*Journal of the Korean Statistical Society (2002),* **31**: *1, pp 109–120*

# Weak Convergence of U-empirical Processes for Two Sample Case with Applications

Hyo-Il Park[1] and Jong-Hwa Na[2]

ABSTRACT

In this paper, we show the weak convergence of U-empirical processes for two sample problem. We use the result to show the asymptotic normality for the generalized Hodges-Lehmann estimates with the Bahadur representation for quantiles of U-empirical distributions. Also we consider the asymptotic normality for the test statistics in a simple way.

*Keywords:* Bahadur representation for quantile, kernel, location translation parameter, U-empirical distribution, U-empirical process, weak convergence.

## 1. Introduction

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples with continuous distribution functions $F$ and $G$, respectively. Let $\Delta$ be any parameter, which represents some relation between $F$ and $G$ such as location translation parameter or measure of difference of scale parameters. Let $h(x_1, \ldots, x_k; y_1, \ldots, y_l)$ be a symmetric kernel for $\Delta$ of degree $(k, l)$. In this paper, we allow that $k$ and $l$ need not be the minimum sample sizes required to obtain an unbiased estimate of $\Delta$. Now we define the U-empirical distribution function on $t \in (-\infty, \infty)$ as

$$H_{mn} = \frac{1}{\binom{m}{k}\binom{n}{l}} \sum_{\alpha \in A} \sum_{\beta \in B} I(h(X_{\alpha_1}, \ldots, X_{\alpha_k}; Y_{\beta_1}, \ldots, Y_{\beta_l}) \leq t),$$

where $A(B)$ is the collection of all subsets of $k(l)$ integers chosen without replacement from the integers $\{1, \ldots, m\}(\{1, \ldots, n\})$. Then the corresponding U-empirical process is defined on $t \in (-\infty, \infty)$ as

$$B_{mn}(t) = \sqrt{N}\{H_{mn}(t) - H(t)\},$$

[1]Department of Statistics, Chong-ju University, Chong-ju, Choong-book 360-764, Korea (e-mail : hipark@chongju.ac.kr)

[2]Department of Statistics, Chungbuk National University, Chong-ju, Choong-book 361-763, Korea (e-mail : cherin@cbucc.chungbuk.ac.kr)

where $N = m+n$ and $H(\cdot)$ is the distribution function of $h(X_1, \ldots, X_k; Y_1, \ldots, Y_l)$.

For one sample case, Silverman (1983) showed the weak convergence of the U-empirical processes on a metric space. Arcones (1993) and Arcones and Gine (1993) considered several asymptotic properties of the U-processes. In this paper, we show the weak convergence of $B_{mn}(t)$ to a normal process $B(t)$ and then use to show the asymptotic normality for the generalized Hodges-Lehmann estimates for the parameters of the difference of locations and that of scales with the Bahadur representation for quantiles of U-empirical distributions. Also we consider the asymptotic normality for the nonparametric test statistics in a simple way.

## 2. Main Result

Before we state our main result, we review the asymptotic covariance function for the process $B_{mn}(t)$. For this purpose, let

$$\begin{aligned}\zeta_{c,d}(s,t) = \; & Cov[I(h(X_1, \ldots, X_c, X_{c+1}, \ldots, X_k; Y_1, \ldots, Y_d, Y_{d+1}, \ldots, Y_l) \leq s), \\ & I(h(X_1, \ldots, X_c, X_{k+1}, \ldots, X_{2k-c}; Y_1, \ldots, Y_d, Y_{l+1}, \ldots, Y_{2l-d}) \leq t)]\end{aligned}$$

for $0 \leq c \leq k$ and $0 \leq d \leq l$. We note that for any $s, t \in (-\infty, \infty), \zeta_{0,0} = 0$ and

$$Cov(B_{mn}(s), B_{mn}(t)) = \frac{N}{\binom{m}{k}\binom{n}{l}} \sum_{c=0}^{k} \sum_{d=0}^{l} \binom{k}{c}\binom{m-k}{k-c}\binom{l}{d}\binom{n-l}{l-d} \zeta_{c,d}(s,t).$$

Form now on, we assume that as $N \to \infty$,

$$m/N \to \lambda \textit{ and } n/N \to 1 - \lambda \tag{2.1}$$

with $0 < \lambda < 1$. Then the following lemma is a well known result from the theory of U-statistics (cf. Randles and Wolfe, 1979).

