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# AN INTEGRAL FORMULA AND ITS APPLICATIONS

Y. D. CHAI AND MOONJEONG KIM

ABSTRACT. In this paper, we obtain an integral formula relating the measure of great spheres $S^{n-2}$ and arc length of a curve on the unit sphere $S^{n-1}$. As an application of the formula, we develop a geometric inequality for a spherical curve and prove generalized version of Fenchel's theorem in $R^n$.

## 1. Introduction

Fenchel's theorem states that $\int \kappa ds \geq 2\pi$, with equality if and only if the curve is a convex plane curve. J. W. Milnor [4] reproved the result with a different method and many other mathematicians generalized the result to $R^n$.

The main purpose of this paper is to provide a simple and shorter proof than those previously known. Also, we present a geometric inequality for a spherical curve. The proofs are based on the Crofton's formula on the measure of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$.

## 2. Preliminaries

Let $\alpha = \{(c_1(s), c_2(s), \cdots, c_n(s)) |\ 0 \leq s \leq l\}$ be a closed curve in $R^n$ with arc length parameters.

For a Frenet frame

$$(\alpha(s), e_1(s), \cdots, e_n(s))$$

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of the curve $\alpha$, the Frenet equations are

$$\begin{aligned}\frac{d\alpha}{ds} &= e_1,\\ \frac{de_i}{ds} &= -\kappa_{i-1}(s)e_{i-1} + \kappa_i(s)e_{i+1}.\end{aligned}$$

Here, $\kappa_i$'s are curvatures. Then the unit tangent vector $e_1(s) = \frac{d}{ds}(c_1(s), c_2(s), \cdots, c_n(s)) = (T_1(s), T_2(s), \cdots, T_n(s))$ defines a tangent curve on the unit sphere $S^{n-1}$. The total curvature of $\alpha$ is defined to be the quantity $\int_0^l \kappa_1(s)ds$, $\kappa_1(s) = \left\|\frac{de_1(s)}{ds}\right\|$. Thus the total curvature of $\alpha$ is the length of its tangent curve.

The following lemma shows how tangent curve of a closed curve ranges.

LEMMA 1. *Tangent curve $e_1(s)$ of a closed curve is not contained in any open hemisphere. $e_1$ is contained in a closed hemisphere if and only if $\alpha$ is in a hyperplane.*

*Proof.* If $e_1$ were contained in a hemisphere, we may assume $T_n(s) \geq 0$ for all $0 \leq s \leq l$. Since $\alpha = (c_1(s), c_2(s), \cdots, c_n(s))$ is a closed curve,

$$0 = c_n(l) - c_n(0) = \int_0^l T_n(s)ds. \tag{1}$$

Thus $T_n(s)$ cannot be strictly positive. Hence, $e_1$ cannot lie in an open hemisphere. From (1), since $T_n(s)$ is nonnegative, it must vanish identically, that is, $0 = T_n(s) = \frac{d}{ds}c_n(s)$. Hence $\alpha$ must be in a hyperplane $c_n(s) =$ constant. Conversely, if $\alpha$ is in a hyperplane, then $e_1$ lies on a great sphere $S^{n-2}$ and hence is contained in a closed hemisphere. □

Every oriented great sphere determines uniquely a pole, the endpoint of the unit vector normal to the unit sphere $S^{n-2}$. So the following definition is meaningful.

DEFINITION 1. The area of the domain of their poles is meant by the measure of a set of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$.

## 3. Result

Now we will give the measure of a set of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$ to develop an integral formula concerning an arc on the unit sphere $S^{n-1}$.

