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ON *AGE* RINGS AND *AM* MODULES WITH RELATED CONCEPTS

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ABSTRACT. In this paper, all rings or (left)near-rings R are associative, and for near-ring R, all R-groups are right R action and all modules are right R-modules. First, we begin with the study of rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms and their properties. This study was motivated by the work on the Sullivan's Problem [14]. Next, for any right R-module M, we will introduce AM modules and investigate their basic properties. Finally, for any near-ring R, we will also introduce MR-groups and study some of their properties.

1. Introduction

Throughout this paper, we start with the study of rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This research was motivated by the work on the Sullivan's Research Problem (that is, characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AE rings*) [14], [2], [3], [5], [6], [7], [8], [9], and [11], and the investigation of LSD-generated rings and SD-generated rings [2] and [4].

Let R be an associative ring (or near-ring) not necessarily with unity, G be an additive group not necessarily abelian and M a right

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R-module. We denote that $End(R,+)$ is the ring of additive endomorphisms of R, $End(R,+,\cdot)$ the monoid of ring endomorphisms of R, $End(M)$ the ring of additive endomorphisms of M and $End_R(M)$ the ring of R-endomorphisms of the right R-module M. For $X \subseteq R$, we use $gp(X)$ for the subgroup of $(R,+)$ generated by X. For each $x \in R$, ${}_x\tau$ denotes the left multiplication mapping (that is, $a \longmapsto xa$, for all $a \in R$). Observe ${}_x\tau \in End(R,+)$. $\mathcal{LGE}(R)$ is the set $\{x \in R \mid {}_x\tau \in gp(End(R,+,\cdot))\}$. Note that $\mathcal{LGE}(R)$ is a subring of R. $\mathcal{L}(R)$ is the set $\{x \in R \mid xab = xaxb\}$. $(\mathcal{L}(R),\cdot)$ is a subsemigroup of $(R,\cdot)$, and $x \in \mathcal{L}(R)$ if and only if ${}_x\tau \in End(R,+,\cdot)$. Also $\mathcal{L}(R) \subseteq \mathcal{LGE}(R)$ and $\mathcal{L}(R)$ contains all one-sided unities and all central idempotents of R. Similarly, we use $\mathcal{RGE}(R)$ and $\mathcal{R}(R)$ for the right sided analogs of $\mathcal{LGE}(R)$ and $\mathcal{L}(R)$, respectively.

Let $(G,+)$ be a group (not necessarily abelian) We will use operations on the right side of the variables in near-ring theory to distinguish ring theory. In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f+g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $x \in G$ (called the *pointwise addition of maps*) and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G),+,\cdot)$ becomes a near-ring It is called the *self map near-ring* of the group G or *near-ring of self maps* on G. Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G),+,\cdot)$ is a zero symmetric near-ring [12, 13]

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if for all $a,\ b \in R$, (i) $(a+b)\theta = a\theta + b\theta$ and (ii) $(ab)\theta = a\theta b\theta$.

We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as for rings ([1])

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R,+,\cdot) \longrightarrow (M(G),+,\cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a,\ b \in R$.

Note that R itself is an R-group called the *regular group*

Moreover, naturally, every group G has an $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

An R-group G with the property that for each $x \in G$ and $a,\ b \in R$, $(x+y)a = xa + ya$ is called a *distributive R-group*. Every distributive near-ring R is a distributive R-group.

A representation θ of R on G is called *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group*, or that R *acts faithfully* on G

A ring R is said to be an *AGE ring* if

$$End(R,+) = gp < End(R,+, > .$$

Clearly, we see that every AE ring is AGE, but not conversely from the following examples. Note if the left regular representation of R into $End(R,+)$ is surjective, then R is an AGE ring.

