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CELLULAR EMBEDDINGS OF LINE GRAPHS AND LIFTS

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ABSTRACT. A cellular embedding of a graph G into an orientable surface $\mathbb{S}$ can be considered as a cellular decomposition of $\mathbb{S}$ into 0-cells, 1-cells and 2-cells and vise versa, in which 0-cells and 1-cells form a graph G and this decomposition of $\mathbb{S}$ is called a map in $\mathbb{S}$ with underlying graph G. For a map $\mathcal{M}$ with underlying graph G, we define a natural rotation on the line graph of the graph G and we introduce the line map for $\mathcal{M}$. We find the genus of the supporting surface of the line map for a map and we give a characterization for the line map to be embedded in the sphere. Moreover we show that the line map for any lift of a map $\mathcal{M}$ is map-isomorphic to a lift of the line map for $\mathcal{M}$.

1. Introduction and preliminaries

The concept of a line graph was first appeared in the paper by Whitney [6] and the name line graph was introduced in Harray and Norman [4]. Various properties and some generalizations related to line graphs have been much studied until now. Recently a relationship between a covering of the line graph of a graph G and the line graph of a covering of G was studied [1]. In this paper we define a line map for a map $\mathcal{M}$ by introducing a natural rotation on the line graph of the underlying graph of $\mathcal{M}$. We find the genus of the supporting surface of the line map for a map. Moreover we show that line maps for derived maps of a map are map-isomorphic to lifts of the line map of the given map. This implies that any line graph of a covering of a graph G covers the line graph of G.

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Throughout this paper, all graphs are assumed to be simple, finite, and connected and all surfaces mean compact and connected 2-dimensional orientable manifold without boundary.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Every edge in $E(G)$ is represented as a set consisting two vertices in $V(G)$. Each edge $\{u, v\}$ gives rise to a pair of two directed edges $x = \langle u, v\rangle, x^{-1} = \langle v, u\rangle$ with opposite directions; $x = \langle u, v\rangle$ joins from u to v and the inverse $x^{-1} = \langle v, u\rangle$ joins from v to u. These directed edges will be called *arcs* (or *darts*). The pair $\{x, x^{-1}\}$ also represents the (undirected) edge $\{u, v\}$. Let $D(G)$ be the set of all arcs of a graph G and call it the *arc set* of G. Let $D^+(v)$ be the set of arcs in $D(G)$ emanating from v and $D^-(v)$ the set of arcs in $D(G)$ terminated at v.

For any arc x incident to (that is, terminated at or initiated at) a vertex v, let

$$_vx = \begin{cases} x & \text{if } x \in D^+(v) \\ x^{-1} & \text{if } x \in D^-(v). \end{cases}$$

Let $\widehat{D}(G)$ be a set of arcs which contains only one arc from each pair of $\{x, x^{-1}\}$, call it a *controlling arc set* of G, and let $\widehat{D}(v)$ denote the set of arcs in $\widehat{D}(G)$ incident to a vertex v in G. In treating line graphs and line maps, we shall use a fixed controlling arc set.

For $x \in D(G)$, let

$$\widehat{x} = \begin{cases} x & \text{if } x \in \widehat{D}(G) \\ x^{-1} & \text{if } x \notin \widehat{D}(G). \end{cases}$$

Then the sets $D^+(v)$ and $\widehat{D}(v)$ are in 1-1 correspondence via $x \mapsto \widehat{x}$, and so the number $|\widehat{D}(v)|$ of arcs in $\widehat{D}(v)$ is the degree of v, denoted by $\deg(v)$.

