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SOME EINSTEIN PRODUCT MANIFOLDS

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ABSTRACT In this paper, we get conditions for the natural projections of some product manifolds with varying metrics of two Riemannian manifolds to be harmonic, and necessary and sufficient conditions for some product manifolds with the harmonic natural projections of two Einstein manifolds to be Einstein manifolds.

1. Introduction

For complete Riemannian manifolds (M, g), (N, h), a smooth map $\phi : M \longrightarrow N$ is said to be harmonic if $tr\nabla(df) = 0$, namely, the tension field $\tau(\phi)$ vanishes identically (cf. [2]).

On the other hand, harmonic maps ϕ between compact Riemannian manifolds (M, g) and (N, h) are the extrema of the energy functional $E(\phi) = \frac{1}{2}\int_M \| \mathrm{d}\phi \|^2 \mathrm{d}v_g$. This suggests a variational approach to finding harmonic mappings.

A Riemannian metric g is called Einstein if its Ricci tensor satisfies $Ric(g) = kg$ for some constant k.

In this paper we get necessary and sufficient conditions for some product manifolds $(B \times F,\ g + f\bar{g})$ of two Einstein manifolds (B, g) and $(F, \bar{g})$ by f to be Einstein manifolds(cf. [1]) And under assumptions that the natural projections $\pi . B \times F \longrightarrow B$ and $\sigma : B \times F \longrightarrow F$ are harmonic, we obtain the complete conditions for product manifolds with varying metrics(warped product manifolds, twisted manifolds, and doubly warped product manifolds) of two Einstein manifolds

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(B, g) and $(F, \bar{g})$ by f to be Einstein manifolds(cf. Proposition 2.4, 3.4, 4.4, and Theorem 2.6, 3.6, 4.6).

2. The warped product manifold

Let (B, g) (resp. $(F, \bar{g})$) be an n-dimensional (resp. p-dimensional) Riemannian manifold and f a positive smooth function on B. *The warped product manifold* $M = B \times_f F$ is the differentiable product manifold $B \times F$ equipped with the metric $\tilde{g}$ defined by $\tilde{g} := g + f^2 \bar{g}$, i.e.,

$$(2.1) \quad \tilde{g}(X, Y) = g(\pi_* X, \pi_* Y) + f^2 \bar{g}(\sigma_* X, \sigma_* Y)$$

for each tangent vector X, Y on M. From now on in this paper, π (resp. σ) is the canonical projection of M onto B (resp. F). The curvature tensor $\widetilde{R}$ of $(M, \tilde{g})$ is given by $\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z$, $(X, Y, Z \in \mathfrak{X}(M))$, where $\widetilde{\nabla}$ is the Levi-Civita connection of (M, g).

For a local coordinate system (u^a) of B, the metric tensor g has the components g_{ab}, where $g_{ab} = g(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^b})$. Similarly, for a local coordinate system (u^x) of F, $\bar{g}$ has the components $\bar{g}_{xy}$.

Throughout this paper, the indices $a, b, c, \dots$ (resp. $x, y, z, \dots$) run over $\{1, 2, \dots, n\}$ (resp. $\{n+1, n+2, \dots, n+p\}$) and the indices $i, j, k, \dots$ run over the range $\{1, 2, \dots, n+p\}$, and the summation convention is used with respect to those systems of indices. Then, for the local coordinate system (u^i) of $M = B \times_f F$, $\tilde{g}$ has the components $\tilde{g}_{ij}$.

If $f = 1$, then $B \times_f F$ reduces to a Riemannian product manifold. B is called the base of $M = B \times_f F$, F a fibre and f a warping function.

In this paper, we denote by ∇_b (resp. $\overline{\nabla}_x$) the components of the covariant derivative with respect to g (resp. $\bar{g}$) and $\{{}_b{}^a{}_c\}$ (resp. $\overline{\{{}_y{}^x{}_z\}}$) the Christoffel symbols of g (resp. $\bar{g}$) on (B, g) (resp. $(F, \bar{g})$).