R is called *LSD* (*LSD-generated*) if $R = \mathcal{L}(R)$ $(R = gp(\mathcal{L}(R)))$, and also R is called *RSD* (*RSD-generated*) if $R = \mathcal{R}(R)$ $(R = gp(\mathcal{R}(R)))$ R is called *SD* (*SD-generated*) if $R = \mathcal{L}(R) \cap \mathcal{R}(R)$ $(R = gp(\mathcal{L}(R) \cap \mathcal{R}(R))$ [2], [4] and [10] The classes of LSD, LSD-generated, SD and SD-generated rings are closed with respect to homomorphisms and direct sums, and the class of AGE rings is not contained in the class of SD-generated rings [2]. Examples are provided to show that the classes of AGE and LSD-generated rings are distinct. Although the class of AE rings is a proper subclass of the class of SD rings.

$\mathcal{I}(R)$ and $\mathcal{N}(R)$ denote the set of idempotent elements of R and the set of nilpotent elements of R, respectively.

Let $G,\ T$ be two additive groups (not necessarily abelian). Then the set

$$M(G,\ T) := \{f \mid f : G \longrightarrow T\}$$

of all maps from G to T becomes an additive group under pointwise addition of maps. Since $M(T)$ is a near-ring of self maps on T, we note that $M(G,\ T)$ is an $M(T)$-group with a scalar multiplition:

$$M(G,\ T) \times M(T) \longrightarrow M(G,\ T)$$

defined by $(f,\ g) \longmapsto f \cdot g$, where $x(f \cdot g) = (xf)g$ for all $x \in G$.

Let G and T be two R-groups. Then the mapping $f : G \longrightarrow T$ is called a *R-group homomorphism* if for all $x,\ y \in G$ and $a \in R$, (i) $(x+y)f = xf + yf$ and (ii) $(xa)f = (xf)a$. In this paper, we call that the mapping $f : G \longrightarrow T$ with the condition $(xa)f = (xf)a$ is an *R-map*.

Also, we can replace R-group homomorphism by R-group momomorphism, R-group epimorphism, R-group isomorphism, R-group endomorphism and R-group automorphism, if these terms have their usual meanings as for modules ([1]).

A near-ring R is called *distributively generated* (briefly, *D.G.*) by S if

$$(R,+) = gp < S >= gp < R_d >$$

where S is a semigroup of distributive elements in R, in particular, $S = R_d$ (this is motivated by the set of all distributive elements of R is multiplicatively closed and contain the unity of R if it exists), where $gp < S >$ is a group generated by S, we denote this D.G near-ring R which is generated by S is (R, S)

On the other hand, the set of all distributive elements of $M(G)$ are obviously the set $End(G)$ of all endomorphisns of the group G, that is,

$$(M(G))_d \ = \ End(G)$$

which is a semigroup under composition, but not yet a near-ring. Here we denote that $E(G)$ is the D G. near-ring generated by $End(G)$, that is,

$$E(G) = (M(G)),\ End(G).$$

Obviously, $E(G)$ is a subnear-ring of $(M_0(G), +,\ )$. Thus we say that $E(G)$ is the *endomorphism near-ring* of the group G.

For the remainder concepts and results on near-rings and R-groups, we refer to J. D. P. Meldrum [12], and G. Pilz [13]

2. Some Examples and Results

Notice that R is an AGE ring if and only if there exists a subset S of $End(R,+,\cdot)$ such that

$$End(R,+) = gp < S > .$$

Sometimes, we will use the notations: $End_{\mathbb{Z}}(R)$ instead of $End(R,+)$, $End(R)$ instead of $End(R,+,\cdot)$ and $GE(R)$ instead of $gp < End(R,+,\cdot) >$

Clearly, GE(R) is a subring of $End_{\mathbb{Z}}(R)$.

For any near-ring R and R-group G, we define the set

$$M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r, \ for \ all \ x \in G, \ r \in R\}$$

of all R-maps on G as defined previously.

