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FINSLER SPACES WITH THE SECOND APPROXIMATE MATSUMOTO METRIC

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ABSTRACT. The present paper is devoted to studying the condition for a Finsler space with the second approximate Matsumoto metric to be a Berwald space and to be a Douglas space.

1. Introduction

The Finsler space $F^n = (M^n, L(x,y))$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. The Douglas space was introduced by S. Bácsó and M. Matsumoto ([2]) as a generalization of the Berwald space from the viewpoint of geodesic equations. The interesting and important examples of (α, β)-metric are Randers metric, Kropina metric and Matsumoto metric. The Matsumoto metric is an exact formulation of the model of Finsler space, and has been studied by M. Hashiguchi et al. ([3]). In the Matsumoto metric, the 1-form $b_i(x)y^i$ was originally to be induced by earth's gravity ([6]). Hence we could regard $b_i(x)$ as the infinitesimals. The present authors ([11]) have investigated the Finsler spaces with the first approximate Matsumoto metric in which all powers ≥ 3 of $b_i(x)$ are neglected to be a Berwald and to be a Douglas space.

Recently, M. Matsumoto ([9]) has found the condition for Matsumoto space F^n to be a Douglas space, and proved that if F^n is a Douglas space, then F^n is a Berwald space.

The present paper is the consecutive study of the above one. We shall find the condition for the Finsler space with the second approximate Matsumoto metric in which all powers ≥ 4 of $b_i(x)$ are neglected to be a Berwald and to be a Douglas space.

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2. Preliminaries

The Matsumoto metric $L = \alpha^2/(\alpha - \beta)$ is expressed as the form

$$L = \alpha\Big\{\sum_{k=0}^{\infty}(\frac{\beta}{\alpha})^k\Big\} \tag{2.1}$$

for $|\beta| < |\alpha|$. We regard $b_i(x)$ as very small numerically. If we neglect all the power ≥ 2 of $b_i(x)$ in (2.1), then $L = \alpha + \beta$, that is a Randers metric. If we neglect all the power ≥ 3 of $b_i(x)$ in (2.1), then L is the first approximate Matsumoto metric ([11]). Hereafter we neglect all the power ≥ 4 of $b_i(x)$ in (2.1), then (α,β)-metric

$$L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} \tag{2.2.}$$

is an approximate metric to the Matsumoto metric. We shall call the (α,β)-metric (2.2) *the second approximate Matsumoto metric.*

On the other hand, the geodesics of a Finsler space $F^n = (M^n, L)$ are given by the system of differential equations including the functions

$$2G^i(x,y) = g^{ij}(y^r\dot{\partial}_j\partial_r F - \partial_j F) = \gamma_j{}^i{}_k y^j y^k,$$

where $\gamma_j{}^i{}_k$ are the Christoffel symbols constructed from the Finsler metric tensor $g_{ij}(x,y)$ with respect to x^i. The space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with $F^n = (M^n, L(\alpha,\beta))$ ([1], [8]). The covariant differentiation with respect to the Levi-Civita connection $\{_j{}^i{}_k\}(x)$ of R^n is denoted by (;). We use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s^i{}_j = a^{ir}s_{rj}, \quad s_j = b_r s^r{}_j.$$

According to [7], the functions $G^i(x,y)$ of F^n with (α,β)-metric are written in the form

$$\begin{aligned} &2G^i = \{_0{}^i{}_0\} + 2B^i, \\ &B^i = \frac{\alpha L_\beta}{L_\alpha}s^i{}_0 + C^*\Big\{\frac{\beta L_\beta}{\alpha L}y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i)\Big\}, \end{aligned} \tag{2.3}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha$, the subscript 0 means contraction by y^i and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \quad b^i = a^{ij}b_j,$$

$$\gamma^2 = b^2\alpha^2 - \beta^2, \quad b^2 = a^{ij}b_i b_j.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity.

