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ON A CLASS OF ANALYTIC FUNCTIONS INVOLVING RUSCHEWEYH DERIVATIVES

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ABSTRACT. Let $A(p,k)$ $(p,k \in N)$ be the class of functions $f(z) = z^p + a_{p+k}z^{p+k} + \cdots$ analytic in the unit disk. We introduce a subclass $H(p,k,\lambda,\delta,A,B)$ of $A(p,k)$ by using the Ruscheweyh derivative. The object of the present paper is to show some properties of functions in the class $H(p,k,\lambda,\delta,A,B)$.

1. Introduction

Throughout this paper we assume that $p,k \in N = \{1,2,3,\cdots\}$, $\delta > 0, -1 \le B < A \le 1$ and $\sigma \ge 1$.

Let $A(p,k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m}z^{p+m}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. The Hadamard product or convolution $(f_1 * f_2)(z)$ of two functions

$$f_j(z) = z^p + \sum_{m=k}^{\infty} a_{p+m,j}z^{p+m} \in A(p,k) \quad (j = 1,2)$$

is given by $(f_1 * f_2)(z) = z^p + \sum\limits_{m=k}^{\infty} a_{p+m,1}a_{p+m,2}z^{p+m}$.

For $\lambda > -p$ and $f(z) \in A(p,k)$, we define

$$(1) \qquad D^{\lambda+p-1}f(z) = \frac{z^p}{(1-z)^{\lambda+p}} * f(z).$$

The symbol $D^{\lambda+p-1}$ when $p = 1$ was introduced by Ruscheweyh [5] and was named the Ruscheweyh derivative.

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A function $f(z) \in A(p,k)$ is said to be in $H(p,k,\lambda,\delta,A,B)$ if it satisfies

$$(1-\delta)\frac{D^{\lambda+p-1}f(z)}{z^p}+\delta\frac{D^{\lambda+p}f(z)}{z^p}\prec\frac{1+Az}{1+Bz}, \tag{2}$$

where $\lambda > -p$ and " $\prec$" stands for subordination.

For the class $H(p,k,\lambda,\delta,A,B)$, in this paper, we derive two sharp inequalities which extend and improve the corresponding results in [1], [2], and [3]. Further, we discuss the partial sums of certain functions in the class and establish a convolution theorem for the class $H(p,k,\lambda,\delta,A,B)$.

In the sequel, we shall write

$$J(p,\lambda,\delta,f,z)=(1-\delta)\frac{D^{\lambda+p-1}f(z)}{z^p}+\delta\frac{D^{\lambda+p}f(z)}{z^p}$$

and

$$\beta(p,k,b,\delta,A,B)=\frac{b+p}{k\delta}\int_0^1 u^{(b+p)/(k\delta)-1}\Big(\frac{1-Au}{1-Bu}\Big)du \quad (b>-p).$$

2. Two inequalities

We shall need the following lemma due to Miller and Mocanu [4].

LEMMA. *Let $h(z)$ be analytic and convex univalent in $E, h(0)=1$, and let $g(z)=1+b_kz^k+b_{k+1}z^{k+1}+\cdots$ be analytic in E. If*

$$g(z)+zg'(z)/c\prec h(z),$$

then for $c\neq 0$ and $\operatorname{Re} c\geq 0$

$$g(z)\prec\frac{c}{k}z^{-c/k}\int_0^z t^{c/k-1}h(t)dt.$$

Applying the above lemma, we derive

THEOREM 1. *Let $f(z)\in H(p,k,\lambda,\delta,A,B)(\lambda>-p)$. Then for $z\in E$*

$$\operatorname{Re}\left\{\left(\frac{D^{\lambda+p-1}f(z)}{z^p}\right)^{1/\sigma}\right\}>(\beta(p,k,\lambda,\delta,A,B))^{1/\sigma}. \tag{3}$$

The result is sharp.

