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LIPSCHITZ TYPE INEQUALITY IN WEIGHTED BLOCH SPACE $\mathcal{B}_q$

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ABSTRACT. Let B be the open unit ball with center 0 in the complex space $\mathbb{C}^n$. For each $q > 0$, $\mathcal{B}_q$ consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z\in B}(1- \| z \|^2)^q \| \nabla f(z) \| < \infty .$$

In this paper, we will show that functions in weighted Bloch spaces $\mathcal{B}_q$ $(0 < q < 1)$ satisfies the following Lipschitz type result for Bergman metric β:

$$|f(z) - f(w)| < C\beta(z, w)$$

for some constant C.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of complex plane $\mathbb{C}$. For $z = (z_1, z_2, \dots , z_n)$ and $w = (w_1, w_2, \dots , w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w_j}$ and the norm by $\| z \|^2 = \langle z, z\rangle$.

Let B be the open unit ball with center 0 in the complex space $\mathbb{C}^n$. The boundary of B is the unit sphere $S = \{z \in \mathbb{C}^n : \| z \|= 1\}$. For $z \in B, \xi \in \mathbb{C}^n$, set

$$b_B{}^2(z, \xi) = \frac{n+1}{(1 - \| z \|^2)^2}[\,(1 - \| z \|^2)\| \xi \|^2 + |\langle z, \xi\rangle|^2\,].$$

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If $\gamma : [0,1] \to B$ is a C^1−curve, the Bergman length of γ is defined by

$$|\gamma|_B = \int_0^1 b_B(\gamma(t), \gamma'(t))dt.$$

For $z, w \in B$, define

$$\beta(z,w) = \inf\{|\gamma|_B : \gamma(0) = z, \gamma(1) = w\},$$

where the infimum is taken over all C^1−curves from z to w. β is called the Bergman metric on B.

For a in B and $r > 0$, let $E(a,r) = \{z \in B : \beta(a,z) < r\}$ be the open ball in the Bergman metric with center a and radius r. Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. Let $|E(a,r)|$ be the $d\nu$-volume measure of $E(a,r)$. Given a function f in $L^2(B, d\nu)$, let

$$\hat{f}_r(z) = \frac{1}{|E(z,r)|}\int_{E(z,r)} f(w)d\nu(w)$$

be the mean of f over $E(z,r)$. The mean oscillation of f in the Bergman metric is the function $MO_r f(z)$ defined on B by

$$MO_r f(z) = \left[\frac{1}{|E(z,r)|}\int_{E(z,r)} |f(w) - \hat{f}_r(z)|^2 d\nu(w)\right]^{\frac{1}{2}}.$$

We define $BMO_r(B)$ to be the space of all f such that $MO_r f$ is bounded on B. We equip $BMO_r(B)$ with the semi-norm

$$\| f \|_r = \sup\{MO_r f(z) : z \in B\}.$$

It was proved in [3] that $BMO_r(B)$ is independent of r and all the semi-norm $\|\|_r$ are mutually equivalent. Thus we simply write BMO for $BMO_r(B)$.

BMO in the Bergman metric was first exhibited in [2, 3] where BMO was used to characterize the boundedness of Hankel operators on the Bergman spaces. Suppose f is in L^1 $(B, d\nu)$. The Berezin transform of f is defined by

$$\tilde{f}(z) = \frac{1}{K(z,z)}\int_B |K(z,w)|^2 f(w)d\nu(w),$$

where $K(z,w)$ is the Bergman reproducing kernel. It was proved in [3] that for f in $L^2\ (B,d\nu)$, we have $f \in BMO$ if and only if the function $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$ is bounded in B. Moreover

$$\| f \|_{BMO} = \sup\{\left[\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2\right]^{\frac{1}{2}} : z \in B\}$$

is a complete and invariant semi-norm on BMO.

Let $H(B)$ be the space of all holomorphic functions on B. In Section 2, we will show that if $f \in L^1(B,d\nu) \cap H(B)$, then $\tilde{f}(z) = cf(z)$ for some constant c.

If $f \in H(B)$, then the quantity Qf is defined by

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\nabla f(z) \cdot \xi|}{b_B(z,\xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n,$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f. The quantity Qf is invariant under the group $\mathrm{Aut}(B)$ of holomorphic automorphisms of B. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in \mathrm{Aut}(B)$. A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function if

$$\sup_{z \in B} Qf(z) < \infty.$$

Bloch functions on bounded homogeneous domains were first studied in [5]. In [12], Timoney showed that the linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B}(1 - \| z \|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space $\mathcal{B}$ of Bloch functions on B.

It was shown in [3] that $BMO \cap H(B) = \mathcal{B}(B)$. Moreover the above seminorm for the Bloch functions is equivalent to the BMO-norm $\| f \|_{BMO}$ for holomorphic functions.

