ij})\,d\Sigma-\int_{C_p}\bar{p}_iu_i dS-\int_{C_u}\sigma_{ij}n_j(u_i-\bar{u}_i)\,dS \qquad (2)$$

where $E(e_{ij})$ is a strain energy density function, and Σ is the region occupied by a body (Fig. 1 (a)). The statement of the variational method is as follows. The actual solution of elasticity can be obtained from the stationary value condition of the functional Π. Note that, in the present case the Clapeyron's theorem takes the form

$$\iint_{\Sigma}E(e_{ij})\,dA=\frac{1}{2}\int_{C_p+C_u}\sigma_{ij}n_ju_i dS \qquad (3)$$

Substituting Eq. (3) into Eq. (2) yields

$$\Pi=\frac{1}{2}\int_{C_p}\sigma_{ij}n_ju_i dS-\frac{1}{2}\int_{C_u}\sigma_{ij}n_ju_i dS-\int_{C_p}\bar{p}_iu_i dS+\int_{C_u}\sigma_{ij}n_j\bar{u}_i dS \qquad (4)$$

In Eqs. (2) and (4), all the physical components u_i, e_{ij} and σ_{ij} are defined in the notched region Σ (Fig. 1(a)).

An elasticity solution with the components of displacements u_i and stresses σ_{ij} is introduced as follows

$$\sigma_{ij}=\sum_{k=1}^{M}X_k\sigma_{ij}^{(k)},\quad u_i=\sum_{k=1}^{M}X_ku_i^{(k)} \qquad (5)$$

where X_k is the undetermined coefficients. Assume that each term in Eq. (5) satisfies (a) all the governing equations of plane elasticity and (b) the boundary condition, Eq. (1a). In the case, the stress and displacement fields defined by Eq. (5) are called the eigenfunction expansion form hereafter. Substituting Eq. (5) into Eq. (4) and letting