Proof. From (1) we easily have

$$z(D^{\lambda+p-1}f(z))'=(\lambda+p)D^{\lambda+p}f(z)-\lambda D^{\lambda+p-1}f(z). \tag{4}$$

Let

$$g(z) = D^{\lambda+p-1}f(z)/z^p \tag{5}$$

for $f(z) \in H(p,k,\lambda,\delta,A,B)$. Then the function $g(z) = 1 + b_k z^k + b_{k+1}z^{k+1} + \cdots$ is analytic in E. Using (4) and (5) we obtain

$$\frac{D^{\lambda+p}f(z)}{z^p} = g(z) + \frac{zg'(z)}{\lambda+p}$$

and so

$$J(p,\lambda,\delta,f,z) = g(z) + \frac{\delta}{\lambda+p}zg'(z).$$

Thus (2) gives

$$g(z) + \frac{\delta}{\lambda+p}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Now an application of the lemma leads to

$$g(z) \prec \frac{\lambda+p}{k\delta}z^{-(\lambda+p)/(k\delta)}\int_0^z t^{(\lambda+p)/(k\delta)-1}\left(\frac{1+At}{1+Bt}\right)dt$$

or

$$\frac{D^{\lambda+p-1}f(z)}{z^p} = \frac{\lambda+p}{k\delta}\int_0^1 u^{(\lambda+p)/(k\delta)-1}\left(\frac{1+Auw(z)}{1+Buw(z)}\right)du, \tag{6}$$

where $w(z)$ is analytic in E with $w(0)=0$ and $|w(z)|<1$ $(z \in E)$.

In view of $-1 \le B < A \le 1$ and $\lambda > -p$, it follows from (6) that

$$\operatorname{Re}\frac{D^{\lambda+p-1}f(z)}{z^p} > \beta(p,k,\lambda,\delta,A,B) > 0 \quad (z \in E). \tag{7}$$

Therefore, with the aid of the elementary inequality $\operatorname{Re}(w^{1/\sigma}) \ge (\operatorname{Re} w)^{1/\sigma}$ for

$$\operatorname{Re} w > 0 \qquad \text{and} \qquad \sigma \ge 1,$$

the inequality (3) follows directly from (7).

To show the sharpness of (3), we take $f(z) \in A(p,k)$ defined by

$$\frac{D^{\lambda+p-1}f(z)}{z^p} = \frac{\lambda+p}{k\delta}\int_0^1 u^{(\lambda+p)/(k\delta)-1}\left(\frac{1+Auz^k}{1+Buz^k}\right)du.$$

For this function we find that

$$J(p,\lambda,\delta,f,z) = \frac{1+Az^k}{1+Bz^k}$$

and

$$\frac{D^{\lambda+p-1}f(z)}{z^p} \to \beta(p,k,\lambda,\delta,A,B) \quad \text{as} \quad z \to e^{i\pi/k}.$$

Hence the proof of the theorem is complete. □

REMARK 1. For $k = \delta = 1$, $\lambda = n + 1$, $n \in N \cup \{0\}$, $A = 1 - 2\alpha$, $B = -1$ and $\sigma = 2$, our result sharpens the main theorem of [1].

REMARK 2. Taking $\lambda = n$ to be any integer greater than $-p, k = \delta = \sigma = 1$, $A = 1 - 2\alpha$, $B = -1$, and replacing $f(z)$ by $zf'(z)/p$, Theorem 1 improves Theorem 1 of [3].

THEOREM 2. *Let* $f(z) \in A(p, k)$ *and let*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \quad (c > -p). \tag{8}$$

If

$$(1-\delta)\frac{D^{\lambda+p-1}F(z)}{z^p} + \delta\frac{D^{\lambda+p-1}f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (\lambda > -p), \tag{9}$$

then for $z \in E$

$$\text{Re}\left\{\left(\frac{D^{\lambda+p-1}F(z)}{z^p}\right)^{1/\sigma}\right\} > (\beta(p, k, c, \delta, A, B))^{1/\sigma}.$$

The result is sharp.

