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# CELLULAR ALGEBRAS AND CENTERS OF HECKE ALGEBRAS

YEON-KWAN JEONG, IN-SOK LEE, HYEKYUNG OH, AND KYUNG-HWAN PARK

ABSTRACT. In this short note, we find bases of the centers of generic Hecke algebras associated with certain finite Coxeter groups. Our bases are described using the notion of cell datum of Graham and Lehrer, and the notion of norm.

## 1. Introduction

For any ring $R$ and any finite Weyl group $W$, the set of conjugacy class sums of a group algebra $R[W]$ forms a basis of the center of $R[W]$. We know that there is the Lusztig isomorphism between the group algebra of $W$ over $\mathbb{Q}(q^{1/2})$ and the Hecke algebra $\mathcal{H}(W)$ over $\mathbb{Q}(q^{1/2})$, where $q$ is an indeterminate over $\mathbb{Q}$ and $W$ is an indecomposable Weyl group (see [9]). Therefore, it is natural to ask if there is a 'generic' analogue (or a $q$-analogue) of the above basis of the center of $R[W]$. In [3], using the class polynomials, Geck and Rouquier found such a basis of the center of $\mathcal{H}(W)$, where $\mathcal{H}(W)$ is the Hecke algebra associated with $W$ over $\mathbb{Z}[q, q^{-1}]$ with $q$ an indeterminate. On the other hand, for the Hecke algebra of the symmetric group $S_n$ over $\mathbb{Q}[q, q^{-1}]$, $q$ an indeterminate, Jones ([7]) found a basis of the center of $\mathcal{H}(S_n)$ over $\mathbb{Q}[q, q^{-1}]$, which is an another generic analogue of conjugacy class sums (see [7]). Finding such a basis, Jones used the concept of norm which is also an important tool in the present paper (see Lemma 2.2). Furthermore, in [10], [1], and [6], a complete set of primitive central idempotents is described for Hecke

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algebra of type $A_n$, $B_n$ and $D_{2n+1}$, respectively, using Jucy-Murphy type elements.

The motivation of this paper is the following question: What are canonical bases of the centers for the Hecke algebras $\mathcal{H}(S_n)$ over $\mathbb{Q}(q)$, $q$ an indeterminate, which not only are $q$-analogues of the conjugacy class sum bases but also can be described in terms of the Kazhdan-Lusztig bases $\{C_w \mid w \in S_n\}$? (see [8] for the Kazhdan-Lusztig bases.)

In Proposition 2.4, we first observe that any basis of the center consists of the elements

$$\mathcal{N}_{S_n}(h) = \sum_{w \in S_n} q^{-\ell(w)} T_{w^{-1}} h T_w$$

for $h \in \mathcal{H}(S_n)$. An answer to the above question can be given as a special case of Theorem 3.7. In fact, if we choose only one standard tableau $S(\lambda)$ for each partition $\lambda$ of $n$, and if $w \sim (S(\lambda), S(\lambda))$ denotes the Robinson-Schensted correspondence (For example, see [11] for the Robinson-Schensted correspondence.) then

$$\{\mathcal{N}_{S_n}(C_w) \mid \lambda \text{ is a partition of } n,\ w \sim (S(\lambda), S(\lambda))\}$$

is a basis of the center of $\mathcal{H}(S_n)$.

So, we have the same question for Hecke algebras of type other than $A_n$. Since the cell structure of Hecke algebra of type other than $A_n$ does not play such a role as that of $\mathcal{H}(S_n)$, we have to modify our question. We found an essential clue in [4], where Graham and Lehrer introduced the concept of cellular algebra and showed that the Kazhdan-Lusztig basis $\{C_w\}$ is a cellular basis for the Hecke algebra of type $A_n$ (see [4]). This observation led us to consider cellular bases instead of the Kazhdan-Lusztig bases in the above question.

In Theorem 3.7, we answer this modified question when $\mathcal{H}(W)$ is the Hecke algebra of type $A_n, B_n, D_{2n+1}$ or dihedral group. These algebras are cellular algebras and their cell data $(\Lambda, M, C, *)$ are described in [10], [1], [6] and [2], respectively.

The most significant aspect of cellular algebra is that one can systematically understand the irreducible representations of Hecke algebra using its cellular basis (see [4]). Our result shows that a basis of the center can be also described in terms of cell datum.