The Finsler space F^n with (α, β)-metric is a Douglas space, if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$ ([2]). From (2.3) we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{2.4}$$

We shall state the following Lemma for the later.

LEMMA ([3]). *If $\alpha^2 \equiv 0$ (mod. β), that is, $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3. The condition to be a Berwald space

The present section is devoted to find the condition for a Finsler space F^n with the second approximate Matsumoto metric (2.2) to be a Berwald space. From (2.3) the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ of F^n with (α, β)-metric is given by

$$G^i{}_j = \dot\partial_j G^i = \gamma_0{}^i{}_j + B^i{}_j,$$
$$G_j{}^i{}_k = \dot\partial_k G^i{}_j = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

where we put $B^i{}_j = \dot\partial_j B^i$ and $B_j{}^i{}_k = \dot\partial_k B^i{}_j$. On account of [7], $B_j{}^i{}_k$ are determined by

$$L_\alpha B_j{}^k{}_i y^j y_k + \alpha L_\beta (B_j{}^k{}_i b_k - b_{j;i}) y^j = 0. \tag{3.1}$$

In F^n with (2.2), we obtain

(3.2)
$$L_\alpha = \frac{\alpha^3 - \alpha\beta^2 - 2\beta^3}{\alpha^3}, \; L_\beta = \frac{\alpha^2 + 2\alpha\beta + 3\beta^2}{\alpha^2}, \; L_{\alpha\alpha} = \frac{2\alpha\beta^2 + 6\beta^3}{\alpha^4}.$$

Substituting (3.2) in (3.1), we have

$$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B_{jki} y^j y^k + \alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)(B_{jki} b^k - b_{j;i}) y^j = 0, \tag{3.3}$$

where $B_{jki} = a_{kr} B_j{}^r{}_i$.

We suppose that F^n is a Berwald space, i.e., $B_j{}^i{}_k$ and $b_{i;j}$ are functions of position alone. Then (3.3) is separated in the rational and irrational terms in y^i as follows:

$$\alpha\{(\alpha^2-\beta^2)B_{jki}y^jy^k+2\alpha^2\beta(B_{jki}b^k-b_{j;i})y^j\}$$
$$-2\beta^3B_{jki}y^jy^k+\alpha^2(\alpha^2+3\beta^2)(B_{jki}b^k-b_{j;i})y^j=0,$$

which is reduced to two equations as follows:

$$(\alpha^2-\beta^2)B_{jki}y^jy^k+2\alpha^2\beta(B_{jki}b^k-b_{j;i})y^j=0, \tag{3.4}$$

$$-2\beta^3B_{jki}y^jy^k+\alpha^2(\alpha^2+3\beta^2)(B_{jki}b^k-b_{j;i})y^j=0. \tag{3.5}$$

Eliminating terms $(B_{jki}b^k-b_{j;i})y^j$ from the equations above, we have

$$(\alpha^2+\beta^2)^2B_{jki}y^jy^k=0,$$

which implies $B_{jki}y^jy^k=0$. Hence we get $B_{jki}+B_{kji}=0$. Since B_{jki} is symmetric in (j,i), we easily get $B_{jki}=0$. Therefore we have from (3.4) or (3.5)

$$b_{j;i}=0. \tag{3.6}$$

On the other hand, Hashiguch, Hōjyō and Matsumoto have shown ([3]) that if (3.6) holds good, then F^n is a Berwald space. Thus we have

THEOREM 3.1. *A Finsler space with the second approximate Matsumoto metric* (2.2) *is a Berwald space if and only if* $b_{j;i}=0$.