For each $q > 0$, the weighted Bloch space of B, denoted by $\mathcal{B}_q$, consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B}(1 - \| z \|^2)^q \| \nabla f(z) \| < \infty.$$

For each $q > 0$, we let $\mathcal{B}_{q,0}$ denote the subspace of $\mathcal{B}_q$ consisting of functions f with

$$\lim_{\|z\|\to 1^-}(1-\| z \|^2)^q \| \nabla f(z) \| = 0.$$

The family of weighted Bloch spaces $\mathcal{B}_q$ is an increasing family with respect to q in the sense that $\mathcal{B}_{q_1} \subset \mathcal{B}_{q_2}$ for $q_1 < q_2$. In particular, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_{1,0} = \mathcal{B}_0$. Let us define a norm on $\mathcal{B}_q$ as follows:

$$\| f \|_q = |f(0)| + \sup\{(1 - \| w \|^2)^q \| \nabla f(w) \|: w \in B\}.$$

It was proved in [7] that the space $\mathcal{B}_q$ is a Banach space with respect to the above norm for each $q > 0$, and that the little Bloch space $\mathcal{B}_{q,0}$ associated with $\mathcal{B}_q$ is a separable subspace of $\mathcal{B}_q$ which is the closure of the polynomials for each $q \geq 1$.

Let D be the open unit disk in the complex plane $\mathbb{C}$. It was proved in [15] that an analytic function f defined on D belongs to the Bloch space if and only if $|f(z) - f(w)| \leq C\delta(z,w)$ for some constant C and all z, w in D where δ is the Bergman distance on D. The purpose of this paper is to extend the above Lipschitz type inequality to the case of n-dimensional complex space.

In particular, in Section 3, we will show that if $f \in \mathcal{B}_q$, $0 < q < 1$, then

$$|f(z) - f(w)| \leq C\beta(z,w)$$

for some constant C where β is the Bergman metric on B.

2. Berezin transform of f in $L^1(B, d\nu) \cap H(B)$

Let $a \in B$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a \quad \text{if} \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a\rangle}.$$

It is easily shown that the mapping φ_a belongs to $\mathrm{Aut}(B)$ where $\mathrm{Aut}(B)$ is the group of all biholomorphic mappings of B onto itself, and satisfies $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a(\varphi_a(z)) = z$. Furthermore, for all $z, w \in \overline{B}$, we have

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \| a \|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}.$$

In particular, for $a \in B$, $z \in \overline{B}$,

$$1 - \| \varphi_a(z) \|^2 = \frac{(1 - \| a \|^2)(1 - \| z \|^2)}{|1 - \langle z, a \rangle|^2}$$

(See [9, Theorem 2.2.2]).

THEOREM 1. *Let ψ be a biholomorphic mapping of B onto itself and $a = \psi^{-1}(0)$. The determinant $J_R\psi$ of the real Jacobian matrix of ψ satisfies the following identity:*

$$J_R\psi(z) = |J\psi(z)|^2 = \left(\frac{1 - \| a \|^2}{|1 - \langle z, a \rangle|^2} \right)^{n+1} = \left(\frac{1 - \| \psi(z) \|^2}{1 - \| z \|^2} \right)^{n+1}.$$

Proof. See [9, Theorem 2.2.6]. □

The measure μ_q is the weighted Lebesgue measure:

$$d\mu_q = c_q (1 - \| z \|^2)^q d\nu(z),$$

where $q > -1$ is fixed, and c_q is a normalization constant such that $\mu_q(B) = 1$.

THEOREM 2. *If $f \in L^1(B, \mu_q) \cap H(B), q > -1$, then*

$$f(z) = c_q \int_B \frac{(1 - \| w \|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} f(w) d\nu(w).$$

Proof. Since $f \in H(B)$, by the mean value theorem,

$$f(0) = \int_S f(r\zeta)d\sigma(\zeta), \quad 0 < r < 1.$$

By integrating both sides of above equality with respect to the measure $2n(1-r^2)^q \, r^{2n-1}dr$ over $[0,1]$, we have

$$2n \int_0^1 \int_S f(r\zeta)(1-r^2)^q r^{2n-1} d\sigma(\zeta)dr = f(0)c_q^{-1}.$$