THEOREM 1. *Let $\gamma$ be a smooth arc on the unit sphere $S^{n-1}$. The measure of the oriented great spheres $S^{n-2}$ of $S^{n-1}$ which meet $\gamma$, each counted a number of times equal to the number of its common points with $\gamma$, is equal to $\frac{Vol(S^{n-1})}{\pi}$ times the length of $\gamma$.*

*Proof.* We suppose $\gamma$ is defined by a unit vector $e_1(s)$ expressed as a function of its arc length $s$. In a certain neighborhood of $s$, we take a frame field $\{e_2(s), \cdots, e_n(s)\}$ as follows; we set $\{e_2(s), e_3(s)\}$ such that $\frac{de_1(s)}{ds} = a_{12}e_2 + a_{13}e_3$, $\{e_2(s), \cdots, e_n(s)\}$ satisfies $e_i \cdot e_j = \delta_{ij}, 1 \le i, j \le n$ and $\det(e_1, e_2, \cdots, e_n) = +1$. From differentiation of $e_i \cdot e_j = \delta_{ij}$ and $\{e_2(s), e_3(s)\}$, we obtain the skew-symmetric matrix of the coefficients as follows;

$$\begin{pmatrix} \frac{de_1}{ds} \\ \frac{de_2}{ds} \\ \frac{de_3}{ds} \\ \vdots \\ \frac{de_i}{ds} \\ \vdots \\ \frac{de_n}{ds} \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & \cdots & 0 \\ -a_{12} & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -a_{2i} & -a_{3i} & -a_{4i} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -a_{2n} & -a_{3n} & -a_{4n} & \cdots & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{pmatrix}.$$

If an oriented great sphere $S^{n-2}$ meets $\gamma$ at the point $e_1(s)$, its pole is of the form

$$\begin{aligned} & \phi(s, \theta_1, \cdots, \theta_{n-2}) \\ = \ & (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2}) e_2(s) + \cdots \\ & + (\cos\theta_{i-2} \sin\theta_{i-1} \cdots \sin\theta_{n-2}) e_i(s) + \cdots + \cos\theta_{n-2} e_n(s), \end{aligned}$$

where $0 \le \theta_1 \le 2\pi, 0 \le \theta_2, \cdots, \theta_{n-2} \le \pi$.

Thus $(s, \theta_1, \cdots, \theta_{n-2})$ serve as local coordinates in the domain of these poles, we wish to find an expression for the element of area of this domain.

$$\begin{aligned}\phi_s &= (\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2})e_2'(s)+\cdots\\ &\quad+(\cos\theta_{i-2}\sin\theta_{i-1}\cdots\sin\theta_{n-2})e_i'(s)+\cdots+\cos\theta_{n-2}e_n'(s)\\ &= -(a_{12}\sin\theta_1\cdots\sin\theta_{n-2}+a_{13}\cos\theta_1\cdots\sin\theta_{n-2})e_1+\\ &\quad(a_{23}\sin\theta_1\cdots\sin\theta_{n-2}+\cdots-a_{2n}\cos\theta_{n-2})e_2+\cdots\\ &\quad+(a_{2i}\sin\theta_1\cdots\sin\theta_{n-2}+\cdots-a_{in}\cos\theta_{n-2})e_i+\cdots\\ &\quad+(a_{2n}\sin\theta_1\cdots\sin\theta_{n-2}+\cdots-a_{(n-1)n}\cos\theta_{n-2})e_n.\end{aligned}$$

$$\begin{aligned}\phi_{\theta_1} &= \cos\theta_1\sin\theta_2\cdots\sin\theta_{n-2}e_2-\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}e_3\\ &\vdots\\ \phi_{\theta_{n-2}} &= \sin\theta_1\sin\theta_2\cdots\cos\theta_{n-2}e_2+\cdots-\sin\theta_{n-2}e_n.\end{aligned}$$

Hence, the element of area of $\phi$ is

$$\begin{aligned}|dA| &= (\det(E_i\cdot E_j))^{\frac{1}{2}}\\ &= \sin^2\theta_2\sin^3\theta_3\cdots\sin^{n-2}\theta_{n-2}|a_{12}\sin\theta_1+a_{13}\cos\theta_1|dsd\theta_1\cdots d\theta_{n-2},\end{aligned}$$

where $0\le\theta_1\le 2\pi, 0\le\theta_2,\cdots,\theta_{n-2}\le\pi$.
On the other hand, since $s$ is the arc length of $\gamma$, we have

$$a_{12}^2+a_{13}^2=1,$$

and we put $a_{12}=\cos\rho(s), a_{13}=\sin\rho(s)$, for some $\rho(s)$. Then

$$\begin{aligned}|dA| &= \sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}|a_{12}\sin\theta_1+a_{13}\cos\theta_1|dsd\theta_1\cdots d\theta_{n-2}\\ &= \sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}|\cos\rho(s)\sin\theta_1+\sin\rho(s)\cos\theta_1|dsd\theta_1\cdots d\theta_{n-2}\\ &= \sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}|\sin(\rho(s)+\theta_1)|dsd\theta_1\cdots d\theta_{n-2}.\end{aligned}$$