## 2. Centers of Hecke algebras

A Coxeter system $(W, S)$ is a pair consisting of a group $W$ and a set of generators $S \subset W$, subject only to the following relations:

$$(ss')^{m(s,s')} = 1,$$

where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in $S$. Since the generators $s \in S$ have order 2 in $W$, each $w$ in $W$ can be written in the form $w = s_1 s_2 ... s_r$ for some $s_i$ in $S$. We define the length $\ell(w)$ of $w$ to be the smallest $r$ for which such an expression exists, and call the expression reduced. By convention, we put $\ell(1) = 0$. (For more details, see, for example, [5].)

DEFINITION 2.1. Let $R$ be an integral domain, $q$ be an invertible element of $R$ and $(W, S)$ be a finite Coxeter system. The *Hecke algebra* $\mathcal{H}(W)$ of $W$ over $R$ with respect to $q$ is defined to be a free $R$-module with basis $\{T_w \mid w \in W\}$ with the following associative $R$-algebra structure:

(1) $T_1$ acting as the identity,
(2) $T_s T_w = T_{sw}$ if $\ell(sw) > \ell(w)$,
(3) $T_s T_w = (q-1)T_w + qT_{sw}$ if $\ell(sw) < \ell(w)$

for $s \in S$, $w \in W$. By convention, we write $\alpha$ to denote $\alpha T_1$ for $\alpha \in R$. Let $*$ be the $R$-linear anti-automorphism of $\mathcal{H}(W)$ for which $T_w^* = T_{w^{-1}}$.

In [7], Jones proved the following result using the concept of Frobenius algebra. In this paper, we provide an elementary proof.

LEMMA 2.2 ([7, Lemma 2.4]). *Let $(W, S)$ be a finite Coxeter system and $\mathcal{H}(W)$ be the Hecke algebra corresponding to $W$. Define the norm $\mathcal{N}_W(h)$ by*

$$\mathcal{N}_W(h) = \sum_{w \in W} q^{-\ell(w)} T_{w^{-1}} h T_w,$$

*for $h \in \mathcal{H}(W)$, then $\mathcal{N}_W(h)$ is contained in the center of $\mathcal{H}(W)$.*

*Proof.* Put $A_s = \{w \in W \mid \ell(ws) < \ell(w)\}$ for $s \in S$ and $B_s = W - A_s$. We may denote $\mathcal{N}_W(h) = \mathcal{N}_{A_s}(h) + \mathcal{N}_{B_s}(h)$, where $\mathcal{N}_{A_s}(h) = \sum_{w \in A_s} q^{-\ell(w)} T_{w^{-1}} h T_w$ and $\mathcal{N}_{B_s}(h) = \sum_{w \in B_s} q^{-\ell(w)} T_{w^{-1}} h T_w$. Then we have

$$
\begin{aligned}
&T_s\mathcal{N}_{A_s}(h)\\
&= T_s \sum_{w\in A_s} q^{-\ell(w)} T_{w^{-1}} h T_w\\
&= \sum_{w\in A_s} q^{-\ell(w)} T_s T_s T_{v^{-1}} h T_w \quad \text{(where } w = vs \text{ and } \ell(w) = \ell(v)+1)\\
&= \sum_{w\in A_s} q^{-\ell(w)} ((q-1)T_s + q) T_{v^{-1}} h T_v T_s\\
&= (q-1)\sum_{w\in A_s} q^{-\ell(w)} T_s T_{v^{-1}} h T_v T_s + \left(\sum_{w\in A_s} q^{-\ell(w)+1} T_{v^{-1}} h T_v\right) T_s\\
&= (q-1)\mathcal{N}_{A_s}(h) + \left(\sum_{v\in B_s} q^{-\ell(v)} T_{v^{-1}} h T_v\right) T_s\\
&= (q-1)\mathcal{N}_{A_s}(h) + \mathcal{N}_{B_s}(h) T_s.
\end{aligned}
$$

Similarly, we get

$$\mathcal{N}_{A_s}(h)T_s = (q-1)\mathcal{N}_{A_s}(h) + T_s\mathcal{N}_{B_s}(h).$$

Thus we conclude that

$$
\begin{aligned}
T_s\mathcal{N}_W(h) &= T_s\mathcal{N}_{A_s}(h) + T_s\mathcal{N}_{B_s}(h)\\
&= (q-1)\mathcal{N}_{A_s}(h) + \mathcal{N}_{B_s}(h)T_s + T_s\mathcal{N}_{B_s}(h)\\
&= \mathcal{N}_{A_s}(h)T_s + \mathcal{N}_{B_s}(h)T_s\\
&= \mathcal{N}_W(h)T_s.
\end{aligned}
$$