Namely,

$$f(0) = c_q \int_B f(w)(1 - \| w \|^2)^q d\nu(w).$$

Replace f by $f \circ \varphi_z$ and apply Theorem 1. Then

$$\begin{aligned}
f(z) &= c_q \int_B f(w)(1 - \| \varphi_z(w) \|^2)^q \left(\frac{(1 - \| z \|^2)}{|1 - \langle w, z \rangle|^2} \right)^{n+1} d\nu(w) \\
&= c_q \int_B f(w) \left(\frac{(1 - \| z \|^2)(1 - \| w \|^2)}{|1 - \langle w, z \rangle|^2} \right)^q \\
&\qquad \times \left(\frac{(1 - \| z \|^2)}{|1 - \langle w, z \rangle|^2} \right)^{n+1} d\nu(w) \\
&= c_q(1 - \| z \|^2)^{n+q+1} \int_B f(w) \frac{(1 - \| w \|^2)^q}{|1 - \langle w, z \rangle|^{2(n+q+1)}} d\nu(w) \\
&= c_q(1 - \| z \|^2)^{n+q+1} \\
&\qquad \times \int_B f(w) \frac{(1 - \| w \|^2)^q}{(1 - \langle w, z \rangle)^{n+q+1}(1 - \langle z, w \rangle)^{n+q+1}} d\nu(w).
\end{aligned}$$

Replacing $f(w)$ again by $f(w)(1 - \langle w, z \rangle)^{n+q+1}$, we get

$$\begin{aligned}
&f(z)(1 - \| z \|^2)^{n+q+1} \\
&= c_q(1 - \| z \|^2)^{n+q+1} \int_B f(w) \frac{(1 - \| w \|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} d\nu(w). \\
&f(z) = c_q \int_B \frac{(1 - \| w \|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} f(w) d\nu(w).
\end{aligned}$$

□

THEOREM 3. *If* $f \in L^1(B, d\nu) \cap H(B)$, *then* $c_{n+1}\tilde{f}(z) = f(z)$. *Here* c_{n+1} *is a normalization constant such that* $\mu_{n+1}(B) = 1$, *where* $d\mu_{n+1} = c_{n+1}(1 - \| z \|^2)^q d\nu(z)$.

Proof. It is well known that

$$K(z,w) = \frac{1}{(1 - \langle z, w\rangle)^{n+1}}$$

in the case of open unit ball B in $\mathbb{C}^n$.

$$\begin{aligned}\tilde{f}(z) &= \frac{1}{K(z,z)} \int_B |K(z,w)|^2 f(w) d\nu(w) \\ &= (1 - |z|^2)^{n+1} \int_B \frac{1}{(1 - \langle z, w\rangle)^{2(n+1)}} f(w) d\nu(w) \\ &= (1 - |z|^2)^{n+1} \int_B \frac{(1 - \| w \|^2)^{n+1}}{(1 - \langle z, w\rangle)^{n+1+n+1}} \frac{f(w)}{(1 - \| w \|^2)^{n+1}} d\nu(w).\end{aligned}$$

Since $f \in L^1(B, d\nu) \cap H(B)$,

$$\frac{f(w)}{(1 - \| w \|^2)^{n+1}} \in L^1(B, \mu_{n+1}) \cap H(B).$$

By Theorem 2,

$$\begin{aligned}&\frac{f(z)}{(1 - \| z \|^2)^{n+1}} \\ &= c_{n+1} \int_B \frac{(1 - \| w \|^2)^{n+1}}{(1 - \langle z, w\rangle)^{n+1+n+1}} \frac{f(w)}{(1 - \| w \|^2)^{n+1}} d\nu(w).\end{aligned}$$

We can see that

$$\begin{aligned}\tilde{f}(z) &= (1 - \| z \|^2)^{n+1} \frac{f(z)}{(1 - \| z \|^2)^{n+1}} \frac{1}{c_{n+1}} \\ &= \frac{1}{c_{n+1}} f(z).\end{aligned}$$

□

3. Lipschitz type result in $\mathcal{B}_q, 0 < q < 1$

THEOREM 4. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1 - \| w \|^2)^t}{|1 - \langle z, w \rangle|^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,

(i) $I_{c,t}(z)$ *is bounded in B if $c < 0$;*
(ii) $I_{0,t}(z) \sim -\log(1 - \| z \|^2)$ *as* $\| z \| \to 1^-$;
(iii) $I_{c,t}(z) \sim (1 - \| z \|^2)^{-c}$ *as* $\| z \| \to 1^-$ *if* $c > 0$.

Proof. See [9, Proposition 1.4.10]. □

Let $0 < p < \infty$ and $s \in \mathbb{R}$. The holomorphic Besov p-spaces $\mathcal{B}_p^s(B)$ with weight s is defined by the space of all holomorphic functions f on the unit ball B such that

$$\| f \|_{p,s} = \left\{ \int_B (Qf)^p(z)(1 - \| z \|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$

Here $d\lambda(z) = (1 - \| z \|^2)^{-n-1} d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on B.