The following Lemmas are easily obtained.

Lemma 2.1. *The Christoffel symbols* $\widetilde{\{{}_i{}^k{}_j\}}$ *of the Livi-Civita con-*

nection $\widetilde{\nabla}$ on the warped product manifold M are given as follows:

$$\left\{\begin{array}{l} \widetilde{\{{}_b{}^a{}_c\}} = \{{}_b{}^a{}_c\}, \quad \widetilde{\{{}_b{}^a{}_z\}} = 0, \quad \widetilde{\{{}_y{}^a{}_z\}} = -ff_b g^{ba}\bar{g}_{yz}, \\ \widetilde{\{{}_b{}^x{}_c\}} = 0, \quad \widetilde{\{{}_b{}^x{}_z\}} = f^{-1}f_b\delta_z{}^x, \quad \widetilde{\{{}_y{}^x{}_z\}} = \overline{\{{}_y{}^x{}_z\}}, \end{array}\right. \tag{2.2}$$

where $f_b := \partial f/\partial u^b$ and $(g^{ab}) := (g_{ab})^{-1}$.

Lemma 2.2. *Let R, $\overline{R}$ and $\widetilde{R}$ be the curvature tensors of (B, g), $(F, \bar{g})$ and the warped product manifold $(M, \tilde{g})$ respectively. Then*

$$\left\{\begin{array}{l} \widetilde{R}_{abc}{}^d = R_{abc}{}^d, \quad \widetilde{R}_{ayc}{}^w = f^{-1}\delta_y{}^w\nabla_a f_c, \\ \widetilde{R}_{ayz}{}^d = -fg^{dc}\bar{g}_{yz}\nabla_a f_c, \\ \widetilde{R}_{xyz}{}^w = \overline{R}_{xyz}{}^w + \| df \|_g^2(\bar{g}_{xz}\delta_y{}^w - \bar{g}_{yz}\delta_x{}^w), \end{array}\right. \tag{2.3}$$

and the others $\widetilde{R}_{ijk}{}^l$ of (M, g) are zero, where $\| df \|_g^2 = f_a f_b g^{ab}$.

We get from Lemma 2.1 and 2.2

Lemma 2.3. *Let S, $\overline{S}$ be $\widetilde{S}$ be the Ricci tensors of (B, g), $(F, \bar{g})$ and the warped product manifold $(M, \tilde{g})$ respectively. Then*

$$\left\{\begin{array}{l} \widetilde{S}_{ab} = S_{ab} - pf^{-1}\nabla_a f_b, \quad \widetilde{S}_{ax} = 0, \\ \widetilde{S}_{xy} = \overline{S}_{xy} + f\bar{g}_{xy}\Delta_g f + (1-p)\| df \|_g^2\bar{g}_{xy}, \end{array}\right. \tag{2.4}$$

where $\Delta_g f := -g^{da}\nabla_a f_d$.

Using (2.1) and Lemma 2.3, we get

Proposition 2.4. *Let (B, g) and $(F, \bar{g})$ be n-dimensional and p-dimensional Einstein manifolds with Einstein constants k_1, k_2 respectively. Then, the warped product manifold $(M = B\times_f F,\ \tilde{g})$ is an Einstein manifold with Einstein constant k if and only if*

$$\left\{\begin{array}{l} (k_1 - k)g_{ab} - pf^{-1}\nabla_a f_b = 0, \\ k_2 - kf^2 + f\Delta_g f + (1-p)\| df \|_g^2 = 0. \end{array}\right. \tag{2.5}$$

The tension field $\tau(\phi)$ of a C^∞-map ϕ between two Riemannian manifolds (M,g) and (N,h) can be expressed using the local coordinates (x^i) on M and (y^α) on N as follows(cf. [2]).

$$(2.6)\quad \tau(\phi) = g^{ij}(\phi_{ij}{}^\alpha - \phi_k{}^\alpha\Gamma_{ij}{}^k + \phi_i{}^\beta\phi_j{}^\gamma\widetilde{\Gamma}_{\beta\gamma}{}^\alpha)\frac{\partial}{\partial y^\alpha}.$$