Any graph G can be regarded as a topological space in the following sense: By regarding the vertices of G as 0-cells and the edges of G as 1-cells, the graph G can be identified with a finite 1-dimensional CW-complex in the Euclidean 3-space $\mathbb{R}^3$. An *embedding* of a graph G in a surface $\mathbb{S}$ is a topological embedding $\imath : G \to \mathbb{S}$. If every component of $\mathbb{S} - \imath(G)$, called a *region*, is homeomorphic to an open disk in the Euclidean plane, then the embedding $\imath : G \to \mathbb{S}$ is called a *2-cell* or *cellular embedding*. A *map* in an orientable surface $\mathbb{S}$ is a cellular decomposition of $\mathbb{S}$ into 0-cells, 1-cells and 2-cells which form a graph, called its *underlying graph*. We shall treat maps in which the underlying graph is simple. A map in an orientable surface $\mathbb{S}$ with underlying graph G can be considered as a cellular embedding $\imath$ of G into $\mathbb{S}$ and vice versa. A *local rotation* ρ_v at v in G is a cyclic permutation of $D^+(v)$. A *rotation* ρ for a graph G is a permutation of the arc set $D(G)$ which may be decomposed into $|V(G)|$ cycles ρ_v in which ρ_v permutes the arcs in $D^+(v)$ cyclically and fixes the arcs except arcs in $D^+(v)$, where $|V(G)|$ denote the number of vertices in G. The permutation λ of $D(G)$ swapping two opposite arcs on every edge in G is called the *arc-reversing involution.*

As usual, we describe a map by means of rotation ρ and arc-reversing involution λ which are permutations of the arc set $D(G)$ of G as above. The permutation ρ is a rotation which at each vertex v cyclically permutes the arcs in $D^+(v)$ in accordance with the orientation of the surface so that for each arc x in $D^+(v)$, the arc $\rho(x)$ is the clockwise next arc in $D^+(v)$ on the surface. It is known that a rotation ρ on G determines a 2-cell embedding of G into an orientable surface $\mathbb{S}_g$. Conversely every 2-cell embedding of G into an orientable surface $\mathbb{S}_g$ gives rise to a rotation ρ on G associated with it ([5]). The regions of the 2-cell embedding associated to a rotation ρ for G are given by the disjoint cycles in the decomposition of $\rho \circ \lambda$ as the product of disjoint cycles. A map $\mathcal{M}$ having underlying graph G, arc set $D(G)$, rotation ρ and arc-reversing involution λ will be denoted by $(D(G), \langle\rho, \lambda\rangle)$.

Let $\mathcal{M} = (D(G), \langle\rho, \lambda\rangle)$ and $\mathcal{M}' = (D(G'), \langle\rho', \lambda'\rangle)$ be two maps.

A *graph-homomorphism* f from G to G' is a mapping $f : D(G) \to D(G')$ such that $f(\lambda(x)) = \lambda'(f(x))$ for any $x \in D(G)$, and $f(x), f(y)$ have the same initial vertex for any two arcs x, y with the same initial vertex. A bijective graph-homomorphism from G onto G' is called a *graph-isomorphism* from G to G'. A graph-homomorphism f from G onto G' is a *covering projection* if for every vertex v of G, f maps $D^+(v)$ bijectively to $D^+(f(v))$. The domain graph G of f is called a *covering* (graph) of G' and G' is said to be *covered by* G. Fibers of any two vertices and arcs in G are of the same cardinality.

A *map-homomorphism* f from $\mathcal{M}$ to $\mathcal{M}'$ is a mapping $f : D(G) \to D(G')$ such that $f(\lambda(x)) = \lambda'(f(x))$ and $f(\rho(x)) = \rho'(f(x))$ for any $x \in D(G)$. A bijective map-homomorphism from $\mathcal{M}$ onto $\mathcal{M}'$ is called a *map-isomorphism* from $\mathcal{M}$ to $\mathcal{M}'$. Note that any map-homomorphism is a graph-homomorphism.

Let G be a graph with arc set $D(G)$ and let S_k denote the symmetric group of $\{1, \cdots, k\}$. A mapping $\phi : D(G) \to S_k$ is a *permutation voltage assignment* on G if $\phi(x^{-1}) = \phi(x)^{-1}$ for each arc $x \in D(G)$. The graph G endowed with a permutation voltage assignment ϕ gives rise to a new graph G^ϕ, called a *derived graph* or a *lift* of G with respect to ϕ. The vertex set of G^ϕ is defined to be the set $V(G^\phi) = V(G) \times \{1, \cdots, k\}$

and a vertex (v, j) in $V(G^\phi)$ will be denoted as v_j. The arc set $D(G^\phi)$ of G^ϕ is defined to be the set $D(G^\phi) = D(G) \times \{1, \cdots, k\}$ and an arc (x, j) in $D(G^\phi)$ will be denoted as x_j. An arc x_j in $D(G^\phi)$ joins u_j to $v_{\phi(x)(j)}$ if x joins u to v and $D^+(v_j) = \{x_j | x \in D^+(v)\}$. It is known that every covering of a given graph arises from some permutation voltage assignment in a symmetric group ([3]).