**Lemma 2.1.** *Under the assumption (2.1), for each $s, t \in (-\infty, \infty)$*

$$\lim_{N \to \infty} Cov(B_{mn}(s), B_{mn}(t)) = k^2 \frac{\zeta_{1,0}(s,t)}{\lambda} + l^2 \frac{\zeta_{0,1}(s,t)}{1-\lambda}.$$

Also we obtain the representation of a U-statistic as an average of averages of *iid* random variables for two sample case (cf. Serfling, 1980) in the following lemma. The U-statistic for $\Delta$ is defined as

$$U_{mn} = \frac{1}{\binom{m}{k}\binom{n}{l}} \sum_{\alpha \in A} \sum_{\beta \in B} h(X_{\alpha_1}, \ldots, X_{\alpha_k}; Y_{\beta_1}, \ldots, Y_{\beta_l}).$$

**Lemma 2.2.** *Let $r = min([m/k], [n/l])$, where $[\cdot]$ is the greatest integer and define*

$$\begin{aligned}W(x_1,\ldots,x_m;y_1,\ldots,y_n) \\ = \frac{1}{r}\{h(x_1,\ldots,x_k;y_1,\ldots,y_l) + h(x_{k+1},\ldots,x_{2k};y_{l+1},\ldots,y_{2l}) \\ + \ldots + h(x_{rk-k+1},\ldots,x_{rk};y_{rl-l+1},\ldots,y_{rl})\}.\end{aligned}$$

Letting $\sum_{m!}\sum_{n!}$ denote summation over all $m!n!$ permutations $(i_1,\ldots,i_m)$ of $(1,\ldots,m)$ and $(j_1,\ldots,j_n)$ of $(1,\ldots,n)$ and $\sum_{c(m,k)}\sum_{c(n,l)}$ denote summation over all $\binom{m}{k}\binom{n}{l}$ combinations $\{i_1,\ldots,i_k\}$ from $\{1,\ldots,m\}$ and $\{j_1,\ldots,j_l\}$ from $\{1,\ldots,n\}$, we have

$$U_{mn} = \frac{1}{m!}\frac{1}{n!}\sum_{m!}\sum_{n!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}).$$

**Proof.** First of all, we note that for any fixed permutation $(j_1,\ldots,j_n)$ of $(1,\ldots,n)$, we have from Serfling (1980),

$$r\sum_{m!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}) = rk!(m-k)!\sum_{c(m,k)} h(x_{i1},\ldots,x_{ik};y_{j1},\ldots,y_{jl}).$$

Also for any fixed permutation $(i_1,\ldots,i_m)$ of $(1,\ldots,m)$, we have

$$r\sum_{n!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}) = rl!(n-l)!\sum_{c(n,l)} h(x_{i1},\ldots,x_{ik};y_{j1},\ldots,y_{jl}).$$

Therefore we have that

$$\begin{aligned}\sum_{m!}\sum_{n!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}) \\ = k!(m-k)!l!(n-l)!\sum_{c(m,k)}\sum_{c(n,l)} h(x_{i1},\ldots,x_{ik};y_{j1},\ldots,y_{jl}).\end{aligned}$$

This implies that

$$\sum_{m!}\sum_{n!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}) = k!(m-k)!l!(n-l)!\binom{m}{k}\binom{n}{l}U_{mn},$$

or

$$U_{mn} = \frac{1}{m!}\frac{1}{n!}\sum_{m!}\sum_{n!} W(x_{i1},\ldots,x_{im};y_{j1},\ldots,y_{jn}).$$

□

We note that $W$ consists of $r$ *iid* random variables. Now we state our main result in the following theorem.