Here $\Gamma_{ij}{}^k$, $\widetilde{\Gamma}_{\beta\gamma}{}^\alpha$ are Cristoffel symbols on $(M,g),(N,h)$ respectively and $\phi_j{}^\alpha$ is the matrix representation of $d\phi$ with respect to the chosen frame fields, and $(g^{ij}) := (g_{ij})^{-1}$ and $\phi_{ij}{}^\alpha := \partial\phi_j{}^\alpha/\partial x^i$.

The following can be obtained by using Lemma 2.1 and (2.6).

Lemma 2.5. *Let π (resp. σ) be the canonical projection of the warped product manifold $(M,\tilde{g})$ onto (B,g) (resp. $(F,\bar{g})$). Then σ is harmonic, and π is harmonic iff f is a constant.*

By virtue of Proposition 2.4 and Lemma 2.5, we get

Theorem 2.6. *Let (B,g) and $(F,\bar{g})$ be Einstein manifolds with Einstein constants k_1, k_2 respectively, and π a harmonic mapping. Then, the warped product manifold $(M = B\times_f F,\ \tilde{g})$ is an Einstein manifold with Einstein constant k if and only if*

$$(2.12)\quad k = k_1 = f^{-2}k_2.$$

3. The twisted manifold

Let (B,g) (resp. $(F,\bar{g})$) be an n-dimensional (resp. p-dimensional) Riemannian manifold and f a positive smooth function on $B\times F$. *The twisted manifold* $M = B\times_f F$ is the differentiable product manifold $B\times F$ equipped with the metric $\tilde{g}$ defined by $\tilde{g} := g + f^2\bar{g}$, i.e.,

$$(3.1)\quad \tilde{g}(X,Y) = g(\pi_*X,\pi_*Y) + f^2\bar{g}(\sigma_*X,\sigma_*Y)$$

for each tangent vector X, Y on M.

The following Lemmas are easily obtained.

Lemma 3.1. *The Christoffel symbols* $\widetilde{\left\{ {}_{i}{}^{k}{}_{j} \right\}}$ *of the Levi-Civita connection* $\widetilde{\nabla}$ *on the twisted manifold* M *are given as follows:*

$$(3.2)\quad \begin{cases} \widetilde{\left\{ {}_{b}{}^{a}{}_{c} \right\}} = \left\{ {}_{b}{}^{a}{}_{c} \right\}, \quad \widetilde{\left\{ {}_{b}{}^{a}{}_{z} \right\}} = 0, \quad \widetilde{\left\{ {}_{y}{}^{a}{}_{z} \right\}} = -f f_b g^{ba} \bar{g}_{yz}, \\ \widetilde{\left\{ {}_{b}{}^{x}{}_{c} \right\}} = 0, \quad \widetilde{\left\{ {}_{b}{}^{x}{}_{z} \right\}} = f^{-1} f_b \delta_z{}^x, \\ \widetilde{\left\{ {}_{y}{}^{x}{}_{z} \right\}} = \overline{\left\{ {}_{y}{}^{x}{}_{z} \right\}} + f^{-1}(f_y \delta_z{}^x + f_z \delta_y{}^x - f_w \bar{g}^{xw} \bar{g}_{yz}). \end{cases}$$