The rotation ρ^ϕ on the derived graph G^ϕ is defined by $\rho^\phi_{v_j}(x_j) = (\rho_v(x))_j$ for any $x_j \in D^+(v_j)$. The arc-reversing involution λ^ϕ on the derived graph G^ϕ is defined by $\lambda^\phi(x_j) = \lambda(x)_{\phi(x)(j)}$ for any $x_j \in D(G^\phi)$. The map $\mathcal{M}^\phi = (D(G^\phi), \langle \rho^\phi, \lambda^\phi \rangle)$ will be called the *derived map* or the *(map-)lift* of a map $\mathcal{M} = (D(G), \langle \rho, \lambda \rangle)$ *with respect to* ϕ. The natural projection $\wp$ from $\mathcal{M}^\phi$ to $\mathcal{M}$ is defined by $\wp(x_j) = x$ for any $x_j \in D(G^\phi)$. Then $\wp$ is a map-homomorphism $\mathcal{M}^\phi$ onto $\mathcal{M}$ and a covering projection from G^ϕ onto G.

2. Line graphs and line maps

In this section we introduce a natural rotation on the line graph of the underlying graph and define the line map for a given map $\mathcal{M}$. Our work will be treated together with fixed controlling arc set.

Let $\mathcal{M} = (D(G), \langle \rho, \lambda \rangle)$ be a map with controlling arc set $\widehat{D}(G)$. Define a cyclic permutation $\widehat{\rho}_v$ of $\widehat{D}(v)$ via

$$\widehat{\rho}_v(x) = \widehat{\rho_v({}_v x)}$$

for $x \in \widehat{D}(v)$.

For $x \in \widehat{D}(G)$, let $\overline{x}$ denote the pair $\{x, x^{-1}\}$. The *line graph* $L(G)$ of a graph G has the vertex set $V(L(G))$ consisting of $\overline{x}$ where $x \in \widehat{D}(G)$, and the arc set $D(L(G))$ consisting of the ordered pairs $\langle x, y \rangle$ where x, y are in $\widehat{D}(G)$ and x, y have a unique common vertex in $V(G)$. For $x \in \widehat{D}(G)$, $D^+(\overline{x}) = \{\langle x, y \rangle \in D(L(G)) | y \in \widehat{D}(G)\}$ (cf. Figure 1).

Now in order to give a *line map* $L(\mathcal{M}) = \big(D(L(G)), \langle L(\rho), L(\lambda) \rangle\big)$ for a map $\mathcal{M} = (D(G), \langle \rho, \lambda \rangle)$, we define the arc-reversing involution $L(\lambda)$ and a natural rotation $L(\rho)$ on $L(G)$.

The arc-reversing involution $L(\lambda)$ on $L(G)$ is given by

$$L(\lambda)\langle x, y \rangle = \langle y, x \rangle$$

for any arc $\langle x, y \rangle \in D(L(G))$.

A *natural* rotation $L(\rho)$ on $L(G)$ is represented as the product of cyclic permutations $L(\rho)_{\overline{x}}$ on $D^+(\overline{x})$, $x \in \widehat{D}(G)$, defined as follows; if x

is an arc in $\widehat{D}(G)$ joining from u to v, define

$$L(\rho)_{\overline{x}}(\langle x,y\rangle)=\begin{cases}\langle x,\widehat{\rho}_u(y)\rangle & \text{if } y\in\widehat{D}(u) \text{ and } \widehat{\rho}_u(y)\neq x\\ \langle x,\widehat{\rho}_v(x)\rangle & \text{if } y\in\widehat{D}(u) \text{ and } \widehat{\rho}_u(y)= x\\ \langle x,\widehat{\rho}_v(y)\rangle & \text{if } y\in\widehat{D}(v) \text{ and } \widehat{\rho}_v(y)\neq x\\ \langle x,\widehat{\rho}_u(x)\rangle & \text{if } y\in\widehat{D}(v) \text{ and } \widehat{\rho}_v(y)= x\end{cases}$$

for any $\langle x,y\rangle\in D^+(\overline{x})$ (cf. Figure 1).