$$\frac{\partial\Pi}{\partial X_m}=0,\quad (m=1,2,\cdots,M) \qquad (6)$$

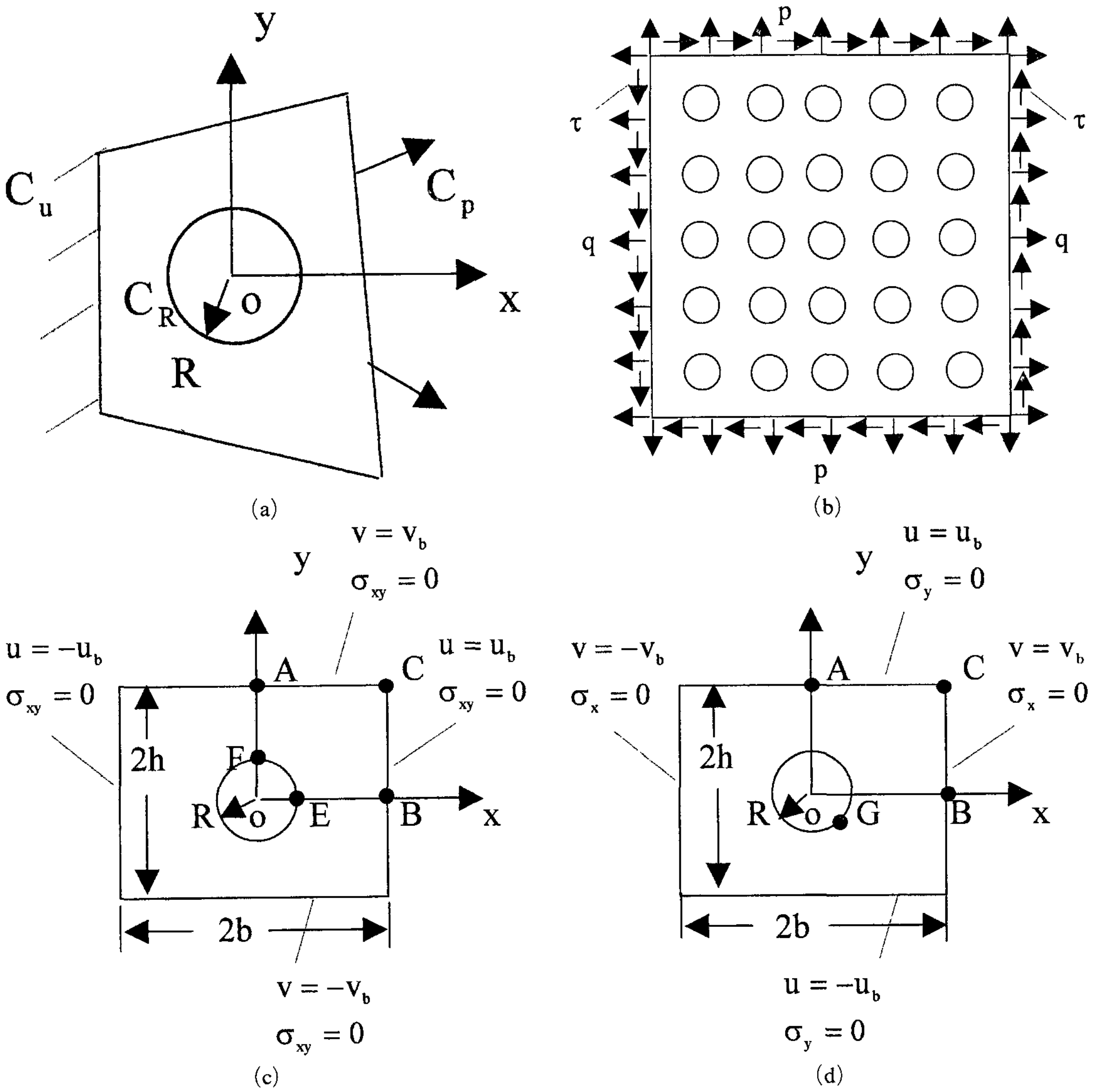

Fig. 1 (a) a finite plate with a circular hole, (b) the loading condition for the doubly periodic hole problem, (c) the boundary condition for the tension loading case, (d) the boundary condition for the shear loading case

the linear algebraic equations for unknowns X_m $(m=1, 2, \cdots, M)$ are obtained as follows

$$\sum_{k=1}^{M} A_{mk} X_k = B_m \quad (m=1, 2, \cdots, M) \tag{7}$$

where

$$A_{mk}=A_{km}=\int_{C_p} \sigma_{ij}^{(k)} n_j u_i^{(m)} dS - \int_{C_u} \sigma_{ij}^{(m)} n_j u_i^{(k)} dS \quad (m, k=1, 2, \cdots, M) \tag{8}$$

$$B_m=\int_{C_p} \bar{p}_i u_i^{(m)} dS - \int_{C_u} \sigma_{ij}^{(m)} n_j \bar{u}_i dS \quad (m=1, 2, \cdots, M) \tag{9}$$

For convenience , it is preferable to write Eqs. (8) and (9) in alternative forms,

$$A_{mk}=A_{km}=\int_{C_p} u_i^{(m)} \sigma_{ij}^{(k)} n_j dS + \int_{C_u} (-\sigma_{ij}^{(m)} n_j u_i^{(k)}) dS \quad (m, k=1, 2, \cdots, M) \tag{8a}$$

$$B_m=\int_{C_p}\bar{p}_i u_i^{(m)}dS + \int_{C_u}(-\sigma_{ij}^{(m)}n_j\bar{u}_i)dS \ (m=1, 2, \cdots, M) \tag{9a}$$

From Eq. (8a) the following rules can be seen for the composition of the integrands in A_{mk}:

(a) If the integration is performed along the boundary C_p, the integrand is the m-th displacement multiplied by the k-th traction in the expansion form. If the condition is the C_p type, the superscript "k" always follows the traction component.

(b) If the integration is performed along the boundary C_u, the integrand is the m-th traction multiplied by the k-th displacement and by (-1). If the condition is the C_u type, the superscript "k" always follows the displacement component.

The complex mixed boundary condition is defined such that on the same boundary one traction component is given and one displacement component is also given beforehand. In the complex mixed boundary condition, we simply decompose one integral in A_{mk} into two integrals. Among them, one belongs to the given boundary traction type (C_p), and the other belongs to the given displacement type (C_u). The detail will be presented in concrete examples.

Similar description can be carried out for B_m values in the complex mixed boundary condition.

In the derivation the undetermined coefficients X_m ($m=1, 2, \cdots, M$) are obtained from the variational principle. Therefore, the method is called the eigenfunction expansion variational method (EEVM) in this paper (Chen, 1983).

The stresses (σ_x, σ_y, σ_{xy}), the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$ such that (Muskhelishvili, 1953)

$$\sigma_x+\sigma_y=4\mathrm{Re}\phi'(z)$$

$$\sigma_y-\sigma_y+2i\sigma_{xy}=2[\bar{z}\phi''(z)+\psi'(z)] \tag{10}$$

$$f=-Y+iX=\phi(z)+z\overline{\phi'(z)}+\overline{\psi'(z)} \tag{11}$$

$$2G(u+iv)=\kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi'(z)} \tag{12}$$

where G is the shear modulus of elasticity, $\kappa=(3-v)/(1+v)$ for the plane stress problem, which is assumed in this paper, and v is the Poisson's ratio. $E(=2G(1+v))$ denotes the Young's modulus of elasticity.

The traction free condition along the inner circular hole may be expressed as

$$\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}=0, \ (z\in C_R) \tag{13}$$

It is easy to see that the condition of vanishing resultant force along the boundary C_R is equivalent to the traction free condition along the same boundary.