Lemma 3.2. *The relations of the local components of the curvature tensors of* (B, g), $(F, \bar{g})$ *and the twisted manifold* $(M, \tilde{g})$ *are as follows*

$$(3.3)\quad \begin{cases} \widetilde{R}_{abc}{}^{d} = R_{abc}{}^{d}, \quad \widetilde{R}_{ayc}{}^{w} = f^{-1} \delta_y{}^w \widetilde{\nabla}_a f_c, \\ \widetilde{R}_{ayz}{}^{d} = -f g^{dc} \bar{g}_{yz} \widetilde{\nabla}_a f_c, \\ \widetilde{R}_{ayz}{}^{w} = f^{-1}(\delta_y{}^w \widetilde{\nabla}_a f_z - \bar{g}^{xw} \bar{g}_{yz} \widetilde{\nabla}_a f_x), \\ \widetilde{R}_{xyc}{}^{w} = f^{-1}(\delta_y{}^w \widetilde{\nabla}_c f_x - \delta_x{}^w \widetilde{\nabla}_c f_y), \\ \widetilde{R}_{xyz}{}^{d} = f g^{bd}(\bar{g}_{xz},\ \widetilde{\nabla}_y f_b - \bar{g}_{yz} \widetilde{\nabla}_x f_b), \\ \widetilde{R}_{xyz}{}^{w} = \overline{R}_{xyz}{}^{w} + \| \mathrm{d}f \|_{\tilde{g}}{}^2 \delta_x{}^w \bar{g}_{zy} - \| \mathrm{d}f \|_{\tilde{g}}{}^2 \delta_y{}^w \bar{g}_{zx} \\ \quad + f^{-1}(\delta_y{}^w \widetilde{\nabla}_z f_x - \delta_x{}^w \widetilde{\nabla}_z f_y + \bar{g}^{wu} \bar{g}_{zx} \widetilde{\nabla}_u f_y - \bar{g}^{wu} \bar{g}_{zy} \widetilde{\nabla}_u f_x), \end{cases}$$

and the others of $(M, \tilde{g})$ *are zero, where* $\partial_a := \partial/\partial u^a$.

We obtain from Lemma 3.1 and 3.2

Lemma 3.3. *Let* S, $\overline{S}$ *and* $\widetilde{S}$ *be the Ricci tensors of* (B, g), $(F, \bar{g})$ *and the twisted manifold* $(M, \tilde{g})$ *respectively Then*

$$(3.4)\quad \begin{cases} \widetilde{S}_{ab} = S_{ab} - p f^{-1} \widetilde{\nabla}_a f_b, \quad \widetilde{S}_{ax} = (1-p) f^{-1} \widetilde{\nabla}_a f_x, \\ \widetilde{S}_{xy} = \overline{S}_{xy} + (p-1) \| df \|_{\tilde{g}}{}^2 \bar{g}_{xy} + (2-p) f^{-1} \widetilde{\nabla}_y f_x \\ \qquad + f \bar{g}_{xy} \Delta_{\tilde{g}} f. \end{cases}$$

Using (3.1) and Lemma 3.3, we get

Proposition 3.4. *Let (B, g) and $(F, \bar{g})$ be n-dimensional and p (≥ 2)-dimensional Einstein manifolds with Einstein constants k_1, k_2 respectively. Then, the twisted manifold $(M = B\times_f F,\ \tilde{g})$ is an Einstein manifold with Einstein constant k if and only if*

$$
\begin{cases}
(k_1 - k)g_{ab} - pf^{-1}\widetilde{\nabla}_a f_b \; = \; 0, \quad f^{-1} f_a f_x \; = \; \partial_a f_x, \\
\{(k_2 - kf^2) \; + \; (p-1)\| \, df \, \|_{\bar{g}}{}^2 \\
\qquad\qquad\qquad + f\Delta_{\bar{g}} f\}\bar{g}_{xy} \; + (2-p)f^{-1}\widetilde{\nabla}_y f_x \; = \; 0.
\end{cases}
\tag{3.5}
$$

The following can be obtained by using (2.6) and Lemma 3.1.