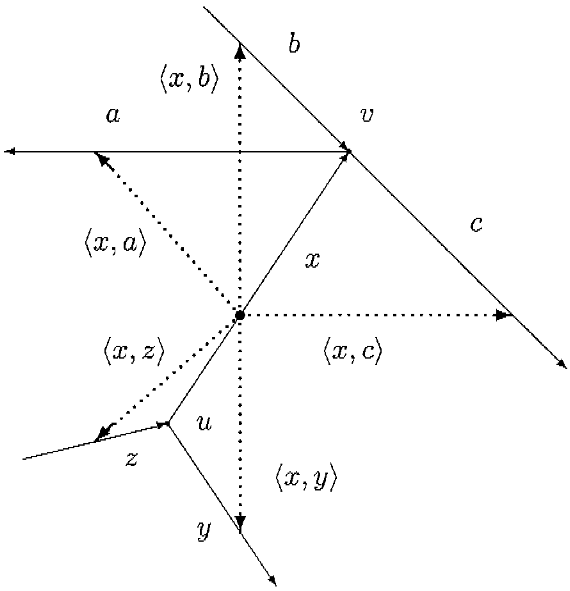

Figure 1. x,y,z,a,b and c are arcs in $\widehat{D}(G)$ and dotted arrows represent arcs in $D^+(\overline{x})$

For a permutation voltage assignment $\phi: D(G)\to S_k$, the derived map $\mathcal{M}^\phi=(D(G^\phi),\langle\rho^\phi,\ \lambda^\phi\rangle)$ of $\mathcal{M}$ is given. Let $\widehat{D}(G^\phi)=\widehat{D}(G)\times\{1,\cdots,k\}$ be the controlling arc set for the derived map $\mathcal{M}^\phi$ and let $\widehat{D}(v_j)=\{x_{[j]}|x\in\widehat{D}(v)\}$, where

$$x_{[j]}=\begin{cases}x_j & \text{if } x\in D^+(v)\\ x_{\phi(x^{-1})(j)} & \text{if } x\in D^-(v).\end{cases}$$

Define a cyclic permutation $\widehat{\rho^{\phi}}_{v_j}$ on $\widehat{D}(v_j)$ via

$$\widehat{\rho^{\phi}}_{v_j}(x_l) = \widehat{\rho}_v(x)_{[l]}$$

for any $x_l \in \widehat{D}(v_j)$.

For $x_j \in \widehat{D}(G^{\phi})$, let $\overline{x_j} = \{x_j, x^{-1}_{\phi(x)(j)}\}$. Then the line map $L(\mathcal{M}^{\phi})$ for $\mathcal{M}^{\phi}$ has the vertex set $V(L(G^{\phi})) = \{\overline{x_j} | x_j \in \widehat{D}(G^{\phi})\}$ and the arc set $D(L(G^{\phi})) = \{\langle x_j, y_l\rangle | x_j, y_l \in \widehat{D}(G^{\phi}),$ and x_j, y_l are adjacent$\}$.

The arc-reversing involution $L(\lambda^{\phi})$ on $L(\mathcal{M}^{\phi})$ is defined by

$$L(\lambda^{\phi})\langle x_j, y_l\rangle = \langle y_l, x_j\rangle$$

for any arc $\langle x_j, y_l\rangle \in D(L(G^{\phi}))$.