4. The condition to be a Douglas space

The present section is devoted to find the condition for a Finsler space F^n with the second approximate Matsumoto metric to be a Douglas space. In F^n with (2.2), we have

$$C^*=\frac{\alpha\{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2s_0\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)\}}{2\beta\{\alpha^3-3\alpha\beta^2-8\beta^3+2b^2\alpha^2(\alpha+3\beta)\}}. \tag{4.1}$$

If $\alpha^3-3\alpha\beta^2-8\beta^3+2b^2\alpha^2(\alpha+3\beta)=0$, then it leads to a contradiction easily. Thus the denominator of C^* is not zero. Substituting (3.2) and (4.1) in (2.4), we have

$$\begin{aligned}&\{\alpha^3-3\alpha\beta^2-8\beta^3+2b^2\alpha^2(\alpha+3\beta)\}\{(\alpha^3-\alpha\beta^2-2\beta^3)B^{ij}\\&\qquad-\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)(s^i{}_0y^j-s^j{}_0y^i)\}\\&-\alpha^2(\alpha+3\beta)\{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)\\&\qquad-2s_0\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)\}(b^iy^j-b^jy^i)=0.\end{aligned}\tag{4.2}$$

Suppose that F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (4.2) in the rational and irrational terms of y^i, we have

(4.3)
$$\begin{aligned}&(\alpha^6-4\alpha^4\beta^2+3\alpha^2\beta^4+16\beta^6+2b^2\alpha^6-2b^2\alpha^4\beta^2-12b^2\alpha^2\beta^4)B^{ij}\\&+(-2\alpha^6\beta+14\alpha^4\beta^3+24\alpha^2\beta^5-10b^2\alpha^6\beta-18b^2\alpha^4\beta^3)\\&\quad\times(s^i{}_0y^j-s^j{}_0y^i)\\&+\{r_{00}(-\alpha^6+\alpha^4\beta^2+6\alpha^2\beta^4)+2s_0(5\alpha^6\beta+9\alpha^4\beta^3)\}\\&\quad\times(b^iy^j-b^jy^i)\\&+\alpha\Big[(-10\alpha^2\beta^3+14\beta^5+6b^2\alpha^4\beta-10b^2\alpha^2\beta^3)B^{ij}\\&\quad+(-\alpha^6+25\alpha^2\beta^4-2b^2\alpha^6-18b^2\alpha^4\beta^2)(s^i{}_0y^j-s^j{}_0y^i)\\&\quad+\{r_{00}(5\alpha^2\beta^3-3\alpha^4\beta)+2s_0(\alpha^6+9\alpha^4\beta^2)\}(b^iy^j-b^jy^i)\Big]\\&=0,\end{aligned}$$

which is reduced to two equations as follows:

$$\begin{aligned}&(\alpha^6-4\alpha^4\beta^2+3\alpha^2\beta^4+16\beta^6\\&\qquad+2b^2\alpha^6-2b^2\alpha^4\beta^2-12b^2\alpha^2\beta^4)B^{ij}\\&+(-2\alpha^6\beta+14\alpha^4\beta^3+24\alpha^2\beta^5-10b^2\alpha^6\beta-18b^2\alpha^4\beta^3)\\&\qquad\times(s^i{}_0y^j-s^j{}_0y^i)\\&+\{r_{00}(-\alpha^6+\alpha^4\beta^2+6\alpha^2\beta^4)+2s_0(5\alpha^6\beta+9\alpha^4\beta^3)\}\\&\qquad\times(b^iy^j-b^jy^i)\\&=0.\end{aligned}\tag{4.4}$$

$$(4.5)\quad \begin{aligned}&(-10\alpha^2\beta^3 + 14\beta^5 + 6b^2\alpha^4\beta - 10b^2\alpha^2\beta^3)B^{ij}\\ &+(-\alpha^6 + 25\alpha^2\beta^4 - 2b^2\alpha^6 - 18b^2\alpha^4\beta^2)(s^i{}_0y^j - s^j{}_0y^i)\\ &+\{r_{00}(5\alpha^2\beta^3 - 3\alpha^4\beta) + 2s_0(\alpha^6 + 9\alpha^4\beta^2)\}(b^iy^j - b^jy^i) = 0.\end{aligned}$$