From the condition Eq. (13) and the fact that $z\bar{z}=R^2(z\in C_R)$, it is easy to obtain the following complex potentials

$$\phi(z)=a_k z^k,$$
$$\psi(z)=-\bar{a}_k R^{2k}z^{-k} - k a_k R^2 z^{k-2}, \ (k=\pm1, \pm2, \pm3, \cdots) \tag{14}$$

which always satisfies the condition Eq. (1a). In Eq. (14), a_k is a constant. Let each term in Eq. (5) be derived from the complex potential shown by Eq. (14). The following term (letting $k=1$ and $a_1=i$ in Eq. (14))

$$\phi(z)=iz, \ \psi(z)=0 \tag{15}$$

should be excluded from the expansion form, simply because it corresponds to a rotation of the body.

2.2 Normal loading case

In the following analysis, an infinite plate with doubly periodic holes is shown in Fig. 1(b), and the plane stress state is assumed. The elastic constants are denoted by v_0, G_0 and E_0($E_0=2G_0(1+v_0)$), and $v_0=0.3$ is used. We first study the case where the remote tensions are $\sigma_x=0$ and $\sigma_y=p$ (Fig. 1(c)). It is convenient to cut a rectangular cell with a hole from the infinite plate (Fig. 1(c)). Clearly, the boundary conditions can be written as

$$v=\bar{v}=\pm v_b, \ \sigma_{xy}=0 \ (-b\le x\le b, \ y=\pm h) \tag{16a}$$

$$u=\bar{u}=\pm u_b, \ \sigma_{xy}=0 \ (x=\pm b, \ -h\le y\le h) \tag{16b}$$

In Eq. (16a, b), v_b and u_b are two undetermined values determined by

$$\int_0^b \sigma_y(x, h)\,dx=bp \tag{17}$$

$$\int_0^h \sigma_x(b, y)\,dy=0 \tag{18}$$

Clearly, the boundary value conditions Eqs.

(16a) and (16b) are complex mixed ones.

Since the symmetric condition exists in the present case, it is suitable to take the terms in the expansion form Eq. (5) by letting $k=2n+1$ and $a_k=1$ in Eq. (14) which becomes

$$\begin{aligned}\phi(z) &= z^{2n+1} \\ \psi(z) &= -R^{4n+2}z^{-(2n+1)} \\ &\quad -(2n+1)R^2 z^{2n-1} \\ &\quad (n=-M_1, -M_1+1, \cdots -1, 0, 1, \cdots, M_1-1)\end{aligned} \quad (19)$$

In Eq. (19) $M_1=M/2$ (M is an even number). Equation (7) is used in the present case to evaluate the undetermined coefficients in the expansion form. However, since the mentioned boundary conditions are complex mixed ones, the components A_{mk} and B_m need to be defined as

$$A_{mk}=\int_{AC_p} f_1 dx+\int_{AC_u} f_2 dx+\int_{BC_p} f_3 dy+\int_{BC_u} f_4 dy \quad (20)$$

$$B_m=\int_{AC_p} g_1 dx+\int_{AC_u} g_2 dx+\int_{BC_p} g_3 dy+\int_{BC_u} g_4 dy \quad (21)$$

where for example AC means that the integration is performed along the line AC in Fig. 1(c). In addition, one integral along the interval AC decomposes into two integrals which are indicated by $\int_{AC_p} f_1 dx$ and $\int_{AC_u} f_2 dx$, respectively.

Since the σ_{xy} component along the line AC is given in Eq. (16a), from C_p type integral in Eq. (8a) we have $f_1=u^{(m)}\sigma_{xy}{}^{(k)}$. Similarly, the displacement v along the line AC is given in Eq. (16a), from C_u type integral in Eq. (8a) we have $f_2=-\sigma_y{}^{(m)}v^{(k)}$. Similarly, we can obtain $f_3=v^{(m)}\sigma_{xy}{}^{(k)}$ and $f_4=-\sigma_x{}^{(m)}u^{(k)}$. Substituting f_1, f_2, f_3 and f_4 into Eq. (20) yields

$$\begin{aligned}A_{mk}=A_{km}&=\int_{AC}[\sigma_{xy}{}^{(k)}u^{(m)}-\sigma_y{}^{(m)}v^{(k)}]dx \\ &+\int_{BC}[\sigma_{xy}{}^{(k)}v^{(m)}-\sigma_x{}^{(m)}u^{(k)}]dy \ (m, k=1, 2, \cdots, M)\end{aligned} \quad (20a)$$