Let $X$ be the oriented great sphere $S^{n-2}$ with $\phi$ as its pole, and let $n(X)$ be the number of points common to $X$ and $\gamma$. Then the measure of oriented great spheres $S^{n-2}$ meeting $\gamma$ in our theorem is given by

$$\begin{aligned}\int n(X)|dA| &= \int_0^l ds\int_0^\pi\cdots\int_0^{2\pi}\\ &\qquad|\sin(\rho(s)+\theta_1)|\sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}d\theta_1\cdots d\theta_{n-2}\\ &= 4l\int_0^\pi\cdots\int_0^\pi\sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}d\theta_2\cdots d\theta_{n-2}.\end{aligned}$$

But

$$\mathrm{Vol}(S^{n-1}) = \int_0^{\pi} \cdots \int_0^{2\pi} |\sin\tau_2| \sin^2\tau_3 \sin^3\tau_4 \cdots \sin^{n-2}\tau_{n-1} d\tau_1 \cdots d\tau_{n-1}$$
$$= 2\pi \cdot 2 \int_0^{\pi} \cdots \int_0^{\pi} \sin^2\tau_3 \sin^3\tau_4 \cdots \sin^{n-2}\tau_{n-1} d\tau_3 \cdots d\tau_{n-1}.$$

Thus

$$\int_0^{\pi} \cdots \int_0^{\pi} \sin^2\tau_3 \sin^3\tau_4 \cdots \sin^{n-2}\tau_{n-1} d\tau_3 \cdots d\tau_{n-1} = \frac{\mathrm{Vol}(S^{n-1})}{4\pi}.$$

Therefore, $\int n(X)|dA| = \dfrac{\mathrm{Vol}(S^{n-1})}{\pi} l.$ □

COROLLARY 1. *Let $\alpha$ be a closed curve in $R^n$. Then $\int \kappa ds \geq 2\pi$, with equality if and only if the curve is a convex plane curve.*

*Proof.* We have known that the total curvature of $\alpha$ is the length of its tangent curve. Furthermore, from Lemma 1, the tangent curve of a closed curve meets every great sphere $S^{n-2}$ at least two points. So $n(X) \geq 2$. It follows that its length is

$$l = \int \kappa ds = \frac{\pi}{\mathrm{Vol}(S^{n-1})} \int n(X)|dA| \geq \frac{\pi}{\mathrm{Vol}(S^{n-1})} \cdot 2 \cdot \mathrm{Vol}(S^{n-1}) = 2\pi.$$

It remains to prove the second part of theorem.

If $\alpha$ is a plane convex curve, then $e_1$ is contained in a closed hemisphere. Furthermore, $e_1$ lies on a great circle. Since $\alpha$ is convex, $e_1$ is a great circle. Thus $n(X)$ is equal to 2. If $\alpha$ is not a plane convex curve, $e_1$ is not contained in any closed hemisphere from Lemma 1. So for some position of $S^{n-1}$, $n(X) > 2$. Hence $\int \kappa ds > 2\pi$. This completes the proof of our corollary. □

A non-oriented geodesic $C$ on $S^2$ can be determined by one of its poles, that is , by either of the extremities of the diameter perpendicular to it. We consider a fixed geodesic $C_0$ and a fixed point $P$ on it. The geodesic $C$ can be determined for the abscissa $t$ of one of the intersection points from $C$ and $C_0$ and the angle $\phi$ between the two circles. From [5], we have the density for measuring sets of geodesics on $S^2$ as follows;

$$dC = \sin\phi \, d\phi \, dt. \tag{2}$$

Let $K$ be a spherical oriented closed curve and $C_0$ be a fixed geodesic. Let $\tau(s)$ be the angle between $C_0$ and the geodesic tangent to $K$ at