For $x_j \in \widehat{D}(G^{\phi})$, $D^+(\overline{x_j}) = \{\langle x_j, y_l\rangle | y \in \widehat{D}(G),\ y_l$ is adjacent to $x_j\}$. A cyclic permutation $L(\rho^{\phi})_{\overline{x_j}}$ on $D^+(\overline{x_j})$ is given as follows; If x is an arc in $\widehat{D}(G)$ joining from u to v, then x_j is an arc in $\widehat{D}(G^{\phi})$ joining from u_j to v_s where $s = \phi(x)(j)$, and

$$L(\rho^{\phi})_{\overline{x_j}}(\langle x_j, y_l\rangle) = \begin{cases} \langle x_j, \widehat{\rho^{\phi}}_{u_j}(y_l)\rangle & \text{if } y_l \in \widehat{D}(u_j) \text{ and } \widehat{\rho^{\phi}}_{u_j}(y_l) \neq x_j \\ \langle x_j, \widehat{\rho^{\phi}}_{v_s}(x_j)\rangle & \text{if } y_l \in \widehat{D}(u_j) \text{ and } \widehat{\rho^{\phi}}_{u_j}(y_l) = x_j \\ \langle x_j, \widehat{\rho^{\phi}}_{v_s}(y_l)\rangle & \text{if } y_l \in \widehat{D}(v_s) \text{ and } \widehat{\rho^{\phi}}_{v_s}(y_l) \neq x_j \\ \langle x_j, \widehat{\rho^{\phi}}_{u_j}(x_j)\rangle & \text{if } y_l \in \widehat{D}(v_s) \text{ and } \widehat{\rho^{\phi}}_{v_s}(y_l) = x_j \end{cases}$$

for any $\langle x_j, y_l\rangle \in D^+(\overline{x_j})$.

3. Genus and lift of line maps

In this section, we find the genus of the line map for a map and we show that line maps for lifts of a map $\mathcal{M}$ are map-isomorphic to lifts of the line map for $\mathcal{M}$.

We gives a lemma relative to a special map on a complete graph, which will be used in finding the genus of the line map for a map.

Let K_n be a complete graph with vertices $v_1, v_2, \cdots, v_n$. Each edge $\{v_i, v_j\}$ in K_n gives rise to a pair of arcs $\langle i,j\rangle, \langle j,i\rangle$ where $\langle i,j\rangle$ is an arc joining from v_i to v_j. Thus $D(K_n)$ is the set $\{\langle i,j\rangle | i,j = 1,2,\cdots,n \text{ and } i \neq j\}$ and $D^+(v_i)$ is the set $\{\langle i,j\rangle |\ j = 1,2,\cdots,n \text{ and } j \neq i\}$.

LEMMA 3.1. *Let ϖ be a permutation $\prod_{i=1}^{n} \varpi_{v_i}$ of $D(K_n)$ in which ϖ_{v_i} is the cyclic permutation $(\langle i,1\rangle \cdots \langle i,i-1\rangle\langle i,i+1\rangle \cdots \langle i,n\rangle)$ of $D^+(v_i)$.*

Then ϖ gives rise to a map $\mathcal{K}_n$ that has $\lceil \frac{n}{2} \rceil$ regions, in which $\lfloor \frac{n-2}{2} \rfloor$ regions are $2n-$sided and the remains(one or two regions) are n-sided.

Proof. The result is directly from checking the cycles in the decomposition of $\varpi \circ \lambda$ as the product of disjoint cycles, where λ is a full involution $\prod_{1 \le i < j \le n} (\langle i, j \rangle, \langle j, i \rangle)$ of $D(K_n)$. □

THEOREM 3.2. *Let $\mathcal{M} = (D(G), \langle \rho, \lambda \rangle)$ be a map in which the genus of the supporting surface is g, and $L(\mathcal{M})$ the line map for $\mathcal{M}$. Then the genus of the supporting surface of $L(\mathcal{M})$ is*

$$g + \frac{1}{4} \sum_{v \in V(G)} \Big\{ \deg(v)^2 - 3\deg(v) - 2\lfloor \frac{\deg(v) - 1}{2} \rfloor + 2 \Big\}.$$