Proof. It follows from (8) that

$$(c+p)D^{\lambda+p-1}f(z) = cD^{\lambda+p-1}F(z) + z(D^{\lambda+p-1}F(z))'. \tag{10}$$

Let

$$G(z) = D^{\lambda+p-1}F(z)/z^p. \tag{11}$$

Then from (10), (11), and (9), we have

$$\begin{aligned} &(1-\delta)\frac{D^{\lambda+p-1}F(z)}{z^p} + \delta\frac{D^{\lambda+p-1}f(z)}{z^p} \\ &= G(z) + \frac{\delta}{c+p} zG'(z) \\ &\prec \frac{1+Az}{1+Bz}. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 1 and hence we omit it. □

REMARK 3. Putting $p = k = \delta = \sigma = 1$, $\lambda = n + 1$, $n \in N \cup \{0\}$, $A = 1 - 2\alpha$ and $B = -1$ in Theorem 2, we refine Theorem 3 (with $c > -1$) of [2].

REMARK 4. Taking $\lambda = n$ to be any integer greater than $-p, k = \delta = \sigma = 1$, $A = 1 - 2\alpha$, $B = -1$, and replacing $f(z)$ by $zf'(z)/p$, Theorem 2 improves Theorem 2 of [3].

3. Partial sums

THEOREM 3. *Let* $f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m} \in A(p,k), s_1(z) = z^p$ *and* $s_n(z) = z^p + \sum_{m=k}^{k+n-2} a_{p+m} z^{p+m} (n \geq 2)$. *Suppose that*

$$\sum_{m=k}^{\infty} c_m |a_{p+m}| \leq 1, \tag{12}$$

where

$$c_m = \frac{1-B}{A-B} \cdot \frac{(1-\delta)(\lambda+p)_m + \delta(\lambda+p+1)_m}{(1)_m} \tag{13}$$

and $(b)_m = b(b+1) \cdot (b+m-1)$.

(i) *If* $\lambda > -p$ *and* $-1 \leq B \leq 0$, *then* $f(z) \in H(p,k,\lambda,\delta,A,B)$.

(ii) *If* $\lambda \geq 1-p$, *then for* $z \in E$

$$\operatorname{Re}\frac{f(z)}{s_n(z)} > 1 - \frac{1}{c_{k+n-1}} \tag{14}$$

and

$$\operatorname{Re}\frac{s_n(z)}{f(z)} > \frac{c_{k+n-1}}{1+c_{k+n-1}}. \tag{15}$$

The estimates are sharp for $n \in N$.

Proof. For $\lambda > -p$, we have $(1-\delta)(\lambda+p)_m + \delta(\lambda+p+1)_m > (\lambda+p)_m > 0$ $(m \geq k)$ and

$$J(p,\lambda,\delta,f,z) = 1 + \sum_{m=k}^{\infty} \frac{(1-\delta)(\lambda+p)_m + \delta(\lambda+p+1)_m}{(1)_m} a_{p+m} z^m. \tag{16}$$

(i) For $-1 \le B \le 0$ and $z \in E$, it follows from (16), (13) and (12) that

$$\begin{aligned}
&\left|\frac{J(p,\lambda,\delta,f,z)-1}{A-BJ(p,\lambda,\delta,f,z)}\right| \\
&= \left|\frac{\sum_{m=k}^{\infty}\frac{(1-\delta)(\lambda+p)_m+\delta(\lambda+p+1)_m}{(1)_m}a_{p+m}z^m}{A-B-B\sum_{m=k}^{\infty}\frac{(1-\delta)(\lambda+p)_m+\delta(\lambda+p+1)_m}{(1)_m}a_{p+m}z^m}\right| \\
&\le \frac{\sum\limits_{m=k}^{\infty} c_m|a_{p+m}|}{1-B+B\sum\limits_{m=k}^{\infty} c_m|a_{p+m}|} \\
&\le 1,
\end{aligned}$$

which implies that $J(p,\lambda,\delta,f,z) \prec (1+Az)/(1+Bz)$. Hence $f(z) \in H(p,k,\lambda,\delta,A,B)$.