Similar derivation can be carried out for B_m defined by Eq. (21). Since the σ_{xy} component along the line AC is given in Eq. (16a) ($\sigma_{xy}=0$), from C_p type integral in Eq. (9a) we have $g_1=0*u^{(m)}=0$. Similarly, the displacement v along the line AC is given in Eq. (16a) ($v=\bar{v}$), from C_u type integral in Eq. (9a) we have $g_2=-\sigma_y{}^{(m)}\bar{v}$. Similarly, we can obtain $g_3=0$ and $g_4=-\sigma_x{}^{(m)}\bar{u}$. Substituing g_1, g_2, g_3 and g_4 into Eq. (21) yields

$$\begin{aligned}B_m=&-\int_{AC}\sigma_y{}^{(m)}\bar{v}dx \\ &-\int_{BC}\sigma_x{}^{(m)}\bar{u}dy \ (m=1, 2, \cdots, \mathrm{M})\end{aligned} \quad (21a)$$

In Eqs. (20a) and (21a), for example, AC means that the integration is performed along the line AC in Fig. 1(c). $\sigma_y{}^{(m)}$ means the σ_y component of the m-th term in the expansion form, and $\bar{v}$ is the given displacement along the line AC shown in Eq. (16a). In fact, since the principle of linear superposition is valid in elasticity, we can instead solve the two boundary value problems defined by

$$v=\bar{v}=\pm 1,\ \sigma_{xy}=0\ (-b\le x\le b,\ y=\pm h) \quad (22a)$$

$$u=\bar{u}=0,\ \sigma_{xy}=0\ (x=\pm b,\ -h\le y\le h) \quad (22b)$$

and

$$v=\bar{v}=0,\ \sigma_{xy}=0\ (-b\le x\le b,\ y=\pm h) \quad (23a)$$

$$u=\bar{u}=\pm 1,\ \sigma_{xy}=0\ (x=\pm b,\ -h\le y\le h) \quad (23b)$$

Finally, the two unknowns u_b and v_b can be determined by using Eqs. (17) and (18).

Clearly, from the deformation response in the y-direction the notched rectangle can be modeled globally by an orthortropic medium without holes. It is known that the constitutive equation of the orthtropic medium takes the form (Lekhnitsky, 1963)

$$\begin{aligned}\varepsilon_x&=\frac{1}{E_1}\sigma_x-\frac{\gamma_{21}}{E_2}\sigma_y, \\ \varepsilon_y&=-\frac{\gamma_{12}}{E_1}\sigma_x+\frac{1}{E_2}\sigma_y, \\ \gamma_{xy}&=\frac{1}{G_{12}}\sigma_{xy}\end{aligned} \quad (24)$$

In Eq. (24), there holds a relation;

$$(E_2\gamma_{12})/(E_1\gamma_{21})=1 \quad (25)$$

From the assumed loading condition and the numerical solution mentioned above, we have

$$\sigma_{x,av}=0,\ \sigma_{y,av}=p,\ \varepsilon_{x,av}=\frac{u_b}{b},\ \varepsilon_{y,av}=\frac{v_b}{h} \quad (26)$$

where the subscript "av" means that the relevant quantity should be understood in the sense of average on some portion of the boundary.

Substituting Eq. (26) into Eq. (24) yields

Table 1 Normalized elastic constant $f_1(h/b, R/c)$, $f_2(h/b, R/c)$ and $f_3(h/b, R/c)$ in the coubly periodic hole problem (see Fig. 1(c) and Eqs. (28), (29) and (30))

$f_1(h/b, R/c)$

R/c=	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	.9907	.9634	.9203	.8634	.7938	.7111	.6:35	.4967
h/b=0.6	.9860	.9461	.8849	.8079	.7192	.6211	.5142	.3969
h/b=0.8	.9815	.9294	.8525	.7602	.6598	.5551	.4477	.3361
h/b=1.0	.9770	.9139	.8244	.7224	.6168	.5117	.4069	.3000
h/b=1.5	.9846	.9422	.8815	.8118	.7397	.6683	.5981	.5286
h/b=2.0	.9885	.9566	.9110	.8586	.8043	.7506	.6977	.6452

$f_2(h/b, R/c)$

R/c=	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	.9908	.9653	.9288	.8868	.8433	.8002	.7579	.7160
h/b=0.6	.9862	.9480	.8933	.8305	.7654	.7010	.6376	.5749
h/b=0.8	.9816	.9308	.8582	.7751	.6892	.6042	.5207	..4381
h/b=1.0	.9770	.9139	.8244	.7224	.6168	.5117	.4069	.3000
h/b=1.5	.9845	.9404	.8737	.7911	.6976	.5965	.4886	.3730
h/b=2.0	.9883	.9547	.9023	.8348	.7547	.6630	.5593	.4409

$f_3(h/b, R/c)$

R/c=	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	1.0102	1.0383	1.0773	1.1193	1.1574	1.1883	1.2121	1.2309
h/b=0.6	1.0154	1.0583	1.1200	1.1894	1.2568	1.3168	1.3692	1.4174
h/b=0.8	1.0206	1.0786	1.1638	1.2626	1.3620	1.4526	1.5305	1.5951
h/b=1.0	1.0257	1.0979	1.2028	1.3228	1.4401	1.5407	1.6174	1.6728
h/b=1.5	1.0171	1.0651	1.1347	1.2141	1.2927	1.3639	1.4270	1.4847
h/b=2.0	1.0128	1.0482	1.0983	1.1533	1.2050	1.2490	1.2855	1.3174