Proof. The number of vertices of $L(\mathcal{M})$ is $\frac{|D(G)|}{2}$ and the number of edges of $L(\mathcal{M})$ is $\frac{1}{2} \sum_{v \in V(G)} \deg(v)(\deg(v) - 1)$. From the definition of $L(\rho)$, each region in $\mathcal{M}$ determines a region in $L(\mathcal{M})$. For each vertex v in G, by Lemma 3.1, the arcs incident to v give rise to $\lceil \frac{\deg(v)}{2} \rceil - 1 = \lfloor \frac{\deg(v)-1}{2} \rfloor$ regions in $L(\mathcal{M})$. Hence the number of regions of $L(\mathcal{M})$ is $r(\mathcal{M}) + \sum_{v \in V(G)} \lfloor \frac{\deg(v)-1}{2} \rfloor$, where $r(\mathcal{M})$ is the number of regions in a map $\mathcal{M}$. From the Euler formula, we see that the genus of the supporting surface of $L(\mathcal{M})$ is given as above. □

For a map $\mathcal{M} = (D(G), \langle \rho, \lambda \rangle)$, iterated line maps are inductively defined by $L^0(\mathcal{M}) = \mathcal{M}$, $L^1(\mathcal{M}) = L(\mathcal{M})$ and $L^{n+1}(\mathcal{M}) = L\big(L^n(\mathcal{M})\big)$ for $n \in \mathbb{N}$ with $L^n(\mathcal{M})$ not empty graph. Let $K_{m,n}$ denote the complete bipartite graph on sets of m vertices and n vertices.

COROLLARY 3.3. *Given a map $\mathcal{M}$ in which the underlying graph G is connected and is not empty.*

(1) *$L(\mathcal{M})$ is embedded in the sphere if and only if $\mathcal{M}$ is embedded in the sphere and G contains no vertex v with $\deg(v) \ge 4$.*
(2) *$L^n(\mathcal{M})$ is embedded in the sphere for every $n \in \mathbb{N}$ with $L^{n-1}(\mathcal{M})$ not empty graph if and only if G is a path or a cycle, or $K_{1,3}$.*

Proof. (1) If $\mathcal{M}$ is not embedded in the sphere, then $L(\mathcal{M})$ is not embedded in the sphere by Theorem 3.2. We suppose that $\mathcal{M}$ is embedded in the sphere. The term $\deg(v)^2 - 3\deg(v) - 2\lfloor \frac{\deg(v)-1}{2} \rfloor + 2$ in theorem 3.2 equal to

$$\begin{cases} (\deg(v) - 1)(\deg(v) - 3) & \text{if } \deg(v) \text{ is odd,} \\ (\deg(v) - 2)^2 & \text{if } \deg(v) \text{ is even.} \end{cases}$$

Hence $L(\mathcal{M})$ is embedded in the sphere if and only if G contains no vertex v with $\deg(v) \geq 4$.

(2) If a map $\mathcal{M}$ has a path or a cycle, or $K_{1,3}$ as the underlying graph G, then $L^n(\mathcal{M})$ is evidently embedded in the sphere for every $n \in \mathbb{N}$ with $L^{n-1}(\mathcal{M})$ not empty graph. For the converse, we suppose that the underlying graph G of a map $\mathcal{M}$ is not a path or a cycle, or $K_{1,3}$. Then G has an edge a with $\deg(u)+\deg(v) \geq 5$, where u, v are the incident vertices of a and so $L(G)$ has a connected subgraph consisting of a 3-cycle and a cut-edge. Hence there exists an edge e in $L^2(G)$ such that the sum of degrees of the incident vertices of e is equal to or greater than 6. Therefore $L^3(G)$ contains a vertex w having $\deg(w) \geq 4$. Hence from (1), $L^3(\mathcal{M})$ is not embedded in the sphere. □

THEOREM 3.4. *The line map of any lift for a map $\mathcal{M}$ is map-isomorphic to a lift of the line map for $\mathcal{M}$.*

Proof. Let $\mathcal{M} = (D(G), \langle\rho, \lambda\rangle)$ be a map with controlling arc set $\widehat{D}(G)$ and $\phi : D(G) \to S_k$ a permutation voltage assignment on G. Let $L(\mathcal{M}) = \big(D(L(G)), \langle L(\rho), L(\lambda)\rangle\big)$ be the line map for $\mathcal{M}$ and $\mathcal{M}^\phi = (D(G^\phi), \langle\rho^\phi, \lambda^\phi\rangle)$ the derived map of the map $\mathcal{M}$, and $L(\mathcal{M}^\phi) = \big(D(L(G^\phi)), \langle L(\rho^\phi), L(\lambda^\phi)\rangle\big)$ the line map for the map $\mathcal{M}^\phi$, which are given in section 2.