Eliminating B^{ij} from these equations, we obtain

$$(4.6)\qquad A(s^i{}_0y^j - s^j{}_0y^i) + B(b^iy^j - b^jy^i) = 0,$$

where

$$(4.7)\quad \begin{aligned}A = {}&4\beta^2(-\alpha^4 + 7\alpha^2\beta^2 + 12\beta^4 - 5b^2\alpha^4 - 9b^2\alpha^2\beta^2)\\ &\times(-5\alpha^2\beta^2 + 7\beta^4 + 3b^2\alpha^4 - 5b^2\alpha^2\beta^2)\\ &-(-\alpha^4 + 25\beta^4 - 2b^2\alpha^4 - 18b^2\alpha^2\beta^2)\\ &\times(\alpha^6 - 4\alpha^4\beta^2 + 3\alpha^2\beta^4 + 16\beta^6 + 2b^2\alpha^6\\ &\quad - 2b^2\alpha^4\beta^2 - 12b^2\alpha^2\beta^4),\end{aligned}$$

$$(4.8)\quad \begin{aligned}B = {}&2\{r_{00}(-\alpha^4 + \alpha^2\beta^2 + 6\beta^4) + 2s_0(5\alpha^4\beta + 9\alpha^2\beta^3)\}\\ &\times(-5\alpha^2\beta^3 + 7\beta^5 + 3b^2\alpha^4\beta - 5b^2\alpha^2\beta^3)\\ &-\{r_{00}(5\beta^3 - 3\alpha^2\beta) + 2s_0(\alpha^4 + 9\alpha^2\beta^2)\}\\ &\times(\alpha^6 - 4\alpha^4\beta^2 + 3\alpha^2\beta^4 + 16\beta^6 + 2b^2\alpha^6\\ &\quad - 2b^2\alpha^4\beta^2 - 12b^2\alpha^2\beta^4).\end{aligned}$$

Transvection of (4.6) by b_iy_j leads to

$$(4.9)\qquad As_0\alpha^2 + B(b^2\alpha^2 - \beta^2) = 0.$$

The term of (4.9) which seemingly does not contain α^2 is $4r_{00}\beta^{11}$. Hence we have $hp(11)$ v_{11} such that

$$(4.10)\qquad r_{00}\beta^{11} = \alpha^2 v_{11}.$$

Then it will be better to divide our consideration in to three cases:

(1^0) $v_{11} = 0$,
(2^0) $v_{11} \neq 0$, $\alpha^2 \not\equiv 0 \pmod{\beta}$,
(3^0) $v_{11} \neq 0$, $\alpha^2 \equiv 0 \pmod{\beta}$.

(1^0) Case of $v_{11} = 0$: From (4.10), $r_{00} = 0$, and (4.9) is reduced to

$$s_0\{A + B_1(b^2\alpha^2 - \beta^2)\} = 0, \tag{4.11}$$

where

$$\begin{aligned}B_1 = {} & 4(5\alpha^2\beta + 9\beta^3)(-5\alpha^2\beta^3 + 7\beta^5 + 3b^2\alpha^4\beta - 5b^2\alpha^2\beta^3)\\ & - 2(\alpha^2 + 9\beta^2)(\alpha^6 - 4\alpha^4\beta^2 + 3\alpha^2\beta^4 + 16\beta^6\\ & \qquad + 2b^2\alpha^6 - 2b^2\alpha^4\beta^2 - 12b^2\alpha^2\beta^4).\end{aligned}$$

