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Optimal Convergence Rate of Empirical Bayes Tests for Uniform Distributions

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ABSTRACT

The empirical Bayes linear loss two-action problem is studied. An empirical Bayes test δ_n^* is proposed. It is shown that δ_n^* is asymptotically optimal in the sense that its regret converges to zero at a rate n^{-1} over a class of priors and the rate n^{-1} is the optimal rate of convergence of empirical Bayes tests.

Keywords: Asymptotically optimal, empirical Bayes, minimum regret, optimal rate of convergence.

1. Introduction

The empirical Bayes approach in statistical decision theory is appropriate when one is confronted repeatedly and independently with the same decision problem. In such instances, it is reasonable to formulate the component problem in the sequence as a Bayes decision problem with respect to an unknown prior distribution on the parameter space, and then use the accumulated past data to improve the decision procedure at each stage. Since Robbins (1956), empirical Bayes procedures have been extensively studied. Many such empirical Bayes procedures have been shown to be asymptotically optimal in the sense that the risk for the $(n+1)$-st decision problem converges to the optimal Bayes risk which would have been obtained if the prior distribution was fully known and the Bayes procedure with respect to this prior distribution was used.

The performance of empirical Bayes procedures in practical applications clearly depends on the convergence rates with which the risks of successive decision problems approach the optimal Bayes risk. Singh (1979) and Singh and Wei (1992) conjectured that the rate n^{-1} is the best possible rate for empirical Bayes procedures dealing with Lebesgue densities. Recently, Karunamuni (1996, 1999)

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raised the following questions: In terms of convergence, what are the best empirical Bayes procedures? What are the optimal rate of convergence? In order to answer these questions, one approach is to find a sharper lower bound for the minimax regret of the empirical Bayes procedures. The importance of such minimax lower bound is that they address the nature of the inherent difficulty of the empirical Bayes decision problem and explain how well can the empirical Bayes decision problem be solved by any empirical Bayes procedure. Karunamuni (1996, 1999) attempted to derive asymptotic minimax lower bounds for the regrets of empirical Bayes tests for normal, exponential and uniform distributions. He claimed having established the optimal rate of convergence for each of the three distributions. However, more studies are needed for finding the optimal rates of convergence.

In this paper, our purpose is to establish the optimal rate of convergence of empirical Bayes tests for uniform distributions. The paper is organized in the following way. The concerned decision problem is introduced in Section 2. A Bayes test is derived. We construct an empirical Bayes test δ_n^* in Section 3. It is shown that the regret of δ_n^* converges to zero at the rate n^{-1} over a class $\mathcal{G}$ of priors. We also establish a lower bound with rate n^{-1} for the minimax regret of empirical Bayes tests in Section 4. Therefore, we conclude that the n^{-1} is the optimal rate of convergence of empirical Bayes tests over the class $\mathcal{G}$.

2. The Decision Problem and a Bayes Test

Let (X, Θ) be a random vector, where Θ is a positive random variable, following an unknown prior distribution G and X, given $\Theta = \theta$, has a uniform distribution with *pdf* $f(x|\theta) = \theta^{-1} I_{[0,\theta]}(x)$. Thus, X has a marginal *pdf* $f_G(x) = \int_x^\infty \theta^{-1} dG(\theta)$. Let θ_0 be a known positive value. Consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ against $H_1 : \theta \geq \theta_0$ based on an observation of X. Let $i, i = 0, 1$, denote the action deciding in favor of H_i. For the parameter θ, the loss of taking action i is

$$\ell(\theta, i) = i(\theta_0 - \theta) I(\theta_0 - \theta) + (1 - i)(\theta - \theta_0) I(\theta - \theta_0), \tag{2.1}$$

where $I(x) = 1(0)$ if $x > 0 (x \leq 0)$.