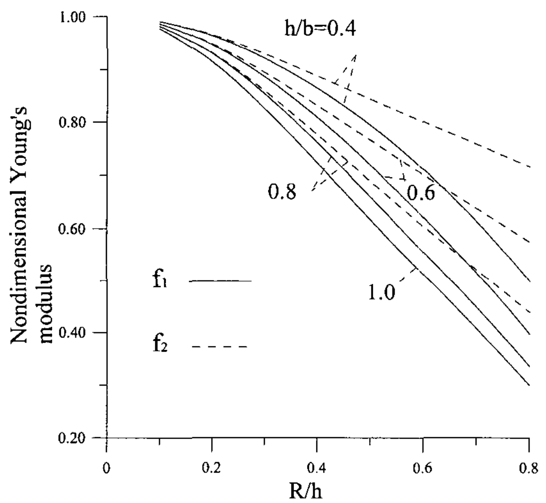

Fig. 2 Normalized elastic constant $f_1(h/b, R/c)$, and $f_2(h/b, R/c)$ in the doubly periodic hole problem (see Fig. 1(c) and Eqs. (28) and (29))

$$E_2=\frac{hp}{v_b},\ \gamma_{21}=-\frac{hu_b}{bv_b} \tag{27}$$

Similarly, we can propose the other boundary condition, with $\sigma_x=q$ and $\sigma_y=0$ at the remote place. From the relevant solution we can evaluate the other two elastic constants E_1 and γ_{12} in a similar way.

Clearly, the results obtained for E_2, γ_{21}, E_1 and γ_{12} may not satisfy the relation Eq. (25) exactly. In fact, in the range for h/b and R/b (or R/h) used in the numerical example, the ratios are found to be $(E_2\gamma_{12})(E_1\gamma_{21})=1.0000$. This is to say that the proposed assumption coincides the physical situation very well.

In case of $M=12$ (M : the number of terms in the expansion form) in Eqs (7), (8) and (9), computation is performed. The calculated elastic constants are expressed as

$$E_1=f_1(h/b, R/c)E_0 \text{ (where } c=\min(b, h)) \tag{28}$$

Table 2 Normalized tangential stress $g_1(h/b, R/c)$ and $g_2(h/b, R/c)$ in the doubly periodic hole problem (see Fig.1(c) and Eqs.(31) and (32))

$g_1(h/b, R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	2.9664	2.8694	2.7440	2.6222	2.5311	2.4844	2.4817	2.5123
h/b=0.6	2.9774	2.9189	2.8489	2.7967	2.7869	2.8331	2.9371	3.0955
h/b=0.8	2.9896	2.9662	2.9500	2.9670	3.0424	3.1969	3.4506	3.8467
h/b=1.0	2.9991	3.0028	3.0298	3.1082	3.2782	3.6091	4.2654	5.7742
h/b=1.5	3.0049	3.0217	3.0585	3.1324	3.2800	3.5811	4.2233	5.7293
h/b=2.0	3.0043	3.0202	3.0564	3.1320	3.2844	3.5926	4.2367	5.7104

$g_2(h/b, R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	−0.9775	−0.9115	−0.8061	−0.6669	−0.5018	−0.3215	−0.1405	0.0273
h/b=0.6	−0.9811	−0.9238	−0.8271	−0.6897	−0.5134	−0.3088	−0.1025	0.0669
h/b=0.8	−0.9838	−0.9347	−0.8504	−0.7268	−0.5597	−0.3529	−0.1327	0.0457
h/b=1.0	−0.9830	−0.9341	−0.8571	−0.7518	−0.6097	−0.4187	−0.1858	0.0348
h/b=1.5	−0.9828	−0.9371	−0.8773	−0.8199	−0.7758	−0.7454	−0.7172	−0.6716
h/b=2.0	−0.9797	−0.9257	−0.8551	−0.7879	−0.7394	−0.7156	−0.7120	−0.7186

$$E_2=f_2(h/b, R/c)E_0 \text{ (where } c=\min(b, h)) \quad (29)$$

$$\gamma_{21}/E_2=\gamma_{12}/E_1 =f_3(h/b, R/c)(\nu_0/E_0) \text{ (where } c=\min(b, h)) \quad (30)$$

The calculated results for $f_1(h/b, R/c)$, $f_2(h/b, R/c)$ and $f_3(h/b, R/c)$ are listed in Table 1 and Fig. 2.

From Table 1 and Fig. 2 we see that when $h/b<1.0$, $E_1<E_2$ or $f_1<f_2$. This result can be explained by the following reason. For example, if $h/b=0.4$ and $R/h=0.4$, the reduction of the section perpendicular to the x-direction is larger than that perpendicular to the y-direction, and it is $R/h=0.4$. However, the reduction of the section perpendicular to the y-direction is lower, and it is $R/b=0.16$. Therefore, in case of $h/b<1.0$, the relation $E_1<E_2$ must exist.