**Theorem 2.1.** *Under the assumption (2.1), $B_{mn}(t)$ converges weakly to a zero-mean normal process $B(t)$ on $D(-\infty,\infty)$, which is the space of functions on $(-\infty,\infty)$ that are right-continuous and have the left-hand limits, with covariance function,*

$$Cov(B(s),B(t)) = k^2\frac{\zeta_{1,0}(s,t)}{\lambda} + l^2\frac{\zeta_{0,1}(s,t)}{1-\lambda}.$$

**Proof.** It is easy to see that for each $t \in (-\infty,\infty)$, $B_{mn}(t)$ converges in distribution to a normal random variable with mean 0 and variance

$$\sigma^2(t) = k^2\zeta_{1,0}(t,t)/\lambda + l^2\zeta_{0,1}(t,t)/(1-\lambda),$$

since $H_{mn}$ is a U-statistic for each $t \in (-\infty,\infty)$ with Lemma 2.1. Thus it only remains to show the tightness to prove the weak convergence of the U-empirical process $B_{mn}(t)$ to a normal process $B(t)$ on $D(-\infty,\infty)$, which is a Brownian bridge. For any given permutations $\alpha$ of $(1,\ldots,m)$ and $\beta$ of $(1,\ldots,n)$, let

$$\begin{aligned} H_{mn}^{\alpha\beta}(t) &= \frac{1}{r}\{I(h(x_{\alpha(1)},\ldots,x_{\alpha(k)};y_{\beta(1)},\ldots,y_{\beta(l)}) \le t) \\ &+ I(h(x_{\alpha(k+1)},\ldots,x_{\alpha(2k)};y_{\beta(l+1)},\ldots,y_{\beta(2l)}) \le t) \\ &+ \ldots + I(h(x_{\alpha(rk-k+1)},\ldots,x_{\alpha(rk)};y_{\beta(rl-l+1)},\ldots,y_{\beta(rl)}) \le t)\}. \end{aligned}$$

Also let

$$B_{mn}^{\alpha\beta}(t) = \sqrt{N}\{H_{mn}^{\alpha\beta}(t) - H(t)\}.$$

Then we note that from Lemma 2.2,

$$B_{mn}(t) = \frac{1}{m!}\frac{1}{n!}\sum_{\alpha}\sum_{\beta} B_{mn}^{\alpha\beta}(t). \tag{2.2}$$

For $0 < y < 1$, define generalized moduli of continuity $\Omega_{mn}$ and $\Omega_{mn}^{\alpha\beta}$ by

$$\Omega_{mn}(y) = sup_{A(y)}|B_{mn}(s) - B_{mn}(t)|$$

and

$$\Omega_{mn}^{\alpha\beta}(y) = sup_{A(y)}|B_{mn}^{\alpha\beta}(s) - B_{mn}^{\alpha\beta}(t)|,$$

where

$$A(y) = \{s,t : |H(s) - H(t)| \le y|\}.$$

Then we note that

$$\Omega_{mn}(y) \leq \frac{1}{m!}\frac{1}{n!}\sum_{\alpha}\sum_{\beta}\Omega_{mn}^{\alpha\beta} \tag{2.3}$$

from Eq. (2.2).

For any $r$, let $D_r$ be the empirical distribution function of $r$ independent random variables uniformly distributed on $[0,1]$. Define

$$V_r(t) = \sqrt{r}(D_r(t) - t)$$

and

$$\omega_r^V(y) = sup_{|s-t|\leq y}|V_r(s) - V_r(t)|,$$

which is the modulus of continuity of $V_r$ over $[0,1]$. The process $B_{mn}^{\alpha\beta}$ is constructed from $r$ independent random variables with distribution function $H$ and therefore the process $B_{mn}^{\alpha\beta} \circ H^{-1}$ and $r^{-1/2}N^{1/2}V_r$ restricted to the set $H(-\infty,\infty)$ have the same distribution. From the definitions of $\Omega_{mn}^{\alpha\beta}$ and $\omega_r^V$, it follows that