Let χ be the sample space of the random variable X. A test δ is defined to be a mapping from χ into the interval $[0, 1]$ so that $\delta(x) = P\{\text{accepting } H_1 | X = x\}$, the probability of accepting H_1 given $X = x$ being observed. Let $R(G, \delta)$ denote the Bayes risk of a test δ when G is the true prior distribution. Suppose that

$\int_0^\infty \theta dG(\theta) < \infty$. Then, by Fubini's theorem, a straightforward computation leads to

$$R(G,\delta) = \int_0^\infty \delta(x) H_G(x)dx + C_G, \tag{2.2}$$

where

$$C_G = \int (\theta - \theta_0) I(\theta - \theta_0) dG(\theta), H_G(x) = (\theta_0 - x) f_G(x) + F_G(x) - 1 \tag{2.3}$$

and F_G is the marginal cumulative distribution of X. Thus, a Bayes test δ_G, which minimizes the Bayes risk among all tests is clearly given by : For each $x > 0$,

$$\delta_G(x) = \begin{cases} 1 & \text{if} \quad H_G \le 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The minimum Bayes risk of the decision problem is

$$R(G,\delta_G) = \int_0^\infty \delta_G(x) H_G(x)dx + C_G. \tag{2.5}$$

Note that as $x \ge \theta_0$, $H_G(x) \le 0$. Thus, $\delta_G(x) = 1$ for $x \ge \theta_0$. Also, $H_G^{(1)}(x) = (\theta_0 - x) f_G^{(1)}(x) \le 0$. That is, $H_G(x)$ is nonincreasing in x for x in $(0, \theta_0)$. Let $A = \{0 < x < \theta_0 | H_G(x) > 0\}$. Define $a_G = \sup A$ if $A \ne \phi$, and 0 otherwise. Note that $H_G(x) > 0$ for $x < a_G$ and $H_G(x) \le 0$ for $x \ge a_G$. Therefore, the Bayes test δ_G can be expressed as :

$$\begin{aligned} \delta_G(x) &= \begin{cases} 1 & \text{if } x \ge a_G, \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} 1 & \text{if } (x \ge \theta_0) \text{ or } (0 < x < \theta_0 \text{ and } H_G(x) \le 0), \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.6}$$

3. Empirical Bayes Testing

3.1. Empirical Bayes framework

Since the prior distribution G is not known, it is impossible to implement the Bayes test δ_G for the underlying testing problem. When a sequence of past data is available, this testing problem has been studied via the empirical Bayes approach by Gupta and Hsiao (1983), Van Houwelingen (1987), Liang (1990) and Karunamuni (1999), respectively. In the empirical Bayes framework, we let (X_i, Θ_i), $i = 1, 2, \ldots$ be *iid* copies of (X, Θ), where X_i, $i = 1, 2\ldots$ are observable, but Θ_i, $i = 1, 2...$ are not observable. At the current stage $n+1$, let θ_{n+1} be a realization of the random variable Θ_{n+1}. We have to make a decision for testing $H_0^{n+1} : \theta_{n+1} \leq \theta_0$ against $H_1^{n+1} : \theta_{n+1} > \theta_0$ with the loss $l(\theta_{n+1}, i)$ of (2.1) based on the present observation $X_{n+1} = x$ and the past data $\underset{\sim}{X}(n) = (X_1, ..., X_n)$.

An empirical Bayes test δ_n is a test on $X_{n+1} = x$ and $\underset{\sim}{X}(n)$ such that $\delta_n(x, \underset{\sim}{X}(n)) = \delta_n(x)$ is the probability of accepting H_1^{n+1}. Given $\underset{\sim}{X}(n)$, the conditional Bayes risk of δ_n is $R(G, \delta_n | \underset{\sim}{X}(n)) = \int \delta_n(x) H_G(x) dx + C_G$, and the (unconditional) Bayes risk of δ_n is $R(G, \delta_n) = \int E_n[\delta_n(x)] H_G(x) dx + C_G$, where the expectation E_n is taken with respect to the probability measure generated by $\underset{\sim}{X}(n)$.

Since $R(G, \delta_G)$ is the minimum Bayes risk, $R(G, \delta_n | \underset{\sim}{X}(n)) - R(G, \delta_G) \geq 0$ for all $\underset{\sim}{X}(n)$ and for all n. Thus, $R(G, \delta_n) - R(G, \delta_G) \geq 0$ for all n. This nonnegative regret $R(G, \delta_n) - R(G, \delta_G)$ is used as a measure of performance of the empirical Bayes test δ_n. An empirical Bayes test δ_n is said to be asymptotically optimal, relative to the prior distribution G, at a rate of convergence ε_n if $R(G, \delta_n) - R(G, \delta_G) = O(\varepsilon_n)$ where $\{\varepsilon_n\}$ is a sequence of decreasing, positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$.