(ii) For $\lambda \ge 1-p$ and $m \ge k$, it is easy to verify that $c_{m+1} > c_m > 1$. Thus

$$\sum_{m=k}^{k+n-2}|a_{p+m}| + c_{k+n-1}\sum_{m=k+n-1}^{\infty}|a_{p+m}| \le \sum_{m=k}^{\infty} c_m|a_{p+m}| \le 1. \tag{17}$$

Let

$$p_1(z) = c_{k+n-1}\left\{\frac{f(z)}{s_n(z)} - \left(1-\frac{1}{c_{k+n-1}}\right)\right\}.$$

Then

$$p_1(z) = 1 + \frac{c_{k+n-1}\sum\limits_{m=k+n-1}^{\infty} a_{p+m}z^m}{1+\sum_{m=k}^{k+n-2} a_{p+m}z^m}$$

and it follows from (17) that

$$\begin{aligned}
&\left|\frac{p_1(z)-1}{p_1(z)+1}\right| \\
&\le \frac{c_{k+n-1}\sum_{m=k+n-1}^{\infty}|a_{p+m}|}{2-2\sum_{m=k}^{k+n-2}|a_{p+m}| - c_{k+n-1}\sum_{m=k+n-1}^{\infty}|a_{p+m}|} \\
&\le 1 \quad (z \in E).
\end{aligned}$$

From this we obtain the inequality (14).

If we take

$$f(z) = z^p - \frac{z^{p+k+n-1}}{c_{k+n-1}}, \tag{18}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^{k+n-1}}{c_{k+n-1}} \to 1 - \frac{1}{c_{k+n-1}} \quad \text{as} \quad z \to 1.$$

This shows that the bound in (14) is best possible for each n.

Similarly, if we put

$$p_2(z) = (1 + c_{k+n-1}) \left(\frac{s_n(z)}{f(z)} - \frac{c_{k+n-1}}{1 + c_{k+n-1}} \right),$$

then we deduce that

$$\begin{aligned} &\left| \frac{p_2(z) - 1}{p_2(z) + 1} \right| \\ &\leq \frac{(1 + c_{k+n-1}) \sum_{m=k+n-1}^{\infty} |a_{p+m}|}{2 - 2\sum_{m=k}^{k+n-2} |a_{p+m}| + (1 - c_{k+n-1}) \sum_{m=k+n-1}^{\infty} |a_{p+m}|} \\ &\leq 1 \quad (z \in E), \end{aligned}$$

which yields (15). The estimate (15) is sharp for each n, with the extremal function $f(z)$ given by (18). The proof is now complete. □

4. Convolution property

Let $P(\alpha, k)$ $(0 \leq \alpha < 1)$ denote the class of functions $p(z) = 1 + b_k z^k + b_{k+1} z^{k+1} + \cdots$ which are analytic in E and satisfy $\text{Re}\,p(z) > \alpha$ $(z \in E)$. It is well known that if $p(z) \in P(\alpha, k)$, then for $z \in E$

$$\text{Re}\,p(z) \geq 2\alpha - 1 + \frac{2(1-\alpha)}{1 + |z|^k}. \tag{19}$$

THEOREM 4. *Let $\lambda > -p$, $-1 \leq B_j < A_j \leq 1$, and $f_j(z) \in H$ $(p, k, \lambda, \delta, A_j, B_j)$ $(j = 1, 2)$. Then*

$$D^{\lambda+p-1}(f_1 * f_2)(z) \in H(p, k, \lambda, \delta, 1 - 2\alpha_0, -1), \tag{20}$$

where

$$\alpha_0 = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{\lambda + p}{k\delta} \int_0^1 \frac{u^{(\lambda+p)/(k\delta)-1}}{1+u} du \right).$$

The result is sharp when $B_1 = B_2 = -1$.