LEMMA 2.1 *Let G be an abelian D.G (R,S)-group. Then the set $M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r, \ for \ all \ x \in G, \ r \in R\}$ is a subnear-ring of $M(G)$.*

PROOF. Let $f, \ g \in M_R(G)$. For any $x \in G$ and $r \in R$, since R is a D.G. near-ring generated by S, consider that

$$r = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $i = 1, \cdots, n$. We have that

$$(xr)(f+g) = (xr)f + (xr)g = (xf)r + (xg)r$$

$$= xf(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) + xg(\delta_1 s_1 + \delta_2 s_2 + \cdots \delta_n s_n)$$

$$= xf\delta_1 s_1 + xg\delta_1 s_1 + xf\delta_2 s_2 + xg\delta_2 s_2 + \cdots + xf\delta_n s_n + xg\delta_n s_n$$

$$= \delta_1 xfs_1 + \delta_1 xgs_1 + \delta_2 xfs_2 + \delta_2 xgs_2 + \cdots + \delta_n xfs_n + \delta_n xgs_n$$

$$= \delta_1(xfs_1 + xgs_1) + \delta_2(xfs_2 + xgs_2) + \cdots + \delta_n(xfs_n + xgs_n)$$

$$= \delta_1(xf + xg)s_1 + \delta_2(xf + xg)s_2 + \cdots + \delta_n(xf + xg)s_n$$

$$= (xf + xg)\delta_1 s_1 + (xf + xg)\delta_2 s_2 + \cdots + (xf + xg)\delta_n s_n$$

$$= (xf + xg)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n)$$

$$= (xf + xg)r = x(f + g)r.$$

Similary, we have the following equalities:

$$(xr)(-f) = -(xr)f = -(xf)r = x(-f)r$$

and

$$(xr)f \cdot g = ((xr)f)g = ((xf)r)g = (xf)gr = x(f \cdot g)r.$$

Thus $M_R(G)$ is a subnear-ring of $M(G)$. □

In ring and module theory, we obtain the following important structure for near-ring and R-group theory:

COROLLARY 2.2 (C. J. MAXSON). *Let R be a ring and V a right R-module. Than $M_R(V) := \{f \in M(V) \mid (xr)f = (xf)r,\ for\ all\ x \in V,\ r \in R\}$ is a subnear-ring of $M(V)$.*

EXAMPLE 2.3 [2]. Rings additively generated by central idempotents and one sided unities are LSD-generated and RSD-generated, so that SD-generated. In particular, since the rings $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by 1, and $End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}, End_{\mathbb{Z}}(\mathbb{Z}_n) \cong \mathbb{Z}_n$, we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both AGE, LSD-generated and SD-generated rings. However, $\mathbb{Z}$ and $\mathbb{Z}_n$ are all not AE rings except the cases $\mathbb{Z}_1$ and $\mathbb{Z}_2$, because any nontrivial endomorphism on $\mathbb{Z}$ or $\mathbb{Z}_n$ is additive but which is not a ring endomorphism. On the other hand, if $x \in \mathcal{L}(R)$ implies $x^3 = x^n$ for $n > 3$, then $\mathcal{L}(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGE ring which is not LSD-generated and SD-generated

PROPOSITION 2.4. *For every AGE ring R, and for any positive integer n, we get that $\oplus_{i=1}^{n} R_i$ is an AGE ring, where $R_i \cong R$, for all i=1,2,...,n.*

PROOF. We prove the case for $n = 2$, that is, $R \oplus R$. Similarly, we can prove for the case $n > 2$. We must show that

$$End_{\mathbb{Z}}(R \oplus R) = GE(R \oplus R).$$

Since $End_{\mathbb{Z}}(R \oplus R) \cong Mat_2(End_{\mathbb{Z}}(R))$, we obtain that

$$End_{\mathbb{Z}}(R \oplus R) \cong \begin{bmatrix} End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \\ End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \end{bmatrix} = \begin{bmatrix} GE(R) & GE(R) \\ GE(R) & GE(R) \end{bmatrix}.$$

Let $f \in End_{\mathbb{Z}}(R \oplus R)$ such that

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ f_{ij} \in GE(R)$$

Then

$$f_{11} = \sum_i \lambda_i h_i, \ f_{12} = \sum_j \lambda_j h_j, \ f_{21} = \sum_k \lambda_k h_k, \ f_{22} = \sum_t \lambda_t h_t,$$

where, $\lambda's \in \mathbb{Z}$ and $h's \in End(R)$. Thus f is expressed of the form

$$f = \sum_i \lambda_i \begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix} + \sum_j \lambda_j \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix} + \sum_k \lambda_k \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix} + \sum_t \lambda_t \begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}$$

Since all $\begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}$ are ring endomorphisms of $R \oplus R$, $R \oplus R$ is an AGE ring. □

From Example 2.3 and Proposition 2.4, there exist numerously many examples of AGE rings and LSD-generated rings.