For a fixed $p \in (0, \infty)$, $\mathcal{B}_p^s(B)$ is an increasing family of function spaces in s; that is, if $-\infty < s \leq t < +\infty$, then $\mathcal{B}_p^s(B) \subset \mathcal{B}_p^t(B)$. Similarly, for a fixed $s \in R$, the family $\mathcal{B}_p^s(B)$ is increasing with respect to $p \in (0, n - s)$. The holomorphic Besov p-space $\mathcal{B}_p^s(B)$ with weight s include many well known spaces as special case. $\mathcal{B}_p^s(B)$ is the usual Hardy p-space $H^p(B)$ for $s = n$, the Bergman space $L_a^p(B)$ for $s = n + 1$ (See [1]). In particular, the diagonal Besov space $\mathcal{B}_p^0(B)$ are shown to be Möbius invariant subsets of the Bloch space.

THEOREM 5. *Let $0 < p < \infty$ and $s \in \mathbb{R}$. Then*

$$\mathcal{B}_q \subseteq \mathcal{B}_p^s,$$

where $q < 1 + \frac{s-n}{p}$.

Proof. From the fact that $Qf(z)$ and $(1-\| z \|^2) \| \nabla f(z) \|$ behave the same within constants as $\| z \| \to 1$ (See [12]), we may replace $Qf(z)$ by $(1-\| z \|^2) \| \nabla f(z) \|$ with a different constant C in the definition of $\| f \|_{p,s}$. Namely,

$$\begin{aligned}
\| f \|_{p,s}^p &= \int_B (Qf)^p(z)(1-\| z \|^2)^s d\lambda(z) \\
&\leq C \int_B \left[(1-\| z \|^2) \| \nabla f(z) \|\right]^p (1-\| z \|^2)^s d\lambda(z) \\
&\leq C \int_B \left[\frac{(1-\| z \|^2)^q \| \nabla f(z) \|}{(1-\| z \|^2)^{q-1}}\right]^p (1-\| z \|^2)^s d\lambda(z) \\
&\leq C \| f \|_q^p \int_B (1-\| z \|^2)^{-pq+p+s-n-1} d\nu(z).
\end{aligned}$$

By Theorem 4, if $q < 1 + \frac{s-n}{p}$, then

$$\| f \|_{p,s} \leq C \| f \|_q$$

which yields the desired result. □

THEOREM 6. *Let $p \in (1, \infty)$ and $-p < s < 0$. Then there exists a positive constant C such that*

$$|f(z) - f(a)| \leq \frac{C}{\| z-a \|^{\frac{s}{p}}} \| f \|_{p,s}, \quad a, z \in B$$

for all M-harmonic functions f on B. In particular, $f \in \mathcal{B}_p^s$ satisfies the Lipschitz condition of order $-s/p$.

Proof. See [6, Theorem 1.4]. □

COROLLARY 7. *Let $q \in (0,1)$. If the function f in $\mathcal{B}_q$, then there exist constants $C > 0$ and $t > 0$ such that for all $z, w \in B$,*

$$|f(z) - f(w)| \leq C \| z-w \|^t \| f \|_q .$$

Proof. If we choose $p \in (1, \infty)$ and s $(-p < s < 0)$ such that $q < 1 + \frac{s-n}{p}$, then

$$|f(z) - f(w)| \leq C \| z-w \|^{-\frac{s}{p}} \| f \|_q$$

follows from Theorem 5 and Theorem 6. □

THEOREM 8. *For any smooth curve $\gamma : I \to B$ and any f in BMO, we have*

$$|\frac{d}{dt}\tilde{f}(\gamma(t))| \leq 2\sqrt{2}\left(\frac{ds}{dt}\right) \| f \|_{BMO(\gamma(I))} .$$

Proof. See [3]. □

COROLLARY 9. *For f in BMO,*

$$|\tilde{f}(a) - \tilde{f}(b)| \leq 2\sqrt{2} \| f \|_{BMO}\ \beta(a,b).$$

Proof. Choose γ in Theorem 8 to be a geodesic joining a to b of length $\beta(a,b)$. □

THEOREM 10. *For f in $\mathcal{B}_q$, $0 < q < 1$,*

$$|f(a) - f(b)| \leq c_{n+1}\beta(a,b).$$

Proof. If $f \in \mathcal{B}_q$, $0 < q < 1$, then

$$|f(z)| \leq |f(0)| + C \| z \|^t \| f \|_q$$

for some constant $C > 0$ and $t > 0$ by Corollary 7. Since

$$|f(z)| \leq |f(0)| + C \| f \|_q$$

for all $z \in B$, $f \in L^1(B, d\nu) \cap H(B)$. By Theorem 3,

$$\begin{aligned} |f(a) - f(b)| &= |c_{n+1}\tilde{f}(a) - c_{n+1}\tilde{f}(b)| \\ &\leq c_{n+1}\beta(a,b). \end{aligned}$$ □

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