On the contrary, when $h/b>1.0$, $E_1>E_2$ or $f_1>f_2$. From Table 1 we see that the normalized $\gamma_{21}/E_2(=f_3(\nu_0/E_0))$ value is generally higher than unity. From the theory of elasticity we know that in the simple tension case the volume increment is proportional to $1-2\nu_0$. Therefore, a higher value of γ_{21}/E_2 means that the average volume increment is less than the case of a solid plate without holes.

For the remote stress, $\sigma_x=0$ and $\sigma_y=p$, the tangential stresses at the points $E(x=R, y=0)$ and $F(x=0, y=R)$ are expressed as

$$\sigma_{t,E}=g_1(h/b, R/c)p \text{ (where } c=\min(b, h)) \quad (31)$$

$$\sigma_{t,E}=g_2(h/b, R/c)p \text{ (where } c=\min(b, h)) \quad (32)$$

The calculated $g_1(h/b, R/c)$ and $g_2(h/b, R/c)$ values are listed in Table 2.

For the case of a single hole in an infinite body it is well known that $\sigma_{t,F}/p=-1.0$. However, in some extreme case of periodic holes, for example, when $h/b=1.0$ and $R/b=0.8$, this value becomes 0.0348 (see Table 2). This result means that in some particular case, the stress distribution around the hole is quite different from the single hole case.

To examine the convergent tendency, in the condition of $h/b=1$ the tangential stresses $\sigma_{t,E}$ and $\sigma_{t,F}$ for $M=8$, $M=10$ and $M=12$ cases are listed in Table 3. From Table 3 we see that in the case of $R/b=0.5$, the difference for the g_1 values obtained from $M=8$, $M=10$, and $M=12$ is less than 0.2%.

Particularly, in case of $h/b=1.0$ and the remote stress $\sigma_x=0$ and $\sigma_y=p$, the stress components σ_t and σ_r at the point E (Fig. 1 (c)) are expressed by

$$\sigma_{t,E}=h_1(R/b)p \quad (33)$$

$$\sigma_{r,E}=h_2(R/b)p \quad (34)$$

For comparison, the results for $h_1(R/b)$ and

Table 3 Comparison results for normalized tangential stresses $g_1(h/b, R/c)$ and $g_2(h/b, R/c)$ in the doubly periodic hole problem in the case of h/b=1.0 (see Fig.1(c) and Eqs.(31) and (32))

$g_1(h/b, R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
M=8	2.9989	3.0023	3.0291	3.1079	3.2806	3.6139	4.2891	5.7736
M=10	2.9991	3.0031	3.0304	3.1098	3.2821	3.6167	4.2726	5.7448
M=12	2.9991	3.0028	3.0298	3.1082	3.2782	3.6091	4.2654	5.7742

$g_2(h/b, R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
M=8	−0.9828	−0.9336	−0.8566	−0.7523	−0.6129	−0.4243	−0.1718	0.1985
M=10	−0.9830	−0.9343	−0.8577	−0.7533	−0.6135	−0.4269	−0.1980	0.0189
M=12	−0.9830	−0.9341	−0.8571	−0.7518	−0.6097	−0.4187	−0.1858	0.0348

Table 4 Normalized tangential and radial stresses $h_1(R/b)$ and $h_2(R/b)$ in the doubly periodic hole problem (see Fig.1(c) and Eqs. (33) and (34))

$R/b=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h_1(present)	2.9991	3.0028	3.0298	3.1082	3.2780	3.6091	4.2654	5.7742
h_1[9]		3.00		3.11		3.61		5.78
h_2(present)	0	0	0	0	0	0	0	0
h_2[9]		0.000		−0.0002		−0.0009		−0.0017

$h_2(R/b)$ from the present study and Fil'stinsky (Fil'stinsky, 1964) are listed in Table 4 . From Table 4 we see that coincidence of the stress component $\sigma_{t,E}$ between the two approaches is pretty good. Also, in the present study, the stress component $\sigma_{r,E}$ coincides with the exact solution ($\sigma_{r,E}=0$), since the traction free condition along the circular hole is always satisfied in the present approach. However, the stress component obtained by Fil'stinsky (Fil'stinsky, 1964) has a slight deviation from the exact solution.