□

In the rest of the present section, we assume that $F$ is a field of characteristic zero. Let $F[q, q^{-1}]$ be the ring of Laurent polynomials over $F$. There is a natural $F$-algebra homomorphism

$$e : \mathcal{H}_{F[q,q^{-1}]}(W) \to F[W],$$

which maps $T_w$ to $w$ for $w \in W$ and $q$ to 1, where $\mathcal{H}_{F[q,q^{-1}]}(W)$ is the Hecke algebra of $W$ over $F[q, q^{-1}]$ and $F[W]$ is the group algebra of $W$ over $F$. When we apply the map $e$ to the elements of $\mathcal{H}_{F[q,q^{-1}]}(W)$, we simply say "set $q = 1$". Also, we write $e(h) = h[1]$ for $h \in \mathcal{H}_{F[q,q^{-1}]}(W)$.

REMARK 2.3. If we consider the $F[q,q^{-1}]$-linear operator $L_h$ on $\mathcal{H}_{F[q,q^{-1}]}(W)$ defined by $L_h(x) = hx$ for $x, h \in \mathcal{H}_{F[q,q^{-1}]}(W)$ and the $F$-linear operator $L_{h[1]}$ on $F[W]$ defined by $L_{h[1]}(a) = h[1]\ a$ for $h \in \mathcal{H}_{F[q,q^{-1}]}(W)$, $a \in F[W]$, then $e$ maps the determinant of $L_h$ to the determinant of $L_{h[1]}$. So, if $h[1] \in F[W]$ is an invertible element, that is, the determinant of $L_{h[1]}$ is not zero, then $h \in \mathcal{H}_{F[q,q^{-1}]}(W)$ is an invertible element in $F(q) \otimes_{F[q,q^{-1}]} \mathcal{H}_{F[q,q^{-1}]}(W) = \mathcal{H}(W)$, where $\mathcal{H}(W)$ is the Hecke algebra of $W$ over $F(q)$.

PROPOSITION 2.4. *Let $W$ be a finite Coxeter group and $\mathcal{H}(W)$ be the Hecke algebra corresponding to $W$ over $F(q)$, $q$ an indeterminate and $F$ a field with characteristic zero. Then the center of $\mathcal{H}(W)$ is equal to*

$$\{\mathcal{N}_W(h) \mid h \in \mathcal{H}(W)\}.$$

*Proof.* Note that the elements $\mathcal{N}_W(h) = \sum_{w \in \mathcal{H}(W)} q^{-\ell(w)} T_{w^{-1}} h T_w$ are contained in the center by Lemma 2.2 and $\mathcal{N}_W(1)[1] = |W| \in F[W]$ is an invertible element. So, $\mathcal{N}_W(1)$ is an invertible element in $\mathcal{H}(W)$ by Remark 2.3 and its inverse element is contained in the center. Let $c = \sum_{w \in W} a_w T_w$ be an element of center where $a_w \in F(q)$, then

$$c\,\mathcal{N}_W(1) = \mathcal{N}_W(c) = \sum_{w \in W} a_w \mathcal{N}_W(T_w).$$

Therefore we get

$$c = \left(\sum_{w \in W} a_w \mathcal{N}_W(T_w)\right) \mathcal{N}_W(1)^{-1} = \sum_{w \in W} a_w \mathcal{N}_W(T_w \mathcal{N}_W(1)^{-1}).$$

This completes the proof. □

Therefore, when we look for a basis of the center of $\mathcal{H}(W)$, Proposition 2.4 forces us to consider the elements $\mathcal{N}_W(h) = \sum_{w \in W} q^{-\ell(w)} T_{w^{-1}} h T_w$ for $h \in \mathcal{H}(W)$.

## 3. A basis of the center of Hecke algebra

In this section, we find a basis of the center of Hecke algebra. We begin with the definition of cellular algebras.

DEFINITION 3.1 ([4]). A *cellular algebra* over $R$ is an associative algebra $A$, together with *cell datum* $(\Lambda, M, C, *)$, where

(C1) $\Lambda$ is a partially ordered set and for each $\lambda \in \Lambda$, $M(\Lambda)$ is a finite set such that $C : \coprod_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \to A$ is an injective map with image an $R$-basis, called a cellular basis, of $A$.

(C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S,T) = C^{\lambda}_{S,T} \in A$. Then $*$ is an $R$-linear anti-automorphism of $A$ such that $(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any element $a \in A$ we have

$$a\, C^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \pmod{A\ (< \lambda)}$$

where $r_a(S', S) \in R$ is independent of $T$ and where $A(< \lambda)$ is the $R$-submodule of $A$ generated by $\{C^{\mu}_{S'',T''} \mid \mu < \lambda; S'', T'' \in M(\mu)\}$.