Define $\psi : D(L(G)) \to S_k$ as follows. For $\langle x, y\rangle \in D(L(G))$, where $x, y \in \widehat{D}(G)$ has a unique common vertex w, one of the following 4 cases holds;

$$\text{case 1} : \mathrm{i}(x) = w = \mathrm{i}(y), \ \text{case 2} : \mathrm{i}(x) = w = \mathrm{t}(y),$$
$$\text{case 3} : \mathrm{t}(x) = w = \mathrm{i}(y), \ \text{case 4} : \mathrm{t}(x) = w = \mathrm{t}(y),$$

where $\mathrm{t}(x)$ means the terminal vertex of x and $\mathrm{i}(x)$ means the initial vertex of arc x. Let

$$\psi(\langle x, y\rangle)(j) = \begin{cases} j & \text{for case 1,} \\ \phi(y^{-1})(j) & \text{for case 2,} \\ \phi(x)(j) & \text{for case 3,} \\ \phi(y^{-1})(\phi(x)(j)) & \text{for case 4.} \end{cases}$$

Then $\psi(\langle y, x\rangle) = \psi(\langle x, y\rangle)^{-1}$ and ψ is a permutation voltage assignment on $L(G)$.

Let $L(\mathcal{M})^\psi = \big(D(L(G)^\psi), \langle L(\rho)^\psi, L(\lambda)^\psi\rangle\big)$ be the lift of the line map $L(\mathcal{M})$ with respect to ψ. The vertex set and arc set of $L(G)^\psi$ are given as

$$V(L(G)^\psi) = \{\overline{x}_j | x \in \widehat{D}(G), j = 1, \cdots, k\},$$

$$D(L(G)^{\psi}) = \{\langle x,y\rangle_j | \langle x,y\rangle \in D(L(G)), j \in \{1,2,\cdots,k\}\}.$$

Let $D^+(\overline{x}_j) = \{\langle x,y\rangle_j | \langle x,y\rangle \in D^+(\overline{x})\}$. The arc-reversing involution $L(\lambda)^{\psi}$ and the rotation $L(\rho)^{\psi}$ are given as

$$L(\lambda)^{\psi}(\langle x,y\rangle_j) = \langle y,x\rangle_{\psi(\langle x,y\rangle)(j)},$$

$$L(\rho)^{\psi}_{\overline{x}_j}(\langle x,y\rangle_j) = \big(L(\rho)_{\overline{x}}(\langle x,y\rangle)\big)_j.$$

Now define $F : D(L(G)^{\psi}) \to D(L(G^{\phi}))$ by

$$F(\langle x,y\rangle_j) = \langle x_j, y_{\psi(\langle x,y\rangle)(j)}\rangle.$$

Then F is a bijection from $D(L(G)^{\psi})$ onto $D(L(G^{\phi}))$ and

$$\begin{aligned} F\big(L(\lambda)^{\psi}(\langle x,y\rangle_j)\big) &= F\big(\langle y,x\rangle_{\psi(\langle x,y\rangle)(j)}\big) \\ &= \langle y_{\psi(\langle x,y\rangle)(j)}, x_j\rangle \\ &= L(\lambda^{\phi})\big(\langle x_j, y_{\psi(\langle x,y\rangle)(j)}\rangle\big) \\ &= L(\lambda^{\phi})\big(F(\langle x,y\rangle_j)\big). \end{aligned}$$