Proof. Since $f_j(z) \in H(p,k,\lambda,\delta,A_j,B_j)$, it follows that

$$p_j(z) \equiv J(p,\lambda,\delta,f_j,z) \in P(\alpha_j,k), \alpha_j = (1-A_j)/(1-B_j),$$

and

$$D^{\lambda+p-1}f_j(z) = \frac{\lambda+p}{\delta} z^{p-(\lambda+p)/\delta} \int_0^z t^{(\lambda+p)/\delta-1} p_j(t)dt \quad (j=1,2). \tag{21}$$

Let $f_0(z) = D^{\lambda+p-1}(f_1 * f_2)(z)$. Then, using (21), a computation shows that

$$D^{\lambda+p-1}f_0(z) = \frac{\lambda+p}{\delta} z^{p-(\lambda+p)/\delta} \int_0^z t^{(\lambda+p)/\delta-1} p_0(t)dt, \tag{22}$$

where

$$p_0(z) = \frac{\lambda+p}{\delta} z^{-(\lambda+p)/\delta} \int_0^z t^{(\lambda+p)/\delta-1} (p_1 * p_2)(t)dt. \tag{23}$$

In view of $p_1(z) \in P(\alpha_1,k)$ and

$$p_2^*(z) = \frac{p_2(z)-\alpha_2}{2(1-\alpha_2)} + \frac{1}{2} \in P(\frac{1}{2},k),$$

we can deduce that $(p_1 * p_2^*)(z) \in P(\alpha_1,k)$ by using the Herglotz formula. Thus

$$(p_1 * p_2)(z) \in P(\alpha_3,k), \quad \alpha_3 = 1-2(1-\alpha_1)(1-\alpha_2). \tag{24}$$

Now, it follows from (22), (23), (24), and (19) that

$$\begin{aligned}
&\mathrm{Re}J(p,\lambda,\delta,f_0,z)\\
&= \mathrm{Re}p_0(z)\\
&= \frac{\lambda+p}{k\delta}\int_0^1 u^{(\lambda+p)/(k\delta)-1}\mathrm{Re}\{(p_1 * p_2)(u^{1/k}z)\}du\\
&\geq \frac{\lambda+p}{k\delta}\int_0^1 u^{(\lambda+p)/(k\delta)-1}\left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u|z|^k}\right)du\\
&> \frac{\lambda+p}{k\delta}\int_0^1 u^{(\lambda+p)/(k\delta)-1}\left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u}\right)du\\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{\lambda+p}{k\delta}\int_0^1 \frac{u^{(\lambda+p)/(k\delta)-1}}{1+u}du\right)\\
&= \alpha_0 \quad (z \in E).
\end{aligned}$$

This proves (20).

When $B_1 = B_2 = -1$ we consider $f_j(z) \in H(p, k, \lambda, \delta, A_j, -1)$ given by

$$\frac{D^{\lambda+p-1} f_j(z)}{z^p} = \frac{\lambda+p}{\delta} z^{-(\lambda+p)/\delta} \int_0^z t^{(\lambda+p)/\delta-1} \left(\frac{1+A_j t^k}{1-t^k}\right) dt \quad (j = 1, 2).$$

Note that

$$\left(\frac{1+A_1 z^k}{1-z^k}\right) * \left(\frac{1+A_2 z^k}{1-z^k}\right) = 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z^k}.$$

Then it follows that

$$J(p, \lambda, \delta, f_0, z) = \frac{\lambda+p}{k\delta} \int_0^1 u^{(\lambda+p)/(k\delta)-1} \times \left\{1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz^k}\right\} du,$$

where $f_0(z) = D^{\lambda+p-1}(f_1 * f_2)(z)$. Therefore

$$J(p, \lambda, \delta, f_0, z) \to 1 - (1+A_1)(1+A_2)\left\{1 - \frac{\lambda+p}{k\delta} \int_0^1 \frac{u^{(\lambda+p)/(k\delta)-1}}{1+u} du\right\}$$

as $z \to e^{i\pi/k}$. The proof is complete. □

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