Obviously, we get the following useful lemma:

LEMMA 2.5. *For any surjective ring endomorphism h, $\mathcal{L}(R)$ and $\mathcal{R}(R)$ are all invariant under h.*

From this lemma, we get the following statement.

PROPOSITION 2.6. *Let R be a ring with unity. If R is an AGE ring with $S \subsetneq End(R)$ such that $End_{\mathbb{Z}}(R) = gp < S >$, and each element of S is onto, then R is LSD-generated, moreover SD-generated.*

PROOF. Let $x \in R$. Consider a left translation mapping $\phi_x : R \longrightarrow R$ by $\phi_x(a) = xa$ for all $a \in R$, which is a group endomorphism. Since R is an AGE ring,

$$\phi_x = \sum_i^n \lambda_i h_i,$$

where $\lambda_i \in \mathbb{Z}$ and $h_i \in End(R)$ such that h_i is onto, for $i = 1, 2, \cdots, n$. Since $1 \in R$, $\phi_x(1) = \sum_i^n \lambda_i h_i(1)$, that is, $x = \sum_i^n \lambda_i h_i(1)$, and since $1 \in \mathcal{L}(R) \cap \mathcal{R}(R)$ by Lemma 2.5, we have $h_i(1) \in \mathcal{L}(R) \cap \mathcal{R}(R)$. Hence R is LSD-generated and RSD-generated, so is SD-generated. □

EXAMPLE 2.7 [2, 4].

(1) If S is an LSD-generated ring, then

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$$

is also LSD-generated by the set

$$\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & y \end{bmatrix} \mid v, x, y \in \mathcal{L}(S)\}.$$

(2) If S is an RSD-generated ring, then

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$$

is also RSD-generated by the set

$$\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} y & y \\ 0 & 0 \end{bmatrix} \mid v, x, y \in \mathcal{R}(S)\}.$$

In particular,

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$$

is an LSD-generated ring with the generators: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and also an RSD-generated ring with the generators: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, but which is not an SD-generated ring. Clearly, $\begin{bmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ is an SD-generated ring.

Similarly,

$$R = \begin{bmatrix} \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & \mathbb{Z}_n \end{bmatrix}$$

is both LSD-generated and RSD-generated, but which is not SD-generated

EXAMPLE 2 8 [4]. Let S be an LSD semigroup (i.e, $xab = xaxb$, for all $x, a, b \in S$). Then the semigroup ring $K[S]$, where K is $\mathbb{Z}$ or $\mathbb{Z}_n$, is an LSD-generated ring. In particular, for a nonempty set S with multiplication $st = t$, for all $s, t \in S$, $\mathbb{Z}[S]$ and $\mathbb{Z}_n[S]$ are LSD-generated rings. Furthermore if $|S| = 2$, then $\mathbb{Z}_2[S]$ is an LSD ring which is not an AGE ring

Obviously, we obtain the following remark: Let R be a ring and $X \subseteq R$ such that $R = gp(X)$ Let I be the ideal generated by $\{bxby - bxy \mid b, x, y \in X\}$ Then R/I is an LSD-generated ring. Analogously, we obtain RSD-generated rings

PROPOSITION 2 9 [2]. *Let* $Y \subseteq End(R, +, \cdot)$ *and* $S \subseteq R$ *such that* $f(S) \subset gp(S)$, *for each* $f \in Y$.

(1) *If* $R = \mathcal{LGE}(R)$ *and for each* $x \in R$, ${}_x\tau = \sum_{i \in I} \pm f_i$, *where each* $f_i \in Y$, *then* $gp(S)$ *is a left ideal of* R.
(2) *If* R *is an AGE ring and* $Y = End(R, +, \cdot)$, *then* $h(S) \subseteq gp(S)$, *for each* $h \in End(R, +)$.