$$E\Omega_{mn}^{\alpha\beta}(y) \leq \sqrt{\frac{N}{r}}E\omega_r^V(y),$$

with equality if $H$ is continuous. Thus substituting the inequality (2.3) gives

$$E\Omega_{mn}(y) \leq \sqrt{\frac{N}{r}}E\omega_r^V(y). \tag{2.4}$$

Now Chebyshev's inequality and (2.4) give

$$\begin{aligned}\lim_{y\downarrow 0}\overline{\lim}_{N\to\infty}P\{\Omega_{mn}(y) \geq \varepsilon\} &\leq \lim_{y\downarrow 0}\overline{\lim}_{N\to\infty}\varepsilon^{-1}E(\Omega_{mn}(y)) \\ &\leq \lim_{y\downarrow 0}\overline{\lim}_{N\to\infty}\varepsilon^{-1}\sqrt{\frac{N}{r}}E(\omega_r^V(y)) \\ &= 0 \quad \text{for all } \varepsilon > 0,\end{aligned}$$

from the tightness of the ordinary processes since $V_r$ consists of $r$ *iid* random variables and the fact that $N/r$ converges to a positive real number. Therefore we may conclude that $B_{mn}(t)$ converges weakly to a zero-mean normal process $B(t)$ on $D(-\infty,\infty)$. □

We note that with the special construction of $B_{mn}$ as in section 23.4 of p.771 in Shorack and Wellner (1985), we may have a stronger conclusion such that almost surely

$$||B_{mn} - B||_{-\infty}^{\infty} \to 0.$$

## 3. Applications

Suppose that $\Delta$ is the location translation parameter between $F$ and $G$ such as

$$G(x) = F(x + \Delta) \tag{3.1}$$

for all $x \in (-\infty, \infty)$. Then the kernel for $\Delta$ is of the form

$$h(X_1; Y_1) = X_1 - Y_1$$

of degree $(1, 1)$. Thus the U-empirical distribution is

$$H_{mn}(t) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_i - Y_j \leq t).$$

Therefore the covariance function of the limiting process $B(t)$ follows easily from Lemma 2.1 with the continuity assumption for distribution for $F$ and $G$ as

$$\begin{aligned} Cov(B(s), B(t)) &= \frac{1}{\lambda} Cov[I(X_1 - Y_1 \leq s), I(X_1 - Y_2 \leq t)] \\ &+ \frac{1}{1-\lambda} Cov[I(X_1 - Y_1 \leq s), I(X_2 - Y_1 \leq t)], \end{aligned}$$

where

$$\begin{aligned} &Cov[I(X_1 - Y_1 \leq s), I(X_1 - Y_2 \leq t)] \\ &= \int_{-\infty}^{\infty} (1 - G(x - \min(s,t)))^2 dF(x) \\ &- \int_{-\infty}^{\infty} (1 - G(x - s)) dF(x) \int_{-\infty}^{\infty} (1 - G(x - t)) dF(x) \end{aligned}$$

and

$$\begin{aligned} &Cov[I(X_1 - Y_1 \leq s), I(X_2 - Y_1 \leq t)] \\ &= \int_{-\infty}^{\infty} F(x - \min(s,t))^2 dG(x) - \int_{-\infty}^{\infty} F(x - s) dG(x) \int_{-\infty}^{\infty} F(x - t) dG(x). \end{aligned}$$

When $\Delta = 0$, since $F = G$, the covariance function can be simplified as

$$\begin{aligned} Cov[B(s), B(t)] &= \frac{1}{\lambda} \{ \int_{-\infty}^{\infty} (1 - F(x - \min(s,t))^2 dF(x) \\ &- \int_{-\infty}^{\infty} (1 - F(x - s)) dF(x) \int_{-\infty}^{\infty} (1 - F(x - t)) dF(x) \} \\ &+ \frac{1}{1-\lambda} \{ \int_{-\infty}^{\infty} F(x - \min(s,t))^2 dF(x) \\ &- \int_{-\infty}^{\infty} F(x - s) dF(x) \int_{-\infty}^{\infty} F(x - t)) dF(x) \}. \end{aligned}$$

Let $0 < p < 1$. Then for any given $p$, we define quantile functions $H^{-1}$ and $H_{mn}^{-1}$ as

$$H^{-1}(p) = \inf\{t : H(t) \geq p\} \text{ and } H_{mn}^{-1}(p) = \inf\{t : H_{mn}(t) \geq p\}.$$