Finally, we show that $F\big(L(\rho)^{\psi}_{\overline{x}_j}(\langle x,y\rangle_j)\big) = L(\rho^{\phi})_{\overline{x_j}}\big(F(\langle x,y\rangle_j)\big)$. To do it we let $u = \mathrm{i}(x)$, $v = \mathrm{t}(x)$ and $s = \phi(x)(j)$. Here we consider the 4th case, that is, $\mathrm{t}(x) = v = \mathrm{t}(y)$. In this case, $y_{\psi(\langle x,y\rangle)(j)} = y_{\phi(y^{-1})(s)} \in \widehat{D}(v_s)$. If $\widehat{\rho}_v(y) \neq x$, then $\widehat{\rho^{\phi}}_{v_s}(y_{\psi(\langle x,y\rangle)(j)}) \neq x_j$ and

$$\begin{aligned} F\Big(L(\rho)^{\psi}_{\overline{x}_j}(\langle x,y\rangle_j)\Big) &= F\Big(\big(L(\rho)_{\overline{x}}(\langle x,y\rangle)\big)_j\Big) \\ &= F\big(\langle x, \widehat{\rho}_v(y)\rangle_j\big) \\ &= \langle x_j, \widehat{\rho}_v(y)_{\psi(\langle x,\widehat{\rho}_v(y)\rangle)(j)}\rangle \\ &= \begin{cases} \langle x_j, \widehat{\rho}_v(y)_s\rangle & \text{if } \mathrm{i}(\widehat{\rho}_v(y)) = v \\ \langle x_j, \widehat{\rho}_v(y)_{\phi(\widehat{\rho}_v(y)^{-1})(s)}\rangle & \text{if } \mathrm{t}(\widehat{\rho}_v(y)) = v \end{cases} \\ &= \langle x_j, \widehat{\rho}_v(y)_{[s]}\rangle \\ &= \langle x_j, \widehat{\rho^{\phi}}_{v_s}(y_{[s]})\rangle \\ &= \langle x_j, \widehat{\rho^{\phi}}_{v_s}(y_{\phi(y^{-1})(s)})\rangle \\ &= \langle x_j, \widehat{\rho^{\phi}}_{v_s}(y_{\psi(\langle x,y\rangle)(j)})\rangle \\ &= L(\rho^{\phi})_{\overline{x_j}}\big(\langle x_j, y_{\psi(\langle x,y\rangle)(j)}\rangle\big) \\ &= L(\rho^{\phi})_{\overline{x_j}}\big(F(\langle x,y\rangle_j)\big). \end{aligned}$$

If $\widehat{\rho}_v(y) = x$, then $\widehat{\rho^{\phi}}_{v_s}(y_{\psi(\langle x,y\rangle)(j)}) = x_j$ and

$$\begin{aligned}
F\Big(L(\rho)^{\psi}_{\overline{x}_j}(\langle x,y\rangle_j)\Big) &= F\Big(\big(L(\rho)_{\overline{x}}(\langle x,y\rangle)\big)_j\Big) \\
&= F\big(\langle x,\widehat{\rho}_u(x)\rangle_j\big) \\
&= \big\langle x_j,\widehat{\rho}_u(x)_{\psi(\langle x,\widehat{\rho}_u(x)\rangle)(j)}\big\rangle \\
&= \begin{cases} \langle x_j,\widehat{\rho}_u(x)_j\rangle & \text{if } \mathrm{i}(\widehat{\rho}_u(x)) = u \\ \langle x_j,\widehat{\rho}_u(x)_{\phi(\widehat{\rho}_u(x)^{-1})(j)}\rangle & \text{if } \mathrm{t}(\widehat{\rho}_u(x)) = u \end{cases} \\
&= \langle x_j,\widehat{\rho}_u(x)_{[j]}\rangle \\
&= \langle x_j,\widehat{\rho^{\phi}}_{u_j}(x_{[j]})\rangle \\
&= \langle x_j,\widehat{\rho^{\phi}}_{u_j}(x_j)\rangle \\
&= L(\rho^{\phi})_{\overline{x_j}}(\langle x_j,y_{\psi(\langle x,y\rangle)(j)}\rangle) \\
&= L(\rho^{\phi})_{\overline{x_j}}\big(F(\langle x,y\rangle_j)\big).
\end{aligned}$$

We can check it similarly for the remaining cases. Therefore $L(\mathcal{M})^{\psi}$ is map-isomorphic to $L(\mathcal{M}^{\phi})$ by the mapping F. □

Since any map-isomorphism is a graph-isomorphism on underlying graphs, we have the following result.

COROLLARY 3.5. *Any line graph of a covering graph of a graph G covers the line graph of G.*

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