COROLLARY 2.10 [2]. *Let $S = \mathcal{I}(R), \mathcal{N}(R)$ or the set of quasiregular elements of R.*

(1) *If R and Y are as in Proposition 2.9 (1), then $gp(S)$ is a left ideal of R.*
(2) *If R and Y are as in Proposition 2.9 (2), then $h(S) \subseteq gp(S)$, for each $h \in End(R,+)$.*

Observe that Example 2.7 is an LSD-generated ring which is not an AGE ring. To see this, let $h : R \longrightarrow R$ be defined by

$$h(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}.$$

Then $h \in End(R,+)$, but $h(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \notin \mathcal{N}(R) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. By Corollary 2.10 (2), R is not an AGE ring.

COROLLARY 2.11.

(1) *If $R = \mathcal{LGE}(R)$ with a unity, then $R = gp(\mathcal{I}(R))$.*
(2) *If R is a division AGE ring, then $R = gp(\mathcal{N}(R))$.*

Next, we introduce a new concept of a right R-module and investigate it's properties. For any ring R, a right R-module M is called an *AM module* over R if

$$End(M) = End_R(M).$$

For example, $\mathbb{Q}$ is an AM module, because of $End_Z\mathbb{Q}=\mathbb{Q}=End_{\mathbb{Q}}\mathbb{Q}$. In particular, R is called an AM ring if R is an AM module as a right R-module R itself, that is, for all $f \in End_{\mathbb{Z}}(R)$, $x, \ r \in R$, we have $f(xr) = f(x)r$.

PROPOSITION 2.12. *Let $\{M_i | i \in \Lambda\}$ be a family of right R-modules. Then each M_i is an AM module if and only if $M := \bigoplus M_i$ is an AM module.*

PROOF. Let $f \in End(M)$, and let $x = (x_i) \in M$, where $x_i = 0$ except finitely many, $r \in R$. Define $f_i \in End(M_i)$ as following

$$f_i(x_i) = f(0, \cdots, x_i, \cdots, 0) \in End(M_i) = End_R(M_i).$$

Thus

$$f((0, \cdots, x_i, \cdots, 0)r) = f_i(x_i)r = f(0, \cdots, x_i, \cdots, 0)r.$$

Consequently, we see that

$$f(xr) = f(\sum(0, \cdots, x_i, \cdots, 0)r) = (\sum f(0, \cdots, x_i, \cdots, 0))r = f(m)r.$$

Conversely, let $g_i \in End(M_i)$ and let $x_i \in M_i$, $r \in R$ Consider, $x = (0, \cdots, x_i, \cdots, 0) \in M$ Define $f(x) = (0, \cdots, g_i(x_i), \cdots, 0)$ Since

$$f \in End(M) = End_R(M),$$

it follows that $g_i \in End_R(M_i)$. □

PROPOSITION 2.13. *Let R be an AM ring. Then for any $r \in R$, rR is also an AM ring.*

PROOF Let $f \in End(rR)$, and $g : R \longrightarrow R$ be defined by $g(x) = f(rx)$ for all $x \in R$ Then $g \in End_{\mathbb{Z}}(R)$ This implies that $g(xry) = g(x)ry$, because $End_{\mathbb{Z}}(R) = End_R(R)$. So we have

$$f(rxry) = f(rx)ry.$$

Hence, for any $r \in R$, rR is an AM ring □

PROPOSITION 2 14 *Let M be a right R-module.*

(1) *If M is a faithful AM module over R, than R is a commutative ring.*
(2) *If M is a simple AM module over R, than $End(M)$ is a division ring.*

PROOF. (1) Let $f \in End(M)$ and let $a, b \in R$, where $f(x) = xa$, for all $x \in M$. Then

$$f(xb) = (xb)a.$$

On the other hand, since $f \in End(M) = End_R(M)$, we have that

$$f(xb) = f(x)b = (xa)b.$$

Hence $(xb)a = (xa)b$ for all $x \in M$. Since M is faithful, we see that $ab = ba$