Since $\Delta$ is a median of $H$, we may take $\hat{\Delta}_{mn} = H_{mn}(1/2)$ as an estimate of $\Delta$. We note that $\hat{\Delta}_{mn}$ is also a Hodges-Lehmann estimate and can be considered as a generalized $L$-estimate in the sense of Serfling (1984). In order to derive the asymptotic normality of $\sqrt{N}(\hat{\Delta}_{mn} - \Delta)$, it would be convenient to consider the following Bahadur representations for the quantiles of the U-empirical distribution for two sample case.

**Theorem 3.1.** *Suppose that there is a real number $\xi_p$ such that $H(\xi_p) = p$, where $H(\cdot)$ is the distribution function of the kernel $h(X_1, \ldots, X_k; Y_1, \ldots, Y_l)$. Also suppose that $H$ is twice differentiable in a neighborhood of $\xi_p$ and $H(\xi_p) > 0$. Then with probability one,*

$$\hat{\xi}_p - \xi_p = \frac{H(\xi_p) - H_{mn}(\xi_p)}{H'(\xi_p)} + O(N^{-3/4}(logN)^{3/4}),$$

*where $\hat{\xi}_p = H_{mn}^{-1}(p)$.*

The proof of Theorem 3.2 will be shown shortly with the following four lemmas (cf. See the proof of Theorem 3.1 of Choudhury and Serfling, 1988).

**Lemma 3.1.** *Let $h(X_1, \ldots, X_k; Y_1, \ldots, Y_l)$ satisfy*

$$\Psi_h(s) = E[exp\{sh(X_1, \ldots, X_k; Y_1, \ldots, Y_l)\}] < \infty, 0 < s \leq s_0.$$

*Then*

$$E[exp\{sU_{mn}\}] \leq \Psi_h^r(\frac{s}{r}), 0 < s \leq s_0 r,$$

*where $r = \min([m/k], [n, l])$, which was defined in Section 2.*

**Lemma 3.2.** *Let $h(x_1, \ldots, x_k; y_1, \ldots, y_l)$ be a kernel for $\Delta$ with $a \leq h(x_1, \ldots, x_k ; y_1, \ldots, y_l) \leq b$. Then for any $t > 0$, we have that*

$$P\{U_{mn} - \Delta \geq t\} \leq e^{-2rt^2/(b-a)^2}.$$

The proofs of Lemma 3.1 and 3.2 follow exactly as those of Theorems A and B in p. 201 of Serfling (1980) by noting that $W(\cdot)$ in Lemma 2.2 is an average of $r$ *iid* random variables.

**Lemma 3.3.** *Suppose that $H$ is differentiable at $\xi_p$ with $H(\xi_p) > 0$. Then with probability one,*

$$|H_{mn}^{-1}(p) - \xi_p| = O(N^{-1/2}(logN)^{1/2})$$

*for all sufficiently large $N$.*

**Proof.** By choosing a sequence of positive constants $\varepsilon_N$ such as

$$\varepsilon_N = \frac{(logN)^{1/2}}{H'(\xi_p)N^{1/2}}$$

in Lemma 3.1 of Choudhury and Serfling (1988) and noting that $H_{mn}$ consists of the indicator functions, we can obtain the result with the application of Lemma 3.2. □

**Lemma 3.4.** *Suppose that $H'$ is bounded in a neighborhood of $\xi_p$ with $H'(\xi_p) > 0$. Let $(a_n)$ be a sequence of positive constants such that*

$$a_n \sim c_0 N^{-1/2}(logN)^{1/2} \quad as\ N \to \infty,$$

*for some constant $c_0 > 0$. Then with probability one, we have as $N \to \infty$*

$$\sup_{|t| \le a_n} |[H_N(\xi_p + t) - H_N(\xi_p)] - [H(\xi_p + t) - H(\xi_p)]| = O(N^{-3/4}(logN)^{3/4}).$$