REMARK 3.2. For an integral domain $R$ and an invertible element $q \in R$, Murphy found a 'standard' basis $\{x_{rs}\}$ which is a cellular basis of $\mathcal{H}(S_n)$ over $R$, and Dipper, James and Murphy also described a cellular basis of the Hecke algebra of type $B_n$ over $R$. (See [10] and [1], respectively.) In [2], Fakiolas investigated the irreducible representations of Hecke algebras associated with the dihedral groups of order $2n$ over $R = \mathbb{Q}(\cos(2\pi/n))(q^{1/2})$, $q$ an indeterminate, and found an $R$-basis which is, in fact, a cellular basis. For $R = \mathbb{Q}(q)$, $q$ an indeterminate, the present authors described a cellular basis of the Hecke algebra of type $D_{2n+1}$ over $R$ in [6]. (In [6], a more general statement is given than stated above.)

The main result of this paper is obtained under the following assumption.

ASSUMPTION 3.3. From now on, we assume that Hecke algebra $\mathcal{H}(W)$ over $R$ is either one of the followings:

(1) $W$ is a Weyl group of type $A_n$,$B_n$ or $D_{2n+1}$ and $R = \mathbb{Q}(q)$, $q$ an indeterminate.

(2) $W$ is a dihedral group of order $2n$ and $R = \mathbb{Q}(\cos(2\pi/n))(q^{1/2})$, $q$ an indeterminate.

One can find the following results in [1], [2], [6] and [10].

REMARK 3.4. (a) The Hecke algebras in Assumption 3.3 are split semisimple cellular algebras and the number of finite dimensional non-isomorphic irreducible representations of $\mathcal{H}(W)$ over $R$ is equal to the cardinality of $\Lambda$ and is equal to the rank of the center of $\mathcal{H}(W)$.

(b) We denote the cellular basis of Hecke algebra in Assumption 3.3 by $\{C^{\lambda}_{S,T}\}$ for simplicity. Note that $T_w C^{\lambda}_{S,T}$ is a $\mathbb{Z}[q,q^{-1}]$-linear combination of $\{C^{\lambda}_{S,T}\}$ for the Hecke algebra of type $A_n, B_n$ or $D_{2n+1}$ and is a $\mathbb{Z}[\cos(2\pi/n), q^{1/2}, q^{-1/2}]$-linear combination of $\{C^{\lambda}_{S,T}\}$ for the Hecke algebra of dihedral group of order $2n$.

We obtain the following lemma using the axioms (C1), (C2) and (C3) in Definition 3.1. If $h = \sum_{\lambda,S,T} r^{\lambda}_{S,T} C^{\lambda}_{S,T}$, where $r^{\lambda}_{S,T} \in R$, we say that $r^{\lambda}_{S,T}$ is the $C^{\lambda}_{S,T}$-coefficient of $h$.

LEMMA 3.5. *Let $\mathcal{H}(W)$ be the Hecke algebra in Assumption 3.3. For $S \in M(\lambda)$ and $w \in W$, assume that the $C^{\lambda}_{S,S}$-coefficient of $T_{w^{-1}} C^{\lambda}_{S,S}$ is $a^{w}_{\lambda,S}$. Then the $C^{\lambda}_{S,S}$-coefficient of $T_{w^{-1}} C^{\lambda}_{S,S} T_w$ is $(a^{w}_{\lambda,S})^2$.*

*Proof.* We denote $r_{(T_{w^{-1}})}$ by $r_w$ for simplicity. Since

$$T_{w^{-1}} C^{\lambda}_{S,S} \equiv a^{w}_{\lambda,S} C^{\lambda}_{S,S} + \sum_{\substack{S' \in M(\lambda) \\ S' \neq S}} r_w(S',S) C^{\lambda}_{S',S} \pmod{A\ (<\lambda)},$$

we get

$$C^{\lambda}_{S,S} T_w \equiv a^{w}_{\lambda,S} C^{\lambda}_{S,S} + \sum_{\substack{S' \in M(\lambda) \\ S' \neq S}} r_w(S',S) C^{\lambda}_{S,S'} \pmod{A\ (<\lambda)}$$

by (C2). Hence,

$$\begin{aligned} & T_{w^{-1}} C^{\lambda}_{S,S} T_w \\ & \equiv a^{w}_{\lambda,S} C^{\lambda}_{S,S} T_w + \sum_{\substack{S' \in M(\lambda) \\ S' \neq S}} r_w(S',S) C^{\lambda}_{S',S} T_w \pmod{A\ (<\lambda)} \\ & \equiv a^{w}_{\lambda,S} \Big( a^{w}_{\lambda,S} C^{\lambda}_{S,S} + \sum_{\substack{S' \in M(\lambda) \\ S' \neq S}} r_w(S',S) C^{\lambda}_{S,S'} \Big) \\ & \quad + \sum_{\substack{S' \in M(\lambda) \\ S' \neq S}} r_w(S',S) \Big( \sum_{T \in M(\lambda)} r_w(T,S) C^{\lambda}_{S',T} \Big) \pmod{A\ (<\lambda)}. \end{aligned}$$