If $A + B_1(b^2\alpha^2 - \beta^2) = 0$ in (4.11), then the term of this equation which does not contain α^2 is $-28\beta^{10}$. Therefore there exists $hp(8)$ v_8 such that $28\beta^{10} = \alpha^2 v_8$. In this case, if $\alpha^2 \not\equiv 0$ (mod. β), then we have $v_8 = 0$, which leads a contradiction. Therefore $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$. If $\alpha^2 \equiv 0$ (mod. β), then $A + B_1(b^2\alpha^2 - \beta^2) = 0$ is written as following

$$\begin{aligned}& 4\beta^2(-\delta^2 + 7\beta\delta + 12\beta^2)(-5\delta + 7\beta)\\ & \quad - (-\delta^2 + 25\beta^2)(\delta^3 - 4\beta\delta^2 + 3\beta^2\delta + 16\beta^3)\\ & -\beta\{4(5\delta + 9\beta)(-5\beta^2\delta + 7\beta^3)\\ & \quad - 2(\delta + 9\beta)(\delta^3 - 4\beta\delta^2 + 3\beta^2\delta + 16\beta^3)\} = 0,\end{aligned} \tag{4.12}$$

provided $b^2 = 0$ and $\alpha^2 = \beta\delta$ by Lemma. The term of (4.12) which does not contain β is δ^5. Therefore there exists $hp(4)$ v_4 such that $\delta^5 = \beta v_4$, which leads a contradiction. Hence $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$.

Therefore, in any case of $\alpha^2 \not\equiv 0$ (mod. β) or $\alpha^2 \equiv 0$ (mod. β), we see that $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$. Thus $s_0 = 0$ from (4.11). Substituting $s_0 = 0$ and $r_{00} = 0$ in (4.6), we have

$$A(s^i{}_0 y^j - s^j{}_0 y^i) = 0. \tag{4.13}$$

If $A = 0$, then we have from (4.7)

$$\begin{aligned}& 4\beta^2(-\alpha^4 + 7\alpha^2\beta^2 + 12\beta^4 - 5b^2\alpha^4 - 9b^2\alpha^2\beta^2)\\ & \quad \times(-5\alpha^2\beta^2 + 7\beta^4 + 3b^2\alpha^4 - 5b^2\alpha^2\beta^2)\\ & -(-\alpha^4 + 25\beta^4 - 2b^2\alpha^4 - 18b^2\alpha^2\beta^2)\\ & \quad \times(\alpha^6 - 4\alpha^4\beta^2 + 3\alpha^2\beta^4 + 16\beta^6 + 2b^2\alpha^6\\ & \qquad - 2b^2\alpha^4\beta^2 - 12b^2\alpha^2\beta^4) = 0.\end{aligned} \tag{4.14}$$

The term of (4.14) which seemingly does not contain α^2 is $-64\beta^{10}$. Thus, there exists $hp(8)$ w_8 such that $-64\beta^{10} = \alpha^2 w_8$. From this equation, we have $w_8 = 0$, which leads a contraction. Therefore $A \neq 0$. Thus we have from (4.13)

$$s^i{}_0 y^j - s^j{}_0 y^i = 0. \tag{4.15}$$

Transvection of (4.15) by y_j gives $s^i{}_0 = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.

(2^0): Case of $v_{11} \neq 0$, $\alpha^2 \not\equiv 0$ (mod. β): In this case, (4.10) shows that there exists a function $h = h(x)$ satisfying

$$r_{00} = h\alpha^2. \tag{4.16}$$

Substituting (4.16) in (4.9), we have

$$\begin{aligned} s_0 A + \Big[& 2\{h(-\alpha^4 + \alpha^2\beta^2 + 6\beta^4) + 2s_0(5\alpha^2\beta + 9\beta^3)\} \\ & \times (-5\alpha^2\beta^3 + 7\beta^5 + 3b^2\alpha^4\beta - 5b^2\alpha^2\beta^3) \\ & - \{h(5\beta^3 - 3\alpha^2\beta) + 2s_0(\alpha^2 + 9\beta^2)\} \\ & \times (\alpha^6 - 4\alpha^4\beta^2 + 3\alpha^2\beta^4 + 16\beta^6 + 2b^2\alpha^6 \\ & - 2b^2\alpha^4\beta^2 - 12b^2\alpha^2\beta^4)\Big](b^2\alpha^2 - \beta^2) = 0. \end{aligned} \tag{4.17}$$