**Proof.** We may prove this on the lines of Lemma 3.2 in Choudhury and Serfling (1988). □

Proof of Theorem 3.1. From Lemma 3.3 and 3.4, we have with probability one, as $N \to \infty$,

$$|[H_N(\hat{\xi}_p) - H_N(\xi_p)] - [H(\hat{\xi}_p) - H(\xi_p)]| = O(N^{-3/4}(logN)^{3/4}).$$

Then by the Young's form of Taylor's expansion (Serfling, 1980), we have with probability one,

$$[H_N(\hat{\xi}_p) - H_N(\xi_p)] - [(\hat{\xi}_p - \xi_p)H'(\xi_p) + (\hat{\xi}_p - \xi_p)^2 H''(\xi_p)/2! + o(N^{-1}logN)] \\ = O(N^{-3/4}(logN)^{3/4})$$

and so we have that with probability one,

$$[H_N(\hat{\xi}_p) - H_N(\xi_p)] - [(\hat{\xi}_p - \xi_p)H'(\xi_p) + (\hat{\xi}_p - \xi_p)^2 H''(\xi_p)/2!] = O(N^{-3/4}(logN)^{3/4}).$$

Also we note that from Lemma 3.3, with probability one,

$$(\hat{\xi}_p - \xi_p)^2 H''(\xi_p)/2! = O(N^{-1} log N).$$

Thus we have with probability one that

$$[H_N(\hat{\xi}_p) - H_N(\xi_p)] - (\hat{\xi}_p - \xi_p)H'(\xi_p) = O(N^{-3/4}(log N)^{3/4}).$$

Since $H(\xi_p) = p$ and $H_N(\hat{\xi}_p) = p + O(N^{-1})$, we have the result.

**Remark.** In the conclusion of Theorem 3.1 of Choudhury and Serfling (1988), $O(\max\{\varepsilon_n^2, \varepsilon_n^{1/2} n^{-1/2}\})$ should be replaced by $O(\max\{\varepsilon_n^2, \varepsilon_n^{1/2} n^{-1/2} (\log n)^{1/2}\})$.

Then from Theorems 2.1 and 3.1, one may easily show that

$$\sqrt{N}(\hat{\Delta}_{mn} - \Delta) = \sqrt{N}(H_{mn}^{-1}(1/2) - \Delta) \xrightarrow{d} Q \sim N(0, \sigma^2),$$

where $\xrightarrow{d}$ means the convergence in distribution and

$$\sigma^2 = Cov(B(0), B(0))\{[\int_{-\infty}^{\infty} f^2(x)dx]^2\}^{-1} = \frac{1}{12}\{\frac{1}{\lambda} + \frac{1}{1-\lambda}\}\{[\int_{-\infty}^{\infty} f^2(x)dx]^2\}^{-1}.$$

For testing $H_0 : \Delta = 0$, we note that the Wilcoxon rank sum statistic $W$ can be written as

$$W = \int_{-\infty}^{\infty} I(0 \le t < \infty) dH_{mn}(t), \tag{3.2}$$

which is the Mann-Whitney form. Therefore the limiting distribution of $W$ can be obtained by noting that

$$\begin{aligned} \sqrt{N}\int_{-\infty}^{\infty} I(0 \le t) d(H_{mn}(t) - H(t)) &= \int_{-\infty}^{\infty} I(0 \le t) dB_{mn}(t) \\ &\xrightarrow{d} \int_{-\infty}^{\infty} I(0 \le t) dB(t). \end{aligned}$$

In order to obtain the variance, first of all, we note that

$$\int_{-\infty}^{\infty} I(0 \le t) dB(t) = B(\infty) - B(0).$$

Since the normal process $B$ has the independent increments, we may obtain the variance as

$$Var(W) = Var(B(\infty) - B(0)) = Var(B(\infty)) + Var(B(0)) = Var(B(0)).$$

Therefore we have that

$$Var(W) = Var(B(0)) = \frac{1}{12}\{\frac{1}{\lambda} + \frac{1}{1-\lambda}\}.$$