So the $C^\lambda_{S,S}$-coefficient of $T_{w^{-1}} C^\lambda_{S,S} T_w$ is $(a^w_{\lambda,S})^2$. □

The following lemma is a consequence of Remark 3.4 and Lemma 3.5.

LEMMA 3.6. *Let $\mathcal{H}(W)$ be the Hecke algebra in Assumption 3.3. For any $\lambda \in \Lambda$ and $S \in M(\lambda)$, the $C^\lambda_{S,S}$-coefficient of*

$$\sum_{w \in W} q^{-\ell(w)} T_{w^{-1}} C^\lambda_{S,S} T_w$$

*is not zero.*

*Proof.* The $C^\lambda_{S,S}$-coefficient of $\sum_{w\in W} q^{-\ell(w)}\ T_{w^{-1}}\ C^\lambda_{S,S}\ T_w$ is $\sum_{w\in W} q^{-\ell(w)}\ (a^w_{\lambda,S})^2$ by Lemma 3.5. If we set $q = 1$, this becomes a sum of squares of real numbers, and hence $\sum_{w\in W} q^{-\ell(w)} (a^w_{\lambda,S})^2 \neq 0$. (Note that $a^1_{\lambda,S} = 1$ and $a^w_{\lambda,S}$ is a Laurent polynomial in $q$ by Remark 3.4.(b).) □

The main result of the present paper is:

THEOREM 3.7. *Let $\mathcal{H}(W)$ be the Hecke algebra in Assumption 3.3. Choose only one $S(\lambda) \in M(\lambda)$ for each $\lambda \in \Lambda$. Then*

$$\{\mathcal{N}_W(C^\lambda_{S(\lambda),S(\lambda)}) \mid \lambda \in \Lambda\}$$

*is an $R$-basis for the center of $\mathcal{H}(W)$.*

*Proof.* Let us denote $N_\lambda = \mathcal{N}_W(C^\lambda_{S(\lambda),S(\lambda)})$ for simplicity. As we mentioned in Remark 3.4.(a), the cardinality of $\Lambda$ is equal to the dimension of the center of $\mathcal{H}(W)$. Therefore it is enough to show that $\{\mathcal{N}_\lambda \mid \lambda \in \Lambda\}$ is linearly independent over $R$. Assume $\sum_{\lambda\in\Lambda} b_\lambda \mathcal{N}_\lambda = 0$ for $b_\lambda \in R$. We have to show that $b_\lambda = 0$ for all $\lambda \in \Lambda$. We show this using the 'induction' with respect to the partial order of $\Lambda$. For a given $\mu_0$, suppose that there is no $\lambda$ such that $\lambda > \mu_0$. Since $\mathcal{N}_\lambda$ is contained in the $R$-submodule of $\mathcal{H}(W)$ generated by $\{C^\mu_{S,T} \mid \mu \leq \lambda\,; S, T \in M(\mu)\}$ by (C2) and (C3), the $C^{\mu_0}_{S(\mu_0),S(\mu_0)}$-coefficient of $\sum_{\lambda\in\Lambda} b_\lambda \mathcal{N}_\lambda$ is the $C^{\mu_0}_{S(\mu_0),S(\mu_0)}$-coefficient of $b_{\mu_0}\mathcal{N}_{\mu_0}$. Thus we get $b_{\mu_0} = 0$ by Lemma 3.6. Next, for a given $\mu$, assume that $b_\lambda = 0$ for all $\lambda > \mu$, then we also get $b_\mu = 0$ by the same argument. This completes the proof. □

The most significant aspect of cellular algebra is that one can systematically understand the irreducible representations of Hecke algebra using its cellular basis (see [4]). It is quite interesting that a basis of the center can be also described in terms of cell datum.

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Department of Mathematics, Seoul National University, Seoul 151-742, Korea

*E-mail*: ykjeong@math.snu.ac.kr
islee@math.snu.ac.kr
hyekyung@math.snu.ac.kr
pkh@math.snu.ac.kr