The terms of (4.17) which seemingly do not contain α^2 are $-4(7s_0 + h\beta)\beta^{10}$. Hence there exists $hp(9)$ w_9 such that $(7s_0 + h\beta)\beta^{10} = \alpha^2 w_9$. Since $\alpha^2 \not\equiv 0$ (mod. β), we must have $w_9 = 0$. Thus we have

$$(7s_0 + h\beta)\beta^{10} = 0, \tag{4.18}$$

which implies $7s_i + hb_i = 0$. Transvecting this equation by b^i, we obtain

$hb^2 = 0$. If $b^2 = 0$, we get from (4.9) and (4.18)

$$\begin{aligned}(-h/7)\Big\{&4\beta^3(-\alpha^4+7\alpha^2\beta^2+12\beta^4)(-5\alpha^2\beta^2+7\beta^4)\\ &-(-\alpha^4+25\beta^4)(\alpha^6-4\alpha^4\beta^2+3\alpha^2\beta^4+16\beta^6)\Big\}\\ -\beta\Big[2\Big\{&h(-\alpha^4+\alpha^2\beta^2+6\beta^4)-(2h/7)\beta(5\alpha^2\beta+9\beta^3)\Big\}\\ &\times(-5\alpha^2\beta^3+7\beta^5)\\ &-\Big\{h(5\beta^3-3\alpha^2\beta)-(2h/7)\beta(\alpha^2+9\beta^2)\Big\}\\ &\times\Big\{\alpha^6-4\alpha^4\beta^2+3\alpha^2\beta^4+16\beta^6\Big\}\Big]=0.\end{aligned} \tag{4.19}$$

The term of (4.19) which seemingly does not contain β is $-(h/7)\alpha^{10}$. Therefore there exists $hp(9)$ u_9 such that $-(h/7)\alpha^{10} = \beta u_9$. Since $\alpha^2 \not\equiv 0$ (mod.β), we have $u_9 = 0$, which leads a contradiction. Thus we have $h = 0$ and hence $s_0 = 0, r_{00} = 0$ from (4.18) and (4.16) respectively. Therefore (4.6) is reduced to $A(s^i{}_0 y^j - s^j{}_0 y^i) = 0$. Since $A \neq 0$, we have $s^i{}_0 y^j - s^j{}_0 y^i = 0$. Transvection of this equation by y_j gives $s^i{}_0 = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i:j} = 0$.

(3^0) Case of $v_9 \neq 0$, $\alpha^2 \equiv 0$ (mod. β): In this case, Lemma shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. Therefore (4.6) gives

$$A'(s^i{}_0 y^j - s^j{}_0 y^i) + B'(b^i y^j - b^j y^i) = 0, \tag{4.20}$$

where

$$\begin{aligned}A' &= (\delta^5 - 4\beta\delta^4 + 2\beta^2\delta^3 - 52\beta^3\delta^2 + 119\beta^4\delta + 64\beta^5),\\ B' &= r_{00}(3\delta^4 - 7\beta\delta^3 - 5\beta^2\delta^2 - 13\beta^3\delta + 4\delta^4)\\ &\quad + 2s_0(-\delta^5 - 5\beta\delta^4 - 17\beta^2\delta^3 - 63\beta^3\delta^2 + 18\beta^4\delta).\end{aligned}$$

Transvection of (4.20) by $b_i y_j$ leads to

$$\begin{aligned}&s_0\delta(\delta^5 - 2\beta\delta^4 + 8\beta^2\delta^3 - 18\beta^3\delta^2 - 7\beta^4\delta + 100\beta^5)\\ &-r_{00}\beta(3\delta^4 - 7\beta\delta^3 - 5\beta^2\delta^2 - 13\beta^3\delta - 4\beta^4) = 0.\end{aligned} \tag{4.21}$$

The term of (4.21) which seemingly does not contain β is $s_0\delta^6$. Thus there exists $hp(6)$ λ_6 such that $s_0\delta^6 = \beta\lambda_6$, which implies $s_0 = k\beta$, where $k = k(x)$.