Lemma 3.5. *Let π (resp. σ) be the canonical projection of the twisted manifold $(M, \tilde{g})$ onto (B, g) (resp. $(F, \bar{g})$). Then π is harmonic iff $f_c = 0$ $(c = 1, 2, \ldots, n)$. Moreover, if $p = 2$, σ is harmonic, and if $p > 2$, σ is harmonic iff $f_z = 0$ $(z = n + 1, n + 2, \ldots, n + p)$.*

By virtue of Proposition 3.4 and Lemma 3.5, we get

Theorem 3.6. *Let (B, g) and $(F, \bar{g})$ be Einstein manifolds with Einstein constants k_1, k_2 respectively. Assume f is a harmonic function, π is a harmonic map, and $\dim F = 2$. Then, the twisted manifold $(M = B\times_f F,\ \tilde{g})$ is an Einstein manifold with Einstein constant k if and only if*

$$
\begin{cases}
k = k_1, \\
\| \, df \, \|_{\bar{g}}{}^2 = kf^2 - k_2.
\end{cases}
\tag{3.6}
$$

4. The doubly warped product manifold

Let (B, g) (resp. $(F, \bar{g})$) be an n-dimensional (resp. p-dimensional) Riemannian manifold and f (resp. h) a positive smooth function on

B (resp. F). *The doubly warped product manifold* $M = B\times_{(h,f)}F$ is the differentiable product manifold $B \times F$ equipped with the metric $\tilde{g}$ defined by $\tilde{g} := h^2 g + f^2 \bar{g}$, i.e.,

$$\tilde{g}(X,Y) = h^2 g(\pi_* X, \pi_* Y) + f^2 \bar{g}(\sigma_* X, \sigma_* Y) \tag{4.1}$$

for each tengent vector X, Y on M , where π (resp. σ) is the canonical projection of M onto B (resp. F).

The following Lemmas are easily obtained.

Lemma 4.1. *The Christoffel symbols* $\widetilde{\left\{ {k \atop i\ j} \right\}}$ *of the Levi-Civita connection* $\widetilde{\nabla}$ *on the doubly warped product manifold* M *are given as follows:*

$$\left\{ \begin{aligned} \widetilde{\left\{ {a \atop b\ c} \right\}} &= \left\{ {a \atop b\ c} \right\}, & \widetilde{\left\{ {a \atop b\ z} \right\}} &= h^{-1} h_z \delta_b{}^a, \\ \widetilde{\left\{ {a \atop y\ z} \right\}} &= -f h^{-2} f_b g^{ba} \bar{g}_{yz}, & \widetilde{\left\{ {x \atop b\ c} \right\}} &= -h f^{-2} h_y \bar{g}^{xy} g_{bc}, \\ \widetilde{\left\{ {x \atop b\ z} \right\}} &= f^{-1} f_b \delta_z{}^x, & \widetilde{\left\{ {x \atop y\ z} \right\}} &= \overline{\left\{ {x \atop y\ z} \right\}}. \end{aligned} \right. \tag{4.2}$$

Lemma 4.2. *Let* R, $\overline{R}$ *and* $\widetilde{R}$ *be the curvature tensors of* (B,g), $(F,\bar{g})$ *and the doubly warped product manifold* $(M,\tilde{g})$ *respectively. Then*

$$\left\{ \begin{aligned} \widetilde{R}_{abc}{}^d &= R_{abc}{}^d + f^{-2} \| \, dh \, \|_{\bar{g}}{}^2 (\delta_b{}^d g_{ca} - \delta_a{}^d g_{cb}), \\ \widetilde{R}_{abc}{}^w &= h f^{-3} h_z \bar{g}^{wz} (f_a g_{bc} - f_b g_{ac}), \\ \widetilde{R}_{abz}{}^d &= (hf)^{-1} h_z (f_b \delta_a{}^d - f_a \delta_b{}^d), \\ \widetilde{R}_{ayc}{}^d &= (hf)^{-1} h_y (f_c \delta_a{}^d - f_e g_{ac} g^{dc}), \\ \widetilde{R}_{ayc}{}^w &= f^{-1} \delta_y{}^w \nabla_a f_c + h f^{-2} g_{ac} \bar{g}^{zw} \overline{\nabla}_y h_z, \\ \widetilde{R}_{ayz}{}^d &= -f h^{-2} g^{dc} \bar{g}_{yz} \nabla_a f_c - h^{-1} \delta_a{}^d \overline{\nabla}_y h_z, \\ \widetilde{R}_{ayz}{}^w &= (hf)^{-1} f_a (h_x \bar{g}^{wx} \bar{g}_{yz} - h_z \delta_y{}^w), \\ \widetilde{R}_{xyc}{}^w &= (hf)^{-1} f_c (h_y \delta_x{}^w - h_x \delta_y{}^w), \\ \widetilde{R}_{xyz}{}^d &= f h^{-3} f_a g^{da} (h_x \bar{g}_{yz} - h_y \bar{g}_{xz}), \\ \widetilde{R}_{xyz}{}^w &= \overline{R}_{xyz}{}^w \ - \ h^{-2} \| \, df \, \|_g^2 (\bar{g}_{yz} \delta_x{}^w \ - \ \bar{g}_{xz} \delta_y{}^w), \end{aligned} \right. \tag{4.3}$$