(2) The proof of this part is easily induced from the Schurs' Lemma. □

Hereafter, we shall treat a special kind of near-ring R and an R-group G, there is a module like concept as follows: Let (R, S) be a D.G. near-ring. Then an additive group G is called a *D.G. (R, S)-group* if there exists a D.G. near-ring homomorphism

$$\theta : (R, S) \longrightarrow (M(G), End(G)) = E(G)$$

such that $S\theta \subseteq End(G)$. If we write that xr instead of $x(r\theta)$ for all $x \in G$ and $r \in R$, then an D.G (R, S)-group is an additive group G satisfying the following conditions:

$$x(rs) = (xr)s$$

and

$$x(r + s) = xr + xs,$$

for all $x \in G$ and all $r, s \in R$,

$$(x + y)s = xs + ys,$$

for all $x, y \in G$ and all $s \in S$.

Such a homomorphism θ is called a *D.G. representation* of (R, S) on G. This D.G. representation is said to be *faithful* if $Ker\theta = \{0\}$ In this case, we say that G is called a *faithful D.G (R, S)-group.*

The following statement is proved very easily, but it is important.

LEMMA 2.15. *Let G be a faithful R-group. Then we have the following conditions:*

(1) *If $(G, +)$ is abelian, then $(R, +)$ is abelian.*
(2) *If G is distributive, then R is distributive.*

From this Lemma, we get the following Proposition:

PROPOSITION 2.16. *If G is a distributive abelian faithful R-group, then R is a ring.*

The following statement which is obtained from Lemma 2.15 and property of faithful D.G. (R, S)-group is a generalization of the Proposition 2.16

PROPOSITION 2.17. *Let (R, S) be a D.G. near-ring. If G is an abelian faithful D.G. (R, S)-group, then R is a ring.*

Finally, we also introduce the MR-property of R-group, which is motivated by the Lemma 2.1.

We denote again that $M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r$, for all $x \in G,\ r \in R\}$ is the set of all R-maps on G. An R-group G is called an *MR-group* over near-ring R, provided that every mapping on G is an R-map of G, that is,

$$M(G) \ = \ M_R(G).$$

For example, every monogenic R-group (see [13]) is an MR-group.

A similar property of Proposition 2.14 (1) for MR-group is obtained, we use operations on the right side of R-group as defined previously.

PROPOSITION 2.18 *If G is a faithful MR-group over near-ring R, than R is a commutative near-ring.*

PROOF. (1) Let $a,\ b \in R$. Define a mapping $f : G \longrightarrow G$ given by $xf \ = \ xa$, for all $x \in G$. Then clearly, $f \in M(G)$ Since G is an MR-group, $f \in M_R(G)$ Thus we have the two equalities:

$$(xb)f \ = (xb)a = x(ba)$$

and since $f \in M(G) = M_R(G)$,

$$(xb)f = (xf)b = (xa)b = x(ab).$$

Since G is a faithful R-group, these two equalities implies that $ab = ba$. Hence R is a commutative near-ring. $\square$

From the Proposition 2.15 and the above Proposition 2.18, we have the following statement.

COROLLARY 2.19. *If G is an abelian faithful MR-group over near-ring R, than R becomes a commutative ring.*

The following are the characterization of MR-groups and AM modules

PROPOSITION 2.20. *Let G be an R-group with the representation*

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Then $R\theta \subseteq$ Center of $M(G)$ if and only if G is an MR-group.

PROOF. We will prove the only if part. Suppose that $R\theta \subseteq$ Center of $M(G)$ To show that $M(G) = M_R(G)$, let $f \in M(G)$, and let $x \in G$, $r \in R$. Then from the definitions of θ and the Center of $M(G)$, we have

$$(xr)f = (xr\theta)f = x(r\theta \circ f) = x(f \circ r\theta) = (xf)r\theta = (xf)r.$$

This implies that $f \in M_R(G)$, that is, $M(G) \subseteq M_R(G)$. Hence G is an MR-group. $\square$

COROLLARY 2.21. *Let M be a right R-module with the right regular representation*

$$\theta . (R, +, \cdot) \longrightarrow (End(M), +, \cdot).$$

Then $\theta(R) \subseteq$ Center of $End(M)$ if and only if M is an AM module.

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