On the other hand, (4.10) gives $r_{00}\beta^{10} = \delta v_{11}$, which must be reduced to

$$r_{00} = \delta v_1, \tag{4.22}$$

where v_1 is a $hp(1)$. Substituting $s_0 = k\beta$ and (4.22) in (4.21), we have

$$\begin{aligned} &k(\delta^5 - 2\beta\delta^4 + 8\beta^2\delta^3 - 18\beta^3\delta^2 - 7\beta^4\delta + 100\beta^5) \\ &\quad -v_1(3\delta^4 - 7\beta\delta^3 + 5\beta^2\delta^2 - 13\beta^3\delta - 4\beta^4) = 0. \end{aligned} \tag{4.23}$$

The terms of (4.23) which do not contain β are $(k\delta - 3v_1)\delta^4$. Therefore there exists $hp(4)$ μ_4 such that $(k\delta - 3v_1)\delta^4 = \beta\mu_4$, which implies $k\delta - 3v_1 = f(x)\beta$. From this we obtain

$$v_1 = \frac{1}{3}(k\delta - f\beta). \tag{4.24}$$

Substituting of (4.24) in (4.23), we get

$$\begin{aligned} &k(\delta^4 + 19\beta\delta^3 - 41\beta^2\delta^2 - 17\beta^3\delta + 300\beta^4) \\ &\quad +f(3\delta^4 - 7\beta\delta^3 - 5\beta^2\delta^2 - 13\beta^3\delta - 4\beta^4) = 0. \end{aligned} \tag{4.25}$$

From (4.25) we see that there exists $hp(3)$ ν_3 such that $(k+3f)\delta^4 = \beta\nu_3$, from which

$$f = -\frac{k}{3}. \tag{4.26}$$

Substituting (4.26) in (4.25), we have

$$k(64\delta^3 - 118\beta\delta^2 - 38\beta^2\delta + 904\beta^3) = 0.$$

If $64\delta^3 - 118\beta\delta^2 - 38\beta^2\delta + 904\beta^3 = 0$, then there exists $hp(2)$ ϕ_2 such that $64\delta^3 = \beta\phi_2$, which leads a contradiction. Thus we have $k = 0$ and hence $s_0 = 0$. From (4.26) and (4.24), we get $f = 0$, $v_1 = 0$ respectively. Therefore we obtain $r_{00} = 0$. Substituting $s_0 = 0$ and $r_{00} = 0$ in (4.20), we get

$$A'(s^i{}_0 y^j - s^j{}_0 y^i) = 0. \tag{4.27}$$

If $A' = \delta^5 - 4\beta\delta^4 - 2\beta^2\delta^3 - 52\beta^3\delta^2 + 119\beta^4\delta + 64 = 0$, then there exists $hp(4)$ ψ_4 such that $\delta^5 = \beta\psi_4$. This leads a contradiction. Thus $A' \neq 0$ in (4.27). Hence we get $s^i{}_0 y^j - s^j{}_0 y^i = 0$. Transvection of this equation by y_j gives $s^i{}_0 = 0$. Thus $s_{ij} = r_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.

Conversely if $b_{i;j} = 0$, then we obtain $B^{ij} = 0$ from (2.4). Hence F^n is a Douglas space. Consequently we have

THEOREM 4.1. *An n-dimensional Finsler space F^n with the second approximate Matsumoto metric (2.2) is a Douglas space, if and only if* $b_{i;j} = 0$.

From Theorem 3.1 and Theorem 4.1, we have

THEOREM 4.2. *If an n-dimensional Finsler space F^n with the second approximate Matsumoto metric (2.2) is a Douglas space, then F^n is a Berwald space.*

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