We note that under the model (3.1), $\Delta$ is also the location translation parameter between the distributions of $(X_1 + \ldots + X_k)/k$ and $(Y_1 + \ldots + Y_k)/k$ for each $k \geq 1$. Therefore we may consider

$$h(X_1, \ldots, X_k; Y_1, \ldots, Y_k) = \frac{X_1 + \ldots + X_k}{k} - \frac{Y_1 + \ldots + Y_k}{k} \tag{3.3}$$

as a generalized kernel for $\Delta$ of degree $(k,k)$. Hollander (1967) considered this generalized kernel for testing $H_0 : \Delta = 0$ with $k = 2$. Based on the generalized kernel, one may obtain the the generalized Hodges-Lehmann estimate for $\Delta$ such as

$$\hat{\Delta} = med\{\frac{X_{i1} + \ldots + X_{ik}}{k} - \frac{Y_{j1} + \ldots + Y_{jk}}{k}\}.$$

Then by calculating $\zeta_{1,0}(s,t)$ and $\zeta_{0,1}(s,t)$, and applying Theorem 3.2 with Bahadur representation, we may derive the asymptotic normality for the generalized Hodges-Lehmann estimate $\hat{\Delta}$. For tesing $H_0 : \Delta = 0$, we may use the following statistic

$$W_k = \int_{-\infty}^{\infty} I(0 \leq t < \infty) dH_{mn}(t),$$

where $H_{mn}$ is the U-empirical distribution based on the kernel (3.3). Then the asymptotic normality for $W_k$ follows easily as for $W$.

As another example, we consider a measure of the difference of scale parameters proposed by Lehmann (1951) such as

$$\Delta = P\{|Y_1 - Y_2| > |X_1 - X_2|\}.$$

Then the corresponding kernel would be of the form

$$h(X_1, X_2; Y_1, Y_2) = |Y_1 - Y_2| - |X_1 - X_2| \tag{3.4}$$

of degree (2,2). Then the corresponding U-empirical distribution becomes

$$H_{mn}(t) = \frac{1}{\binom{m}{2}\binom{n}{2}} \sum\sum_{1 \leq i < j \leq m} \sum\sum_{1 \leq h < k \leq n} I(|Y_h - Y_k| - |X_i - X_j| \leq t).$$

We note that the test statistic for testing $H_0 : \Delta = 1/2$ is also

$$T = \int_{-\infty}^{\infty} I(0 \leq t < \infty) dH_{mn}(t),$$

where $H_{mn}$ is the U-empirical distribution based on the kernel (3.4). Therefore the asymptotic normality for $T$ follows easily with the the same arguments for $W$. Also for the estimation for $\Delta$, $H_{mn}^{-1}(1/2)$ is an estimate for $\Delta$ and also a generalized Hodges-Lehmann estimate. The asymptotic normality becomes obvious.

## Acknowledgement

The authors wish to express their sincere appreciations to the referees for pointing out errors.

## References

Arcones, M. A. and Gine, E. (1993). "Limit theorems for U-processes", *The Annals of Probability*, **21**, 1494-1542.

Arcones, M. A. (1996). "The Bahadur-Kiefer representation for U-quantiles", *The Annals of Statistics*, **24**, 1400-1422.

Choudhury, J. and Serfling, R. J. (1988). "Generalized order statistics, Bahadur representations and sequential nonparametric fixed-width confidence intervals", *Journal of Statistical Planning and Inference*, **19**, 269-282.

Hollander, M. (1967). "Asymptotic efficiency of two nonparametric competitors of Wilcoxon's two test", *Journal of the American Statistical Association*, **62**, 939-949.

Lehmann, E. L. (1951). "Consistency and unbiasedness of certain nonparametric tests", *The Annals of Mathematical Statistics*, **22**, 165-179.

Randles, R. H. and Wolfe, D. A. (1979). *Introduction to the Theory of Nonparametric Statistics*, John Wiley and Sons, Inc.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley and Sons, Inc.

Serfling, R. J. (1984). "Generalized L-, M- and R-statistics", *The Annals of Statistics*, **12**, 176-86.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*, John Wiley and Sons, Inc.

Silverman, B. W. (1983). "Convergence of a class of empirical distribution functions of dependent random variables", *The Annals of Probability*, 11, 745-751.