2.3 Shear loading case

The response of the notched plate to shear loading can be investigated in a similar manner (Fig. 1(b)). In this case we assume that the remote shear traction is $\sigma_{xy}=\tau$. Similar to the previous case, the boundary conditions (Fig. 1 (d)) for the rectangular cell are

$$u=\bar{u}=\pm u_b,\ \sigma_y=0\ (-b\le x\le b,\ y=\pm h) \quad (35a)$$

$$v=\bar{v}=\pm v_b,\ \sigma_x=0\ (x=\pm b,\ -h\le y\le h) \quad (35b)$$

In Eq. (35a, b) u_b, and v_b are two undetermined values determined by

$$\int_0^b \sigma_{xy}(x, h)\,dx=b\tau \quad (36)$$

$$\int_0^h \sigma_{xy}(b, y)\,dy=h\tau \quad (37)$$

Clearly, the boundary conditions Eqs. (35a) and (35b) are complex mixed ones.

In the shear loading case, it is suitable to take the terms in the expansion form Eq. (5) by letting $k=2n+1$ and $a_k=i$ in Eq. (14) which becomes

$$\begin{aligned}&\phi(z)=iz^{2n+1}\\&\psi(z)=i[R^{4n+2}z^{-(2n+1)}-(2n+1)R^2z^{2n-1}]\\&(n=-M_1, -M_1+1, \cdots, -1, 1, \cdots, M_1)\end{aligned} \quad (38)$$

In Eq. (38) $M_1=M/2$ (M is an even number). Equation (7) is used in the present case to evaluate the undetermined coefficients in the expansion form. However, the components A_{mk} and B_m need to be defined as

$$\begin{aligned}A_{mk}=A_{km}=&\int_{AC}[\sigma_y^{(k)}v^{(m)}-\sigma_{xy}^{(m)}u^{(k)}]\,dx\\&+\int_{BC}[\sigma_x^{(k)}u^{(m)}-\sigma_{xy}^{(m)}v^{(k)}]\,dy\\&(m, k=1, 2, \cdots, M)\end{aligned} \quad (39)$$

$$\begin{aligned}B_m=&-\int_{AC}\sigma_{xy}^{(m)}\bar{u}\,dx\\&-\int_{BC}\sigma_{xy}^{(m)}\bar{v}\,dy\ (m=1, 2, \cdots, M)\end{aligned} \quad (40)$$

The notations used in Eqs. (39) and (40) are similar to the previous case.

Table 5 Normalized elastic constant $f_4(h/b,\ R/c)$ and normalized tangential stress $g_3(h/b,\ R/c)$ in the doubly periodic hole problem (see Fig.1(d) and Eqs. (45) and (46))

$f_4(h/b,\ R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	.9903	.9603	.9081	.8309	.7262	.5945	.4411	.2800
h/b=0.6	.9855	.9416	.8680	.7652	.6368	.4906	.3388	.1965
h/b=0.8	.9807	.9233	.8300	.7059	.5606	.4082	.2647	.1439
h/b=1.0	.9759	.9049	.7916	.6454	.4826	.3237	.1883	.0886
h/b=1.5	.9839	.9355	.8552	.7451	.6107	.4622	.3129	.1777
h/b=2.0	.9879	.9509	.8876	.7967	.6787	.5378	.3835	.2311

$g_3(h/b,\ R/c)$

$R/c=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h/b=0.4	4.0649	4.2667	4.6285	5.1936	6.0398	7.3140	9.3300	12.8788
h/b=0.6	4.0667	4.2740	4.6448	5.2207	6.0726	7.3281	9.2417	12.3975
h/b=0.8	4.0707	4.2911	4.6870	5.3037	6.2138	7.5370	9.4864	12.4495
h/b=1.0	4.0825	4.3416	4.8126	5.5568	6.6701	8.3071	10.7412	14.4491
h/b=1.5	4.0674	4.2772	4.6525	5.2355	6.0963	7.3578	9.2961	12.2910
h/b=2.0	4.0660	4.2711	4.6382	5.2094	6.0582	7.3214	9.2842	12.6553

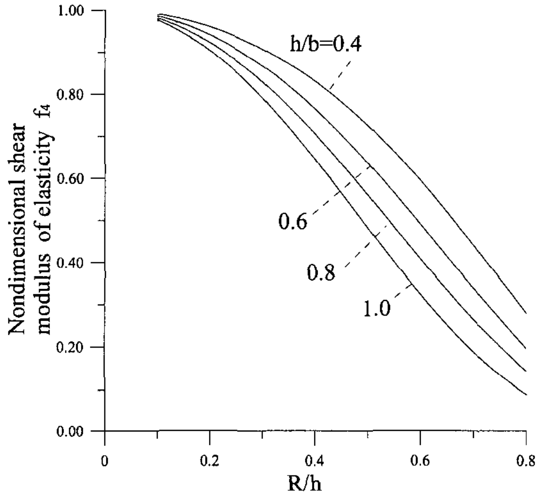

Fig. 3 Normalized elastic constant $f_4(h/b,\ R/c)$ in the doubly periodic hole problem (see Fig. 1 (d) and Eq. (45))

Since the principle of linear superposition is valid in elasticity, we can solve the two boundary value problems defined by

$$u=\bar{u}=\pm 1,\ \sigma_y=0\ (-b\leq x\leq b,\ y=\pm h) \quad (41a)$$
$$v=\bar{v}=0,\ \sigma_x=0\ (x=\pm b,\ -h\leq y\leq h) \quad (41b)$$

and

$$u=\bar{u}=0,\ \sigma_y=0\ (-b\leq x\leq b,\ y=\pm h) \quad (42a)$$
$$v=\bar{v}=\pm 1,\ \sigma_x=0\ (x=\pm b,\ -h\leq y\leq h) \quad (42b)$$

Finally, the two unknowns u_b and v_b can be determined by using Eqs. (36) and (37).