3.2. Construction of an empirical Bayes test

We consider a kernel function K satisfying the following K-conditions: **[K1]** Support of $K \subset [0, 1]$. **[K2]** $|K(x)| \leq k_0$ for all x. **[K3]** $\int_0^1 K(x) dx = 1$. **[K4]** $\int_0^1 x^l K(x) dx = 0, l = 1, 2, ..., r$.

Motivated by the form (2.6), for constructing empirical Bayes tests, we need to have estimates for the function $H_G(x) = (\theta_0 - x) f_G(x) + F_G(x) - 1$. For each n and $x > 0$, define

$$f_n(x) = \frac{1}{nb} \sum_{j=1}^{n} K\left(\frac{X_j - x}{b}\right), \quad F_n(x) = \frac{1}{n} \sum_{j=1}^{n} I(x - X_j),$$
$$H_n(x) = (\theta_0 - x) f_n(x) + F_n(x) - 1, \tag{3.1}$$

where b is a positive value. By mimicking the form (2.6) of the Bayes test δ_G, we propose an empirical Bayes test δ_n^* as follows :

$$\delta_n^*(x) = 1 \quad \text{if} \quad ((x \geq \theta_0) \text{ or } (0 < x < \theta_0 \text{ and } H_n(x) < 0)); \text{ and 0 otherwise.} \tag{3.2}$$

The Bayes risk of the empirical Bayes test δ_n^* is

$$R(G,\delta_n^*) = \int E_n[\delta_n^*(x)]H_G(x)dx + C_G. \tag{3.3}$$

3.3. Asymptotic optimality of δ_n^*

Let $\mathcal{G}$ be the class of priors G for which the following conditions hold : [G1] $\int_0^\infty \theta dG(\theta) < \infty$, [G2] $0 < a_G < \theta_0$, [G3] $f_G(x)$ is a polynomial of degree s for x in $(0, \theta_0 + m)$ for some $m > 0$ where $1 \le s \le r$, and [G4] $f_G^{(1)}(x) \le -B_0 < 0$ in a neighborhood $(a_G - c, a_G + c)$. Without loss of generality, it is assumed that $c > 0$ is small enough so that $a_G - c > 0$ and $a_G + c < \theta_0$.

From (2.5)-(2.6) and (3.2)-(3.3), the regret of the empirical Bayes test δ_n^* can be written as :

$$\begin{aligned}
&R(G,\delta_n^*) - R(G,\delta_G) \\
&= \int_0^{a_G} P_n\{H_n(x) < 0\} H_G(x)dx + \int_{a_G}^\infty P_n\{H_n(x) \ge 0\}[-H_G]dx \qquad (3.4)\\
&= \int_0^{a_G - c} P_n\{H_n(x) < 0\} H_G(x)dx + \int_{a_G-c}^{a_G} P_n\{H_n(x) < 0\} H_G(x)dx \\
&\quad + \int_{a_G}^{a_G+c} P_n\{H_n(x) \ge 0\}[-H_G(x)]dx + \int_{a_G+c}^{\theta_0} P_n\{H_n(x) \ge 0\}[-H_G(x)]dx \\
&= I_n + II_n + III_n + IV_n.
\end{aligned}$$

Let $\beta_G = \min\left\{\left|H_G^{(1)}(x)\right| |a_G - x| \le c\right\}$. Thus, for each G in $\mathcal{G}$, $\beta_G \ge [\theta_0 - (a_G + c)]B_0 > 0$. Let b be a fixed positive value such that $0 < b < m$. Let

$$\begin{aligned}
c_1 &= 2^{-1}\left[2\theta_0^2 b^{-1} k_0^2 f_G(a_G - c) + \frac{1}{2} + \frac{2}{3}(\theta_0 b^{-1} k_0 + 1)H_G(a_G - c)\right]^{-1}, \\
c_2 &= 2^{-1}\left[2\theta_0^2 b^{-1} k_0^2 f_G(a_G) + \frac{1}{2} + \frac{2}{3}(\theta_0 b^{-1} k_0 + 1)\left|H_G(a_G + c)\right|\right]^{-1}.
\end{aligned}$$

Lemma 3.1. *For each prior distribution G in $\mathcal{G}$, the following inequalities hold:*

(a) $I_n \le \dfrac{2\theta_0^2 k_0^2}{nbH_G(a_G - c)} + \dfrac{a_G}{anH_G(a_G - c)}$ *(b)* $II_n \le (2n\beta_G c_1)^{-1}$

(c) $III_n \le (2n\beta_G c_2)^{-1}$ *(d)* $IV_n \le \dfrac{2\theta_0^2 k_0^2}{nb|H_G(a_G + c)|} + \dfrac{\theta_0}{2n|H_G(a_G + c)|}$.