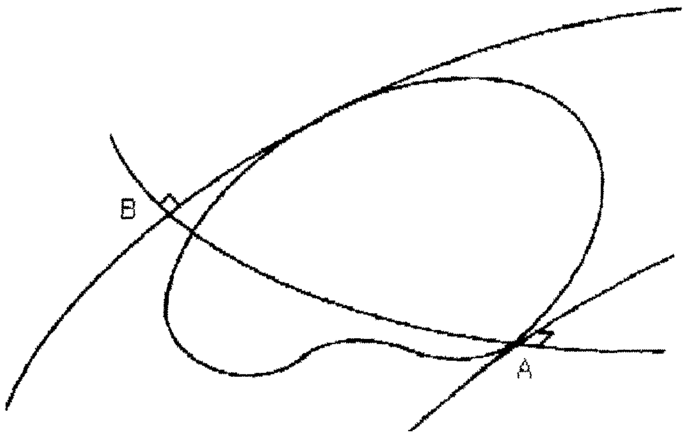


FIGURE 1.

$K(s)$ parametrized by arc length $s$ of $K$. Then the curvature of $K$ at $s$ is defined by $\kappa = \frac{d\tau}{ds}$ and the *absolute total curvature* is defined by the integral

$$c_a = \int_K |\kappa| ds = \int_K |d\tau|. \tag{3}$$

The curvature is assigned a magnitude, measuring the rate of deviation from geodesic-aheadness. The absolute total curvature is a quantity which measures the total turning of the tangent geodesic.

The *breadth* corresponding to a point $A$ of an oriented closed curve $K$ on $S^2$ is equal to the length of arc $AB$ (Figure 1) of the geodesic orthogonal to $K$ at the point $A$ which is comprehended between $A$ and the point of intersection with another geodesic tangent to $K$ also orthogonal to $AB$ and contain $K$ between two tangent geodesics.

The following inequality is a generalization to the sphere of the inequality by Fáry [3] obtained for plane curves.

COROLLARY 2. *If a closed curve of length $L$ on the sphere $S^2$ with absolute total curvature $c_a$ can be enclosed by a spherical circle of radius $\rho$, then $L \leq \rho c_a$.*

*Proof.* Let $K$ be a spherical oriented closed curve, so that there is a prescribed sense of rotation. Let $C_0$ be a fixed geodesic. Let $s$ denote the arc length of $K$ and let $\tau(s)$ be the angle between the tangent geodesic to $K$ and a fixed geodesic. Let $v(\tau)$ denote the number of unoriented tangents to $K$ that has the direction $\tau$. Since each direction $\tau$ appears $v(\tau)$ times, the equation (3) can be written

$$c_a = \int_0^\pi v(\tau)|d\tau|.$$

On the other hand, if a geodesic $C$ has the direction $\tau$ and meets $K$ in $n$ points $p_i$, then there are at least $n$ tangents to $K$ that has the direction $\tau$ ( one for each of the arcs $p_1p_2, p_2p_3, \cdots, p_{n-1}p_n, p_np_1$) and thus $n(\tau) \leq v(\tau)$. From our theorem for non-directed geodesics and (2), we have

$$\begin{aligned} 2L = \int_{C\cap K\neq\emptyset} ndC &= \int dt \int n|\sin\tau d\tau| \\ &\leq \triangle_m \int_0^\pi v|d\tau| = \triangle_m c_a \end{aligned}$$

where $L$ is the length of $K$ which lies inside a spherical circle of radius $\rho$. Since $\triangle_m$ is the maximal breadth of $K$, $\triangle_m \leq 2\rho$. So $L \leq \rho c_a$. □

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Department of Mathematics, Sungkyunkwan University, Suwon 440–746, Korea

*E-mail*: ydchai@yurim.skku.ac.kr
kmj@math.skku.ac.kr