As before, the notched infinite plate can be modeled by an orthortropic medium without holes. From the numerical solution mentioned above, the following average stress and strain are obtained.

$$\sigma_{xy,av}=\tau,\ \gamma_{xy,av}=\frac{u_b}{h}+\frac{v_b}{b} \quad (43)$$

where the subscript "av" means that the relevant quantity should be understood in the sense of average on some portion of the boundary.

Substituting Eq. (43) into Eq. (24) yields

$$G_{12}=\tau\left(\frac{u_b}{h}+\frac{v_b}{b}\right)^{-1} \quad (44)$$

Similarly, the calculated results can be expressed as

$$G_{12}=f_4(h/b,\ R/c)E_0\ (\text{where } c=\min(b,\ h)) \quad (45)$$

The calculated results for $f_4(h/b,\ R/c)$ are listed in Table 5 and shown in Fig. 3. From Table 5 we see that $f_4(h/b,\ R/c)$ values are always less than unity.

Comparing Table 1 with Table 5 we see that, generally, $f_1(h/b,\ R/c)>f_4(h/b,\ R/c)$ and $f_2(h/b,\ R/c)>f_4(h/b,\ R/c)$. For example, in case of $h/b=1.0$, $R/b=0.5$, $f_1(h/b,\ R/c)=f_2(h/b,\ R/c)=0.6168$ and $f_4(h/b,\ R/c)=$

0.4826. This means that the relative reduction of the stiffness is higher in the shear loading than in the tension loading.

In case of the remote stress $\sigma_{xy}=\tau$, the tangential stresses at the point $G(x=R/\sqrt{2}, y=-R/\sqrt{2})$ (Fig. 1(d)) are expressed as

$$\sigma_{t,G}=g_3(h/b,\ R/c)\,\tau \text{ (where } c=\min(b,\ h)) \quad (46)$$

The calculated $g_3(h/b,\ R/c)$ values are listed in Table 5.

From the above analyses we see that the following two bodies have the same resistance with respect to the applied loading.

(a) The isotropic plate containing holes with elastic constants E_0 and $G_0(E_0$ or $v_0)$.

(b) The othotropic plate without holes with elastic constants E_1, E_2, γ_{21} andd G_{12}.

As mentioned above, the elastic constants E_1, E_2, γ_{21} and G_{12} depend on E_o and G_o as well as the cell configuration (b, h and R).

Clearly, the mentioned equivalence for the strain-stress relation holds only with respect to the global deformation for the notched medium. For the local stresses and strains of the notched medium, one should refer to the concrete solution obtained previously.

3. Conclusion

A remarkable advantage of using the eigenfunction expansion form (EE) is that the stress field derived from the EE is not only an elasticity solution but also satisfies the traction free condition along the circular boundary.

There exists a difficult point in the present study. For example in Eqs. (8) and (9), the boundaries were defined of two kinds, namely C_u (the displacement boundary) and C_p (the traction boundary). However, in this paper, the situation is that on the same boundary AC (Fig. 1(c)) one condition belongs to the displacement and other to the traction, which were shown by Eqs. (16a) and (16b). This difficulty was overcome by the following way. For example, in Eq. (16a) we meet the complicated mixed boundary conditions. The first condition ($v=\bar{v}=v_b$, $-b\leq x\leq b$, $y=\pm h$) belongs to the displacement condition, and second condition ($\sigma_{xy}=0$, $-b\leq x\leq y=\pm h$) belongs to the traction condition. In this case, it is simple to divide the boundary AC in Fig. 1(c) into two part C_u and C_p.

Furthermore, when considering the first condition on boundary AC, the type of integration on C_u in Eqs. (8) and (9) will be used. Otherwise, when considering the second condition on boundary AC, the type of integration on C_p in Eqs. (8) and (9) will be used. Actual computation proves that the mentioned procedure yields a reasonable result.

Finally, since all the interested quantities such as the stress concentration factor at the hole edge and the effective moduli of elasticity can be obtained from the suggested eigenfunction expansion variational method (EEVM), the EEVM provides an effective way to solve the doubly periodic problem.

It is found that an infinite plate with doubly periodic holes is equivalent to an orthotropic plate without holes with respect to their global deformation. If the relative radius of the holes is larger, the stress concentration factor and the Young's modulus of elasticity are more affected.

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