The proof of Lemma 3.1 is provided in the Appendix. The following theorem is the main result of the paper, which is a direct consequence of (3.4) and Lemma 3.1.

Theorem 3.1. *Let δ_n^* be the empirical Bayes test constructed through (3.1)-(3.2). Then for each prior distribution G in $\mathcal{G}$, the empirical Bayes test δ_n^* is asymptotically optimal in the sense that*

$$\begin{aligned}
&R(G,\delta_n^*) - R(G,\delta_G)\\
&\leq \frac{2\theta_0^2k_0^2}{nb}\left(\frac{1}{H_G(a_G-c)}+\frac{1}{|H_G(a_G+c)|}\right)+\frac{1}{2n}\left(\frac{a_G}{H_G(a_G-c)}+\frac{\theta_0}{|H_G(a_G+c)|}\right)\\
&\quad+\frac{1}{2n\beta_G}\left(\frac{1}{c_1}+\frac{1}{c_2}\right)\\
&= O(n^{-1}).
\end{aligned}$$

4. A Lower Bound of Minimax Regret

Consider two prior distributions G_1 and G_2 in $\mathcal{G}$ such that $0 < a_{G_1} < a_{G_2} < \theta_0$. Their corresponding marginal densities are f_{G_1} and f_{G_2}, respectively. The Hellinger distance between f_{G_1} and f_{G_2} is defined as $\mathcal{H}(f_{G_1}, f_{G_2}) = \{\int[f_{G_1}^{1/2}(x) - f_{G_2}^{1/2}(x)]^2dx\}^{1/2}$. It is known that $\mathcal{H}$ is bounded above by $\sqrt{2}$. A convenient computational formula is $\mathcal{H}^2(f_{G_1}, f_{G_2}) = 2 - 2\int[f_{G_1}(x)f_{G_2}(x)]^{1/2}dx$. Interested readers are referred to Chen (1997) and the references cited there for more recent developments relating to the Hellinger distance. The following lemma can be obtained by following a discussion analogous to Theorem 2.1 of Liang (2000).

Lemma 4.1. *Let G_1 and G_2 be two prior distributions in $\mathcal{G}$ such that $0 < a_{G_1} < a_{G_2} < \theta_0$ and $\mathcal{H}(f_{G_1}, f_{G_2}) \leq \sqrt{\frac{2}{n}}$. Let C be the class of all empirical Bayes tests δ_n. Then,*

$$\inf_{\delta_n \epsilon C}\sup_{G \epsilon \mathcal{G}}[R(G,\delta_n) - R(G,\delta_G)] \geq d(a_{G_2} - a_{G_1})^2,$$

for some positive value d.

Consider a prior distribution G with density $g(\theta) = \frac{b}{M^3}\theta(M-\theta), 0 \leq \theta \leq M$. Then, $f_G(x) = 3(M-x)^2M^{-3}, F_G(x) = 1-(M-x)^3M^{-3}$ for $0 < x \leq M$. For $M/3 < \theta_0 < M, a_G = (3\theta_0 - M)/2$. Now we choose prior G_1 and G_2 as follows: G_1 is with $M = 1$, G_2 is with $M = 1 + n^{-1/2}$. Also, let $\theta_0 = \frac{1}{2}$. Then, $a_{G_1} =$