and the others $\widetilde{R}_{ijk}{}^{l}$ of (M, g) are zero.

The following Lemma can be obtained by using Lemma 4.2.

Lemma 4.3. *Let S, $\overline{S}$ and $\widetilde{S}$ be the Ricci tensors of (B,g), $(F,\bar{g})$ and the doubly warped product manifold $(M,\tilde{g})$ respectively. Then*

$$(4.4)\quad \begin{cases} \widetilde{S}_{ab} = S_{ab} + (1-n)f^{-2}\| \, dh \, \|_{\bar{g}}^{2} g_{ab} + hf^{-2} g_{ab}\Delta_{\bar{g}} h - pf^{-1}\nabla_a f_b, \\ \widetilde{S}_{ax} = (n+p-2)(hf)^{-1} h_x f_a, \\ \widetilde{S}_{xy} = \overline{S}_{xy} + (1-p)h^{-2}\| \, df \, \|_{g}^{2} \bar{g}_{xy} + fh^{-2}\bar{g}_{xy}\Delta_g f - nh^{-1}\overline{\nabla}_x h_y. \end{cases}$$

We get from (4.1) and Lemma 4.3

Proposition 4.4. *Let (B,g) and $(F,\bar{g})$ be n-dimensional and p-dimensional Einstein manifolds with Einstein constants k_1, k_2 respectively. Then, the doubly warped product manifold $(M = B\times_{(h,f)}F, \tilde{g})$ is an Einstein manifold with Einstein constant k if and only if*

$$(4.5)\quad \begin{cases} (k_1 - kh^2)g_{ab} + (1-n)f^{-2}\| \, dh \, \|_{\bar{g}}^{2} g_{ab} + hf^{-2}g_{ab}\Delta_{\bar{g}} h \\ -pf^{-1}\nabla_a f_b \ = \ 0, \quad h_x f_a \ = \ 0, \\ (k_2 - kf^2)\bar{g}_{xy} + (1-p)h^{-2}\| \, df \, \|_{g}^{2}\bar{g}_{xy} \\ \qquad\qquad + fh^{-2}\bar{g}_{xy}\Delta_g f - nh^{-1}\overline{\nabla}_x h_y \ = \ 0 \end{cases}$$

Using (2.6) and Lemma 4.1, we have

Lemma 4.5. *Let π (resp. σ) be the canonical projection of the doubly warped product manifold $(M,\tilde{g})$ onto (B,g) (resp. $(F,\bar{g})$). Then σ is harmonic iff h is constant, and π is harmonic iff f is a constant.*

By virtue of Proposition 4.4 and Lemma 4.5, we get

Theorem 4.6. *Let (B,g) and $(F,\bar{g})$ be Einstein manifolds with Einstein constants k_1, k_2 respectively, and π and σ harmonic maps.*

Then, the doubly warped product manifold $(M = B \times_{(h,f)} F,\ \tilde{g})$ *is an Einstein manifold with Einstein constant* k *if and only if*

$$(4.6) \quad k = h^{-2} k_1 = f^{-2} k_2 .$$

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