$(3\theta_0 - 1)/2, a_{G_2} = [3\theta_0 - (1 + n^{-1/2})]/2, a_{G_1} - a_{G_2} = 2^{-1}n^{-1/2}$. $f_{G_1}(x) = 3(1 - x)^2, 0 \le x \le 1$; $f_{G_2}(x) = 3(1 + n^{-1/2} - x)/(1 + n^{-1/2})^3,\ 0 \le x \le 1 + n^{-1/2}$. $\int_0^1 [f_{G_1}(x) f_{G_2}(x)]^{1/2} dx = (1 + \frac{3}{2}n^{-1/2})/(1 + n^{-1/2})^{3/2}$. Thus,

$$\mathcal{H}^2(f_{G_1}, f_{G_2}) = \frac{2(1 + n^{-1/2})^{3/2} - (2 + 3n^{-1/2})}{(1 + n^{-1/2})^{3/2}} = \frac{0.75n^{-1} + O(n^{-3/2})}{(1 + n^{-1/2})^{3/2}} \le 2n^{-1}.$$

Therefore, $\mathcal{H}(f_{G_1}, f_{G_2}) \le \sqrt{\frac{2}{n}}$. Note that G_1 and G_2 are in $\mathcal{G}$. Then, by Lemma 4.1, we conclude the following theorem.

Theorem 4.1. $\inf_{\delta_n \epsilon C} \sup_{G \epsilon \mathcal{G}} [R(G, \delta_n) - R(G, \delta_G)] \ge d^* n^{-1}$ *for some positive constant* d^*.

5. Concluding Remarks

The performance of an empirical Bayes procedure is often evaluated by its associated rate of convergence. For discrete exponential families, Liang (1988, 1999) has established the exponential type convergence rate $\exp(-nc), c > 0$, for the two-action problem. Singh (1979) and Singh and Wei (1992) conjectured that the rate n^{-1} is the best possible rate of convergence for empirical Bayes procedures dealing with Lebesgue densities. Recently, Karunamuni (1996, 1999) attempted to find optimal rate of convergence for empirical Bayes tests dealing with normal, exponential and uniform distributions. However, more further studies are needed for the claimed optimal rates of convergence.

In this paper, we study empirical Bayes testing for uniform distributions and investigate the performance of an empirical Bayes test δ_n^* over a class $\mathcal{G}$ of prior distributions. Theorem 3.1 states that for any prior G in $\mathcal{G}$, δ_n^* is asymptotically optimal, and its regret converges to zero at a rate n^{-1}. The lower bound of minimax regret of empirical Bayes tests given in Theorem 4.1 explains that no empirical Bayes test has regret that converges to zero faster than the rate n^{-1} over the class $\mathcal{G}$. Combining the two results, we may conclude that the rate n^{-1} is the optimal rate of convergence over the class $\mathcal{G}$ of priors and δ_n^* is an optimal empirical Bayes test in the sense that it achieves the optimal rate of convergence over the class $\mathcal{G}$.

6. Appendix

In the following, the analysis is made based on the assumption that G is in $\mathcal{G}$. From (3.1),

$$H_n(x) = \frac{1}{n}\sum_{j=1}^{n} V(X_j, x, b), \tag{A.1}$$

where $V(X_j, x, b) = (\theta_0 - x)b^{-1}K\left(\frac{X_j - x}{b}\right) + I(x - X_j) - 1, j = 1, ..., n$, are *iid* random variables, with

$$|V(X_j, x, b)| \le |\theta_0 - x|\, b^{-1}\left|K\left(\frac{X_j - x}{b}\right)\right| + 1 - I(x - X_j) \le |\theta_0 - x|\, b^{-1}k_0 + 1.$$

Thus,

$$|V(X_j, x, b) - E_n V(X_j, x, b)| \le 2(|\theta_0 - x|\, b^{-1}k_0 + 1). \tag{A.2}$$

By Taylor series expansion, for $0 < x < \theta_0$, we have

$$\begin{aligned} & E_n b^{-1}K\left(\frac{X_j - x}{b}\right) \\ &= \int_0^1 K(t) f_G(x + tb)dt \\ &= \int_0^1 K(t)\left[f_G(x) + \sum_{l=0}^{r} f_G^{(l)}(x)\frac{(tb)^l}{l!} + f_G^{(r+1)}(x^*)\frac{(tb)^{r+1}}{(r+1)!}\right]dt = f_G(x). \end{aligned} \tag{A.3}$$

In (A.3), the last equality is obtained by [K3]-[K4] and by noting that $f_G^{(r+1)}(x^*) = 0$ since $f_G(x)$ is a polynomial of degree s with $s \le r$ for x in $(0, \theta_0 + m)$. According to (A.3), we have

$$E_n V(X_j, x, b) = H_G(x) \text{ and } E_n H_n(x) = H_G(x). \tag{A.4}$$

Also, for $x < x < \theta_0$,

$$\begin{aligned} Var(V(X_j, x, b)) &\le 2\theta_0^2 b^{-2} Var\left(K\left(\frac{X_j - x}{b}\right)\right) + 2Var(I(x - X_j)) \\ &\le 2\theta_0^2 b^{-2} E_n\left[K^2\left(\frac{X_j - x}{b}\right)\right] + 2F_G(x)[1 - F_G(x)] \\ &\le 2\theta_0^2 b^{-1} k_0^2 f_G(x) + 2F_G(x)[1 - F_G(x)], \end{aligned} \tag{A.5}$$

and therefore, from (A.1),

$$Var(H_n(x)) \le 2\theta_0^2 b^{-1} k_0^2 f_G(x) n^{-1} + 2F_G(x)[1 - F_G(x)]n^{-1}. \tag{A.6}$$

Proof of Lemma 3.1(a)

For x in $(0, a_G - c)$, $H_G(x) > 0$. By Markov inequality and from (A.4)-(A.6), we have

$$\begin{aligned} &P_n\{H_n(x) < 0\} \\ &= P_n\{H_n(x) - H_G(x) < -H_G(x)\} \le Var(H_n(x))/H_G^2(x) \qquad \text{(A.7)} \\ &\le 2\theta_0^2 b^{-1} k_0^2 f_G(x) n^{-1} + 2F_G(x)[1 - F_G(x)]n^{-1}. \end{aligned}$$

Substituting (A.7) into I_n, and noting that $H_G(x) \ge H_G(a_G - c) > 0$ for $0 < x < a_G - c$ since $H_G(x)$ is decreasing in x for x in $(0, \theta_0)$, it follows that

$$\begin{aligned} I_n &\le \int_0^{a_G - c} \frac{2\theta_0^2 b^{-1} k_0^2 f_G(x) n^{-1}}{H_G(x)} dx + \int_0^{a_G - c} \frac{2F_G(x)[1 - F_G(x)]n^{-1}}{H_G(x)} dx \\ &\le \frac{2\theta_0^2 k_0^2}{nbH_G(a_G - c)} + \frac{a_G}{2nH_G(a_G - c)}. \qquad \square \end{aligned}$$

Proof of Lemma 3.1(b)

For x in $(a_G - c, a_G)$, by (A.2), (A.5) and Bernstein inequality,

$$\begin{aligned} &P_n\{H_n(x) < 0\} \\ &= P_n\{H_n(x) - H_G(x) < -H_G(x)\} \\ &\le \exp\left\{-\frac{n}{2}\frac{H_G^2(x)}{Var(V(X_1, x, b)) + 2(\theta_0 k_0 b^{-1} + 1)H_G(x)/3}\right\} \\ &\le \exp\left\{-\frac{n}{2}\frac{H_G^2(x)}{2\theta_0^2 k_0^2 b^{-1} f_G(x) + 2F_G(x)[1 - F_G(x)] + 2(\theta_0 k_0 b^{-1} + 1)H_G(x)/3}\right\} \\ &\le \exp\left\{-\frac{n}{2}\frac{H_G^2(x)}{2\theta_0^2 k_0^2 b^{-1} f_G(a_G - c) + \frac{1}{2} + 2(\theta_0 k_0 b^{-1} + 1)H_G(a_G - c)/3}\right\} \\ &= \exp\left\{-nc_1 H_G^2(x)\right\}, \qquad \text{(A.8)} \end{aligned}$$

where c_1 is given in Subsection 3.3.

Replacing (A.8) into II_n, we obtain

$$\begin{aligned} II_n &\le \int_{a_G - c}^{a_G} \exp(-nc_1 H_G^2(x)) H_G(x) dx \\ &\le \frac{1}{\beta_G} \int_{a_G - c}^{a_G} \exp(-nc_1 H_G^2(x)) H_G(x) \left[-H_G^{(1)}(x)\right] dx \\ &\le \frac{1}{2n\beta_G c_1}. \qquad \square \end{aligned}$$

The proof of Lemma 3.1 (c) and (d) are similar to that of Lemma 3.1 (b) and (a), respectively. Hence, the details are omitted.

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